Partition of Sparse Multigraphs into a Forest and a Forest with Restrictions

Alexandr Kostochka

University of Illinois at Urbana-Champaign

joint work with Ilkyoo Choi and Matthew Yancey

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We consider partitions of vertex sets of multigraphs and simple graphs into two forests.

The *vertex arboricity* $va(G)$ of a multigraph G is the minimum k such that there is a vertex partition $(V_1, \ldots, V_k)$ of $V(G)$ such that the subgraph $G[V_i]$ of G induced by V_i has no cycles for each $1 \leq i \leq k$. (Introduced by Chartrand, Kronk, and Wall in 1968.)

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Since each forest is 2-colorable, $\chi(G) \leq 2va(G)$ for every G.
In a series of papers it was proved that broad subclasses of planar graphs have vertex arboricity at most 2. More general classes of sparse (multi)graphs also have been considered.

Meaningful measures of sparseness of a (multi)graph G are maximum degree, *maximum average degree*, and its refinement, (a, b)-sparseness.

For $a > 0$ and an arbitrary real b, call a multigraph H (a, b)-*sparse* if for every $A \subseteq V(H)$ with $|A| \geq 2$, the induced subgraph $H[A]$ has at most $a|A| + b$ edges. By definition, forests are exactly $(1, -1)$-sparse multigraphs.

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A result of Borodin from 1976, reproved by Bollobás and Manvel in 1979 and Catlin and Lai in 1995, implies that every simple graph G with $\Delta(G) \leq 4$ not containing K_5 satisfies $va(G) \leq 2$.

A corollary: every graph G with $mad(G) \leq 4$ (i.e. $(2, 0)$-sparse) not containing K_5 satisfies $va(G) \leq 2$.

For an integer D, an *$(\mathcal{F}_D^\Delta, \mathcal{F})$-coloring* of a multigraph G is a partition (M, F) of $V(G)$ such that $G[M]$ is a forest with maximum degree at most D and $G[F]$ is a forest.

The case of $D = 0$ (i.e. M is an independent set) of $(\mathcal{F}_D^\Delta, \mathcal{F})$-coloring has attracted much attention.
It was proved that deciding whether a graph has an $(\mathcal{F}_0^\Delta, \mathcal{F})$-coloring is an NP-complete problem even in some classes of graphs with moderate average degree.
In particular, Dross, Montassier, and Pinlou proved this for the class of planar graphs, and Yang and Yuan proved this for the class of graphs with maximum degree at most 4.

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On the other hand, Borodin and Glebov proved that every planar graph of girth at least 5 has an $(\mathcal{F}_0^\Delta, \mathcal{F})$-coloring. Dross, Montassier, and Pinlou conjectured that this holds already for planar graphs of girth at least 4.

Borodin's result from 1976 yields that every simple graph G with $\Delta(G) \leq 3$ not containing K_4 has an $(\mathcal{F}_0^\Delta, \mathcal{F})$-coloring.

Cranston and Yancey (2020) proved that every $(1.5, 0.5)$-sparse multigraph containing neither K_4 nor the *Moser Spindle* has an $(\mathcal{F}_0^\Delta, \mathcal{F})$-coloring. They proved further that there is a finite set $\mathcal{H}$ of graphs such that every $(1.6, 0.8)$-sparse simple graph not containing a subgraph in $\mathcal{H}$ also has an $(\mathcal{F}_0^\Delta, \mathcal{F})$-coloring. Both bounds are exact.

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We consider $(\mathcal{F}_D^\Delta, \mathcal{F})$-colorings for $D \geq 1$.

Theorem 1 (C-K-Y): For each $D \geq 1$, every $\left(\frac{4D+3}{2(D+1)}, \frac{1}{2(D+1)}\right)$-sparse multigraph has an $(\mathcal{F}_D^{\Delta}, \mathcal{F})$-coloring.

Theorem 2 (C-K-Y): For each $D \geq 2$, every $\left(\frac{6D+5}{3(D+1)}, \frac{2}{3(D+1)}\right)$-sparse simple graph has an $(\mathcal{F}_D^{\Delta}, \mathcal{F})$-coloring.

