

# Equitable DP colorings of graphs

Alexandr Kostochka

University of Illinois Urbana-Champaign

joint work with **Henry A. Kierstead** and **Zimu Xiang**



# Definitions

An **equitable coloring** of a graph is a proper vertex coloring such that the sizes of any two color classes differ by at most 1.

## APPLICATIONS:

1. Scheduling, partitioning, and load balancing problems.
2. Deviation bounds for sums of random variables with limited dependence [Alon-Füredi, Janson-Ruciński, Pemmaraju].
3.  $H$ -factors in graphs [Alon-Yuster].
4. A shorter proof of Blow-Up Lemma [Rödl-Ruciński].
5. Bounds on order statistics of independent random variables [Jacobson-Kaul].

A graph may have an equitable  $k$ -coloring but have no equitable  $(k + 1)$ -colorings.

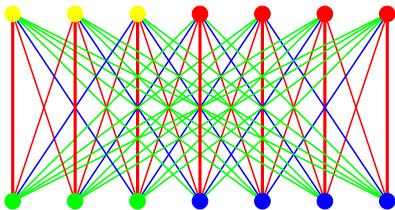


Figure: An equitable 4-coloring of  $K_{7,7}$ .

To decide whether a graph has an equitable  $k$ -coloring is an *NP-complete problem* even for  $k = 3$ .

Conjecture 1 [Erdős, 1964] If  $H$  is a graph on  $n = km$  vertices with  $\delta(H) \geq n(1 - \frac{1}{m})$ , then  $V(H)$  has a  $k$ -partition into  $m$ -cliques.

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**Grünbaum** in 1968 reformulated the conjecture as a statement about equitable coloring and proved the case  $k = 4$ .

Hajnal and Szemerédi in 1970 proved the conjecture:

**Theorem 1 [Hajnal and Szemerédi]:** If  $\Delta(G) \leq r$ , then  $G$  is equitably  $(r + 1)$ -colorable.

The proof did not yield a polynomial algorithm for an equitable  $(r + 1)$ -coloring of graphs  $G$  with  $\Delta(G) \leq r$ . In 2010, **Kierstead-A.K.-Mydlarz-Szemerédi** presented an  $O(rn^2)$  time algorithm for such coloring.

Conjecture 2 [Chen, Lih and Wu, 1994]: Let  $G$  be a **connected** graph with  $\Delta(G) \leq r$ . Then  $G$  has no equitable  $r$ -coloring if and only if either (1)  $G = K_{r+1}$ , or (2)  $r = 2$  and  $G$  is an odd cycle, or (3)  $r$  is **odd** and  $G = K_{r,r}$ .

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**Conjecture 2** was proved:

- (1) for  $r \leq 3$  and for  $r \geq n/2$  [Chen-Lih-Wu],
- (2) for  $r \leq 4$  and for  $r \geq n/4$  [Kierstead-K],
- (3) for **bipartite** graphs [Lih-Wu],
- (4) for **interval** graphs [Chen-Lih-Yan],
- (5) for **split** graphs [Chen-Ko-Lih],
- (6) for **outerplanar** graphs [Yap-Zhang],
- (7) for **planar** graphs  $G$  with  $\Delta(G) \geq 8$  [Yap-Zhang, Nakprasit, K-Lin-Xiang].

## Fewer colors

A direction of study is trying to color "sparse" graphs  $G$  with fewer than  $\Delta(G)$  colors.

**Theorem 2 [Bollobás and Guy]:** A tree  $T$  is equitably 3-colorable if  $|T| \geq 3\Delta(T) - 8$  or  $|T| = 3\Delta(T) - 10$ .

Given a graph  $G$  and a vertex  $v \in V(G)$ , let  $\alpha_v = \alpha_v(G)$  denote the size of the maximum independent set in  $G$  containing  $v$ . If a graph  $G$  has an equitable  $k$ -coloring, then by definition,  $\alpha_v(G) \geq \lfloor n/k \rfloor$  for every  $v \in V(G)$ .

**Theorem 3 [Chen-Lih and Miyata-Tokunaga-Kaneko, 1994]:** For a forest  $T$  of order  $n$  and integer  $k \geq 3$ ,  $T$  is equitably  $k$ -colorable if and only if  $\alpha_v \geq \lfloor n/k \rfloor$  for every vertex  $v \in T$ .

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**Cranston and Mahmood** proved the similar result for outerplanar graphs and  $k \geq 6$ .

