Reconstruction of n-vertex trees from the set of (5n-11)/6-vertex induced subgraphs

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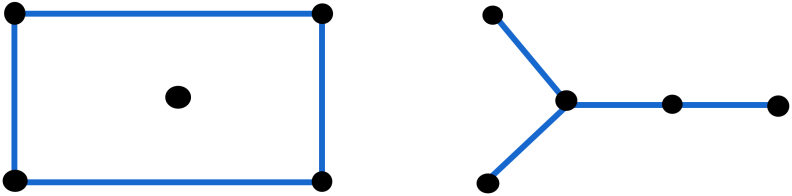
Conjecture: Manvel, 1964, 1969. $\forall \ell \in \mathbf{N}\ \exists M_\ell \in \mathbf{N}$: each graph with $n \geq M_\ell$ vertices is determined by its $n - \ell$-deck.

$M_2 \geq 6$



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3. Kelly, 1957: Trees with at least 3 vertices are 1-reconstructible.
Giles, 1976: Trees with at least 6 vertices are 2-reconstructible.

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Theorem 1 [Groenland, Johnston, Scott, and Tan, 2022^+]: If $n \geq 9\ell + 24\sqrt{2\ell} + o(\sqrt{\ell})$, then all n-vertex trees are ℓ-reconstructible.

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Theorem 2 [K-N-W-Z]: If $n \geq 2\ell + 1$ and $(n, \ell) \neq (5, 2)$, then n-vertex acyclic graphs are ℓ-recognizable; in particular, n-vertex trees are ℓ-recognizable.

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The proof is constructive. We consider an $(n - \ell)$-deck $\mathcal{D}$. By Theorem 2, we can recognize whether $\mathcal{D}$ is the deck of an n-vertex tree.
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Among important parameters are the diameter of T, the number k which is roughly the minimum radius of a connected card, and the number s_ℓ of the centers of spiders $S^{\ell+1}$ with 3 legs of length $\ell + 1$ in T.
We also introduce so called Exclusion Argument for determining important subtrees of our T.

A big case is when the diameter of T is at least $n - 3\ell - 1$.
In this case, our parameter k is at least $\ell + 1$, and we see in the cards all connected subgraphs of T with "not too large" diameter.
Our strategy will depend on whether T contains the spider $S^{\ell+1}$ or not.

When the diameter of T is at most $n - 3\ell - 2$, then we separately consider the cases when after deleting the edges of a longest path there are "large" components or not.