Spanning jellyfishes in graphs

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based on joint work with Jaehoon Kim and Ruth Luo

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Finding longest or just long cycles and paths in a graph is a famous computationally difficult problem. Since this problem is NP-hard, there is no hope to find fast exact algorithms.

One of the approaches to attack the problem is to study extremal problems: we study how short can be a longest cycle/path in a graph if some other parameter is bounded.

Popular parameters are the number of edges and the minimum degree of a graph.

Dirac's Theorem

Theorem 1 [Dirac 1952]: Let $n \geq 3$. Every n-vertex graph G with $\delta(G) \geq n/2$ has a hamiltonian cycle.

Theorem 1 has many strengthenings and generalizations, e.g. in terms of $\sigma_2(G) = \min_{uv \notin E(G)} d(u) + d(v)$:

Theorem 2 [Ore, 1960]: Let $n \geq 3$. Every n-vertex graph G with $\sigma_2(G) \geq n$ has a hamiltonian cycle.

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The bounds of Dirac's Theorem and Ore's Theorem are sharp. So, a natural question is: what kind of subgraphs can we provide if the minimum degree of an n-vertex graph is high, but less than $(n-1)/2$ (respectively, $n/2$)?

One direction is seeking for longest possible cycles and paths. Let $c(G)$ (respectively, $p(G)$) denote the number of vertices in a longest cycle (respectively, path) in G.

Theorem 3 [Dirac 1952]: Let $n \geq k \geq 3$. For every n-vertex graph G with $\delta(G) \geq k$, $c(G) \geq k+1$. If in addition G is 2-connected, then $c(G) \geq \min\{n, 2k\}$.

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This also implies a lower bound on the length of a longest path in a connected graph:

For every n-vertex graph G with $\delta(G) \geq k \geq 1$, $p(G) \geq k+1$. If in addition G is connected, then $p(G) \geq \min\{n, 2k+1\}$.

Another direction is looking for specific spanning trees or spanning unicyclic subgraphs.

Win in 1975 showed that each n-vertex connected graph with $\delta(G) \geq (n-1)/k$ contains a spanning tree T with $\Delta(T) \leq k$.

Broersma and Tuinstra in 1998 proved that every n-vertex connected graph with $\delta(G) \geq (n-k+1)/2$ contains a spanning tree with at most k leaves.

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Some results have been obtained on the existence of spanning trees with bounded number of *branching vertices*, i.e., vertices of degree at least 3.

Conjecture 4, Ozeki and Yamashita, 2011. For all $s \geq 1$, if G is an n-vertex connected graph with $\delta(G) \geq \frac{n-s}{s+3}$, then G contains a spanning tree with at most s branch vertices.

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Gargano, Hell, Stacho and Vaccaro in 2004 proved that every connected n-vertex graph G with $\delta(G) \geq (n-1)/3$ contains a spanning *spider*, i.e. a tree with at most one branching vertex.

They also observed that it is an NP-complete problem to decide whether a graph has a spanning spider.

DeBiasio and Lo in 2019 proved Conjecture 4 asymptotically:

Theorem 5 [DeBiasio and Lo]: Let $s \geq 1$ and $\gamma > 0$. Then there exists $n_0 = n_0(\gamma, s)$ such that for each $n > n_0$ every connected n-vertex graph G with $\delta(G) \geq \left(\frac{1}{s+3} + \gamma\right) n$ contains a spanning tree with at most s branch vertices.

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Chen, Ferrara, Hu, Jacobson and Liu in 2014 sharpened the result of Gargano et. al. as follows:

Theorem 6 [Chen, Ferrara, Hu, Jacobson and Liu] If G is a connected graph of order $n \geq 56$ with $\delta(G) \geq \frac{n-2}{3}$, then G contains a spanning broom.

The restriction $\delta(G) \geq \frac{n-2}{3}$ in Theorem 6 is sharp. ASK FOR A PICTURE.

Chen et al. also considered a unicyclic subgraph similar to a broom: a *jellyfish* is obtained from a broom B by addition of an edge connecting the end of the longest leg to the branching vertex of B.
Sidorenko used the word *keyring* for a jellyfish.

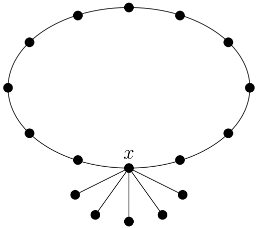

Figure: A jellyfish graph

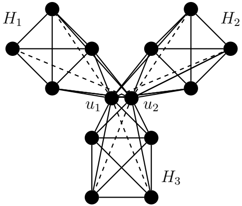

Figure: A 14-vertex graph with no spanning jellyfish

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Theorem 7 [Chen, Ferrara, Hu, Jacobson and Liu.] Let G be a 2-connected graph of order n such that $\delta(G) \geq \frac{n-2}{3}$ and $p(G) - c(G) \leq 1$. Then G contains a spanning jellyfish.

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The inequality $p(G) - c(G) \leq 1$ is a serious restriction. In particular, if we drop this restriction with no other changes, then the conclusion of Theorem 7 does not hold.

Theorem 8 [Kim, A.K. and Luo] For each $n \geq 13$, every 2-connected n-vertex graph G with $\sigma_2(G) \geq \frac{2n-3}{3}$ contains a spanning jellyfish.