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Both bounds are exact. When $D = 1$, the bound of Theorem 2 does not hold as there are more sparse "critical" graphs.

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We will derive the result of Liu and Wang from Theorem 2. Indeed, suppose G is a simple, plane graph with neither a 3-cycle nor a chorded 6-cycle that is $(\mathcal{F}_2^\Delta, \mathcal{F})$-critical. If G has a vertex v of degree at most 2, then an $(\mathcal{F}_2^\Delta, \mathcal{F})$-coloring of $G - v$ can be extended to G by adding v to M if all neighbors of v are in F, and adding v to F otherwise. So $\delta(G) \geq 3$.

Since G is a simple graph without 3-cycles, chorded 6-cycles, or 2-vertices, there are no adjacent faces with length 4. We now apply discharging.

Give each face an initial charge equal to its length. Each face with length at least 5 gives charge $1/9$ to each face that shares an edge with it (with multiplicity, it shares more than one edge).

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By the above, each face ends with charge at least 40/9.
Let n, e, f denote the number of vertices, edges, and faces of G.
By Euler's formula we have $n - e + f = 2$. By above, $f \leq 9e/20$.
Thus $e \leq \frac{20}{11}(n-2) < 1.82n$. This contradicts Theorem 2 which states that $e \geq \frac{17n+3}{9} > 1.88n$.

A similar problem:

An $(\mathcal{F}_D^e, \mathcal{F})$*-coloring* of a multigraph G is a partition (M, F) of $V(G)$ such that $G[M]$ is a forest in which every component has at most D edges and $G[F]$ is a forest.

For $D \in \{0, 1\}$, an $(\mathcal{F}_D^e, \mathcal{F})$-coloring is simply an $(\mathcal{F}_D^\Delta, \mathcal{F})$-coloring, but for $D \geq 2$, an $(\mathcal{F}_D^e, \mathcal{F})$-coloring is significantly more restrictive than an $(\mathcal{F}_D^\Delta, \mathcal{F})$-coloring.

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For $D \in \{0, 1\}$, an $(\mathcal{F}^e_D, \mathcal{F})$-coloring is simply an $(\mathcal{F}^\Delta_D, \mathcal{F})$-coloring, but for $D \geq 2$, an $(\mathcal{F}^e_D, \mathcal{F})$-coloring is significantly more restrictive than an $(\mathcal{F}^\Delta_D, \mathcal{F})$-coloring.

Theorem 3 (C-K-Y): (i) For each odd $D \geq 1$, every $\left(\frac{4D+3}{2D+2}, \frac{1}{2(D+1)}\right)$-sparse multigraph has an $(\mathcal{F}^e_D, \mathcal{F})$-coloring.

(ii) For each even $D \geq 2$, every $\left(\frac{4D+1}{2D+1}, \frac{1}{2D+1}\right)$-sparse multigraph has an $(\mathcal{F}^e_D, \mathcal{F})$-coloring.

Both parts of Theorem 3 are sharp.

Set up

We generalize the concepts of $(\mathcal{F}_D^{\Delta}, \mathcal{F})$-coloring and $(\mathcal{F}_D^{e}, \mathcal{F})$-coloring to weighted graphs:

For a multigraph G, let $w : V(G) \to \{1, \ldots, D+2\}$ denote a weighting of the vertices.
For a weighted (multi)graph G, we say that a vertex partition (M, F) of $V(G)$ is an $(\mathcal{F}_D^{\Delta}, \mathcal{F})$-*coloring* if $G[M]$ and $G[F]$ are forests and for each $v \in M$ we have $w(v) + |E(v, M)| \leq D + 1$.

We consider multi-edges as a 2-cycle, and therefore multi-edges cannot exist in $G[M]$ or $G[F]$. Similarly, (M, F) is an $(\mathcal{F}_D^{e}, \mathcal{F})$-*coloring* if $G[M]$ and $G[F]$ are forests and each component C of $G[M]$ satisfies $\sum_{v \in C} w(v) \leq D + 1$.

Note that both definitions reduce to the original definitions when $w \equiv 1$.