# Equitable list coloring

A *k-uniform list assignment*  $L$  for a graph  $G$  assigns to each vertex  $v \in V(G)$  a set  $L(v)$  of  $k$  allowable colors.

Given a *k-uniform* list assignment  $L$  for an  $n$ -vertex graph  $G$ , we say that  $G$  is *equitably  $L$ -colorable* if  $G$  has a proper coloring  $f$  such that  $f(v) \in L(v)$  for all  $v \in V(G)$  and no color is used on more than  $\lceil n/k \rceil$  vertices.

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A graph  $G$  is *equitably  $k$ -choosable* if  $G$  is equitably  $L$ -colorable for each *k-uniform* list assignment  $L$  for  $G$ .

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In contrast to ordinary equitable coloring, in this setting it is possible that the sizes of two color classes differ significantly, as some color may appear only in the lists of few vertices.

# Equitable list coloring

Conjecture 2 [A.K.–Pelsmajer–West, 2003]: Every graph  $G$  is equitably  $(1 + \Delta(G))$ -choosable.

Conjecture 2 was proved

- (1) for  $\Delta(G) = 3$  [Pelsmajer, Lih–Wang],
- (2) for  $\Delta(G) \geq n/2$  [A.K.–Pelsmajer–West],
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- (3) for  $\Delta(G) \leq 7$  [Kierstead–A.K.]

A graph may be **equitably  $k$ -choosable** but not **equitably  $k$ -colorable**, and vice versa.

In particular, for each integer  $k \geq 3$ , the star  $K_{1,2k}$  is **equitably  $k$ -choosable** but not equitably  $k$ -colorable.

## A variation

Given an  $n$ -vertex graph  $G$  with a  $k$ -uniform list  $L$ , where  $n = sk + j$ ,  $0 \leq j < k$ , an equitable  $L$ -coloring  $f$  of  $G$  is **strongly equitable** (SE, for short) if the number of **color classes of size  $s + 1$**  is at most  $j$ .

So, if  $G$  is **SE  $k$ -choosable**, then it is **equitably  $k$ -colorable** and **equitably  $k$ -choosable**.

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**Theorem 6 [Kierstead–A.K.–Xiang, 2024]:** Every planar graph  $G$  is **SE  $k$ -choosable**, if  $k \geq \max\{9, \Delta(G)\}$ .

**Corollary 7 [K.–K.–X., 2024]:** Conjecture 2 holds **for every planar graph**.

About 2015, Dvořák and Postle in order to solve a problem on list coloring, introduced a more general concept.

We call it **DP coloring**.

Given a graph  $G$  with list assignment  $L$ , let  $\tilde{L}(v) := (v, \alpha)$  for all  $v \in V(G)$  and  $\alpha \in L(v)$ .

Also for every  $uv \in E(G)$ , let  $M(uv)$  be some **matching** between  $\tilde{L}(u)$  and  $\tilde{L}(v)$  (possibly, empty).

Then a **cover**  $H = H(L, M)$  of  $G$  is a graph with

$V(H) = \bigcup_{v \in V} \tilde{L}(v)$  and  $E(H) = \bigcup_{uv \in E} M(uv)$ .

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An  $H$ -coloring of  $G$  is a function  $f : V \rightarrow \Gamma = \bigcup_{v \in V(G)} L(v)$  with  $f(v) \in H(v)$  such that  $\{(v, f(v)) : v \in V\}$  is independent in  $H$ .

The graph  $G$  is **DP  $k$ -colorable** if it has an  $H$ -coloring for every  $k$ -cover  $H$  of  $G$ .

Given a  $k$ -cover  $H$  for an  $n$ -vertex graph  $G$ , we say that  $G$  is *equitably  $H$ -colorable* if  $G$  has an  $H$ -coloring  $f$  such that no color is used on more than  $\lceil n/k \rceil$  vertices.

A graph  $G$  is *EDP  $k$ -choosable* if  $G$  is equitably  $H$ -colorable for each  $k$ -cover  $H$  of  $G$ .

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A graph  $G$  is *EDP  $k$ -choosable* if  $G$  is equitably  $H$ -colorable for each  $k$ -cover  $H$  of  $G$ .

Similarly, if  $H$  is a  $k$ -cover for an  $n$ -vertex graph  $G$ , we say that  $G$  is *SE  $H$ -colorable* if  $G$  has an equitable  $H$ -coloring  $f$  such that the number of color classes of size  $s + 1$  is at most  $j$ .