This Ore-type result immediately implies:

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Both bounds are sharp. Here, 2-connectedness is required: the connected graph G consisting of disjoint cliques of order $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$ with a single additional edge has $\delta(G) = \lfloor n/2 \rfloor - 1$, $\sigma_2(G) = n - 2$ and no spanning jellyfish.

One of the main ingredients of the proof is a modification of the well-known Hopping Lemma by Woodall from 1972.

Unsolved questions

1. The example in Fig. 2 is 2-connected but not 3-connected. It is natural to ask how higher connectivity affects the minimum degree threshold. What is the minimum $f_k(n)$ such that every n-vertex k-connected graph contains a spanning jellyfish? In particular, what is $f_3(n)$?

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1. The example in Fig. 2 is 2-connected but not 3-connected. It is natural to ask how higher connectivity affects the minimum degree threshold. What is the minimum $f_k(n)$ such that every n-vertex k-connected graph contains a spanning jellyfish? In particular, what is $f_3(n)$?

Let $n = 4k$. Consider an n-vertex graph F with vertex partition $V_1 \cup V_2 \cup V_3 \cup V_4$ with $|V_1| = ... = |V_4| = k$, where $F[V_i, V_{i+1}]$ is a complete bipartite graph for each $i \in \{1, 2, 3\}$. As this graph F lacks a spanning jellyfish, $f_k(n) > n/4$ for any $k \leq (n-2)/4$. Determining the precise values of $f_k(n)$ would be an interesting open problem.

2. Call a graph obtained from a spider by replacing a longest path starting from the branching vertex with a cycle an **octopus**. In particular, an octopus is a subdivision of jellyfish.

An interesting problem would be to find the minimum $g_k(n)$ such that each n-vertex k-connected graph G with $\delta(G) \geq g_k(n)$ contains a spanning octopus.
Let $n = 4k + 2$. Consider an n-vertex graph F' with vertex partition $V_1 \cup V_2 \cup V_3 \cup V_4$ with $|V_1| = |V_4| = k + 1$ and $|V_3| = |V_2| = k$. This F' is highly connected with minimum degree $(n - 2)/4$ but lacks a spanning octopus. Determining $g_k(n)$ for $k \in \{1, 2\}$ would already be interesting.

Proof ideas for Theorem 8

For an n-vertex graph G with $\sigma_2(G) \geq \frac{2n-3}{3}$, let
$L = L(G) = \{v \in V(G) : d(v) \leq \frac{n-2}{3}\}$.

For such G, a cycle C in G is ***L*-maximal** if C is a longest cycle and modulo this, $|C \cap L|$ is as large as possible.

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A big part of the proof is a proof of the following lemma.

Lemma 10: If G is an n-vertex 2-connected graph with $\sigma_2(G) \geq \frac{2n-3}{3}$ and G does not contain a spanning jellyfish, then for every $L(G)$-maximal cycle C, $G - C$ has no edges.

Now, all components of $G - C$ are singletons. Let $C = v_1, v_2, \ldots, v_c, v_1$.
We iteratively define the sets $Y_0 (= Y_0(C)), Y_1, Y_2, \ldots$ and $X_1, X_2, \ldots$ as follows.

(X1) $Y_0 = V(G - C)$,
(X2) For $i \geq 1$, $X_i = N_C(Y_{i-1})$, and
(X3) For $i \geq 1$, $Y_i = \{v_i \in V(C) : v_{i-1}, v_{i+1} \in X_i\} \cup Y_{i-1}$.

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Then $Y_0 \subseteq Y_1 \subseteq Y_2 \subseteq \ldots$ and $X_1 \subseteq X_2 \subseteq \ldots$. As n is finite, we may define $X(= X(C)) = \lim_{i \to \infty} X_i$ and $Y(= Y(C)) = \lim_{i \to \infty} Y_i$.

For $W \in \{X, Y\}$, if $u \in W$, then the ***W*-height** of u is defined as $h_W(u) = \min_{i \in \mathbb{N}} \{i : u \in X_i\}$. If $u \notin W$, then we set $h_W(u) = \infty$.

The **height** of an x, x'-path P is $h(P) = \max\{h_X(x), h_X(x')\}$.

A path $P = w_1 \dots w_c$ is a **C-hopping path** if the following hold.

(H1) $w_1, w_c \in X$,
(H2) P does not contain any consecutive vertices in X_1,
(H3) $V(P) = V(C)$, and
(H4) if $j < h(P)$ and $w_s \in Y_j - \{w_1, w_c\}$, then $w_{s-1}, w_{s+1} \in X_j$.

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A path $P = x_1, x_2, \dots, x_p$ is a **good path** if $N(x_1), N(x_p) \subseteq V(P)$ and each of $N(x_1)$ and $N(x_p)$ does not contain consecutive vertices in P.

Lemma 11: If G does not contain a good path with $c + 2$ vertices, then G has no C-hopping path.

Lemma 12 (Modified Hopping Lemma): Suppose G has no good path with $c+2$ vertices. Then

(M1) X does not contain consecutive vertices in C,
(M2) $X \cap Y = \emptyset$ and $N(Y) \subseteq X$, and
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Claim 1: $c \leq n-3$.