Matchings in bipartite graphs

Lecture 13
Matchings

A matching in a graph is a set of non-loop edges that are pairwise disjoint. The size of a maximum matching in $G$ is denoted by $\alpha'(G)$.

Given a matching $M$ in a graph $G$, an $M$-alternating path in $G$ is a path that alternates between edges in $M$ and not in $M$.

An $M$-augmenting path is an $M$-alternating path whose endpoints are not in any edge of $M$.

Theorem 3.1 (Berge)
(A) A matching $M$ in a graph $G$ is maximum if and only if (B) $G$ does not contain any $M$-augmenting path.
Bipartite graphs

Given a bipartite graph $G = (X, Y; E)$, certainly, $\alpha'(G) \leq \min\{|X|, |Y|\}$. But it can be smaller.

The fundamental result for bipartite graphs is the Hall Theorem.
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Proof. The ”only if” part is evident. We prove the ”if” part by induction on $|E(G)|$. Let a bigraph $G = (X, Y; E)$ satisfy (1). Then $d(x) \geq 1$ for each $x \in X$. 


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**Base of induction:** $|E(G)| = 1$. Since $d(x) \geq 1$ for each $x \in X$, this means $|X| = 1$, and the unique edge of $G$ forms a matching covering $X$. 


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Induction Step. Suppose the theorem is true for all graphs with less than $m$ edges. Let $G = (X, Y; E)$ have $m$ edges.
**Case 1:** $|N(S)| = |S|$ for some $\emptyset \neq S \subsetneq X$. Define induced subgraphs $G_1$ and $G_2$ of $G$: $V(G_1) = S \cup N_G(S)$ and $G_2 = G - V(G_1)$.

**Claim 1.** (1) holds for $G_1$. 

**Claim 2.** (1) holds for $G_2$. Indeed, if there is $T \subset X - S$ with $|N_{G_2}(T)| < |T|$, then $|N_{G_2}(S \cup T)| = |N_G(S)| + |N_{G_2}(T)| < |S| + |T| = |S \cup T|$, a contradiction.
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Claim 1. (1) holds for \(G_1\).

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Indeed, if there is \(T \subset X - S\) with \(|N_{G_2}(T)| < |T|\), then

\[
|N_G(S \cup T)| = |N_G(S)| + |N_{G_2}(T)| < |S| + |T| = |S \cup T|,
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a contradiction.
In view of Claims 1 and 2, by the induction assumption, \( G_1 \) has a matching \( M_1 \) covering \( S \) and \( G_2 \) has a matching \( M_2 \) covering \( X - S \). Now \( M_1 \cup M_2 \) covers \( X \).
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**Case 2:**

\[ |N_{G}(S)| \geq |S| + 1 \quad \forall \emptyset \neq S \subset X. \quad (2) \]

Choose any $x_0 \in X$. Since $d(x_0) \geq 1$, there is $y_0 \in N(x_0)$. Let $G' = G - x_0 - y_0$. 
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By (2), for each $\emptyset \neq S \subset X - x_0$,

$$|N_{G'}(S)| \geq |N_G(S)| - 1 \geq (|S| + 1) - 1 = |S|.$$
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So (1) holds for \( G' \), and by IH, \( G' \) has a matching \( M' \) covering \( X - x_0 \).

Then matching \( M' \cup \{x_0y_0\} \) covers \( X \), as claimed. \( \square \)
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**Corollary 3.3 (Marriage Theorem)** For each \( k \geq 1 \) every \( k \)-regular bipartite graph has a perfect matching.

**Proof.** Let \( B = (X, Y; E) \) be a \( k \)-regular bipartite graph. Since each edge of \( B \) has exactly one endpoint in \( X \), and exactly one in \( Y \),

\[
|E(B)| = \sum_{v \in X} d(v) = k|X|,
\]

and

\[
|E(B)| = \sum_{v \in Y} d(v) = k|Y|,
\]

so \( |X| = |Y| \).

Thus each matching that covers \( X \) is perfect. Let us check that Hall’s condition is satisfied.
Let $S \subseteq X$. There are exactly $k|S|$ edges incident with vertices in $S$, so there are at least $k|S|$ edges incident with $N(S)$, and the total number of edges incident with $N(S)$ is $k|N(S)|$, so

$$k|S| \leq k|N(S)|,$$

which is equivalent to Hall’s condition. Thus, we are done by Hall’s Theorem.

Systems of distinct representatives.
Vertex covers

A vertex cover of a graph $G$ is a set $S$ of vertices in $G$ such that each edge of $G$ has at least one end in $S$.

Trivially, $V(G)$ is a vertex cover of $G$. The problem is to find a vertex cover of the minimum cardinality.

The minimum cardinality of a vertex cover of $G$ is denoted by $\beta(G)$.
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Observation A: A set $S \subseteq V(G)$ is a vertex cover if and only if $V(G) - S$ is an independent set.

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Observation C: For each graph $G$, $\alpha'(G) \leq \beta(G) \leq 2\alpha'(G)$. 
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Proof. Let $G = (X, Y; E)$ be a bipartite graph with parts $X$ and $Y$. By Observation C, we need only to prove $\alpha'(G) \geq \beta(G)$.

Let $Q$ be a vertex cover of $G$ with $|Q| = \beta(G)$.
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Claim: (i) $\forall A \subseteq Q \cap X$, \[ |N(A) - Q \cap Y| \geq |A|. \]
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Proof of Claim (i). If for some $A \subseteq Q \cap X \ |N(A) - Q \cap Y| < |A|$, then the set $(Q - A) \cup N(A)$ is a smaller vertex cover. The proof of (ii) is symmetric.
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By the claim and Hall's Theorem, graph $G[(Q \cap X) \cup (Y - Q)]$ has a matching $M_X$ covering $Q \cap X$ and graph $G[(Q \cap Y) \cup (X - Q)]$ has a matching $M_Y$ covering $Q \cap Y$. 