A graph  $G$  is *SEDP  $k$ -choosable* if  $G$  is SE  $H$ -colorable for each  $k$ -cover  $H$  of  $G$ .

**Example 1:** Every path is DP 2-colorable and SEL 2-colorable.  
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**Figure:** Forests  $T$  (left) and  $F_3$  (right) are not EDP 3-colorable.

**Example 2:** All cycles are DP and SEL  $k$ -colorable for all  $k \geq 3$ . But for all cycles  $C_n = v_1 \dots v_n v_1$ :

- (a)  $C_3$  is neither EDP 3-colorable nor EDP 4-colorable;
- (b)  $C_4$  is neither SEDP 3-colorable nor EDP 4-colorable; and
- (c)  $C_6$  is not EDP 3-colorable.

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**Example 2a:** For every  $n \geq 3$  divisible by 3, there is a 2-regular graphs that is not EDP 3-colorable.

We think we proved that for each  $n \geq 7$  cycle  $C_n$  is SEDP  $k$ -colorable.

**Example 3:** For every  $k \geq n$  every  $n$ -vertex graph is SEL  $k$ -choosable and DP  $k$ -colorable. However: the  $n$ -vertex,  $d$ -degenerate graph  $G_{n,d} := K_n - E(K_{n-d})$ , is not EDP  $k$ -colorable for

$$n \leq k \leq n + d - 1 - ((n + d - 1) \bmod 2).$$

In particular, if  $n \leq k \leq 2n - 2$ , then  $K_n$  is not EDP  $k$ -colorable; if  $n = 2s + 1$  then  $K_{1,2s}$  also is not EDP  $n$ -colorable.

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**Proposition 9 [K.–K.–X., 2025+]:** Let  $0 \leq d < n$ . For every  $k \geq n + d$  each  $n$ -vertex  $d$ -degenerate graph  $G$  is SEDP  $k$ -colorable. Furthermore, only  $G_{n,d}$  is not SEDP  $(n + d - 1)$ -colorable.

**Example 4:** For all positive integers  $n$  and all  $k \geq n + 1$ , the complete bipartite graph  $K_{n,n}$  is DP  $k$ -colorable and equitably  $k$ -colorable. It is conjectured that it is also SEL  $k$ -colorable. But  $G := K_{n,n}$  is not EDP  $2n$ -colorable.

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**Example 5:** For any  $s \geq 1$ , the star  $K_{1,2s}$  is SEL  $(s + 1)$ -colorable and DP  $(s + 1)$ -colorable. But  $K_{1,n-1}$  is not SEDP  $k$ -colorable if  $k \leq n$ .

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**Proposition 10 [K.–K.–X., 2025+]:** For every  $\frac{n+2}{2} \leq k < n$  each  $n$ -vertex forest  $G$  is EDP  $k$ -colorable.

**Example 6:** For every  $k \geq 1$  every graph  $G$  with  $\Delta(G) \leq k$  is DP  $(k + 1)$ -colorable. By Hajnal-Szemerédi Theorem, every such graph is equitably  $(k + 1)$ -colorable. It is conjectured that every such graph is also SEL  $(k + 1)$ -colorable. However for all positive integers  $k$ , there are infinitely many graphs  $G$  with  $\Delta(G) \leq k$  that are not EDP  $(k + 1)$ -colorable.

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**Theorem 11 [K.–K.–X., 2025+]:** Every graph  $G$  is SEDP  $k$ -colorable for every  $k \geq 3(\Delta(G))^2$ .

## Questions/Problems

1. Let  $g(D)$  (respectively,  $g_{se}(D)$ ) be the minimum  $k_0$  such that for every  $k \geq k_0$  every graph with maximum degree at most  $D$  is EDP (respectively, SEDP)  $k$ -colorable. It would be interesting to determine whether  $g(D)$  and/or  $g_{se}(D)$  are  $O(D)$ .

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2. It looks that the differences between the bounds for EDP coloring and EL coloring are more significant when the graphs are dense. In particular, we think that every  $n$ -vertex graph with maximum degree less than  $n/2$  is EDP  $n$ -colorable.
3. As observed by Dvořák and Postle, in DP  $k$ -coloring it is enough to consider plain  $k$ -covers. In all cases we considered, if a graph  $G$  with a  $k$ -cover  $H$  is not SE  $H$ -colorable, then also there is a plain  $k$ -cover  $H'$  such that  $G$  is not SE  $H'$ -colorable. It would be interesting to determine whether this always holds for SEDP colorings. This is not the case for SEL coloring.