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ON LARGE SYSTEMS OF SETS WITH NO LARGE WEAK Δ-SUBSYSTEMS

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A family of sets is called a *weak Δ-system* if the cardinality of the intersection of any two sets is the same. We elaborate a construction by Rödl and Thoma [9] and show that for large n, there exists a family $\mathcal{F}$ of subsets of $\{1,\dots,n\}$ without weak Δ-systems of size 3 with $|\mathcal{F}|\geq 2^{c(n\log n)^{1/3}}$.

1. Introduction

The notion of a weak Δ-system was introduced and studied by Erdős, Milner and Rado [5] in 1974. A *weak Δ-system* is a family of sets where all pairs of sets have the same intersection size. Erdős and Szemerédi [7] investigated the behavior of the function $F(n,r)$—the largest integer so that there exists a family $\mathcal{F}$ of subsets of an n-element set which does not contain a Δ-system of r sets. Answering a question of Abbott, they proved that $F(n,3)$ is superpolinomial in n:

$$F(n,3) \geq n^{\log n/4\log\log n}. \tag{1}$$

They also conjectured that for some $\varepsilon>0$,

$$F(n,3) \leq (2-\varepsilon)^n.$$

This conjecture was proved by Frankl and Rödl [8] for $\varepsilon=0.01$. Recently, Rödl and Thoma [9] substantially improved (1) by showing that for sufficiently large n,

$$F(n,r) \geq 2^{\frac{1}{3}n^{1/5}\log_2^{4/5}(r-1)}. \tag{2}$$

In this note, we elaborate the construction of [9] to improve (2) further.

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Theorem 1. *For sufficiently large* n,

$$F(n,3) \geq 2^{0.01(n\ln n)^{1/3}}. \tag{3}$$

With a more careful counting, the factor 0.01 can be replaced by a larger one.

2. Az inequality

We shall use the following form of Chernoff-Hoeffding type inequality (cf. [3], Appendix A).

Theorem 2. *Let* Y *be the sum of mutually independent indicator random variables,* $\mu = \mathbf{E}(Y)$. *For each* $0<\varepsilon<1$,

$$\mathbf{P}\{Y < \mu(1-\varepsilon)\} < \exp\{-\varepsilon^2\mu/2\}. \tag{4}$$

For each $\varepsilon>0$,

$$\mathbf{P}\{Y \geq \mu(1+\varepsilon)\} < \exp\{(\varepsilon-(1+\varepsilon)\ln(1+\varepsilon))\mu\}. \tag{5}$$

Remark. An equivalent form of (5) is: *for every* $a>0$,

$$\mathbf{P}\{Y \geq \mu + a\} < \exp\{a-(\mu+a)\ln(1+a/\mu)\}. \tag{6}$$

3. A random construction

Let n be sufficiently large, k be a power of 2 such that $k \leq n^{1/6}/6 < 2k$ and q be the largest power of 2 such that $q^3 \leq \ln k$. By the definition,

$$0.25\ln^{1/3} n < 0.5\ln^{1/3} k < q \leq \ln^{1/3} k. \tag{7}$$

Let further $l = 6qk^2$ and $m = 6^5k^4/q$. By the definition, m is an integer and $lm = (6k)^6 \leq n$. For $i = 1,\ldots,l$, let $N_i = \{1+(i-1)m, 2+(i-1)m,\ldots,im\}$. For each $i=1,\ldots,l$ and each (0-1)-vector $(\alpha_1,\ldots,\alpha_i)$ of length i, we consider a random subset $\mathbf{A}(\alpha_1,\ldots,\alpha_i)$ of N_i, where elements of N_i are chosen independently with probability

$$p_i = \mathbf{P}\{x \in \mathbf{A}(\alpha_1,\ldots,\alpha_i)\} = qk^{-1-i/2l} \quad \forall x \in N_i.$$

For each (0-1)-vector $(\alpha_1,\ldots,\alpha_l)$ of length l, let

$$\mathbf{B}(\alpha_1,\ldots,\alpha_l) = \bigcup_{i=1}^{l} \mathbf{A}(\alpha_1,\ldots,\alpha_i).$$

Note that for any distinct $(\beta_1,\dots,\beta_s)$ and $(\gamma_1,\dots,\gamma_s)$,

$$\lambda_s = \mathbf{E}\{|\mathbf{A}(\beta_1,\dots,\beta_s)|\} = p_s|N_s| = 6^5 k^{3-s/2l}, \tag{8}$$

$$\nu_s = \mathbf{E}\{|\mathbf{A}(\beta_1,\dots,\beta_s)\cap\mathbf{A}(\gamma_1,\dots,\gamma_s)|\} = p_s^2|N_s| = 6^5 q k^{2-s/l}. \tag{9}$$

Lemma 1. *For a vector* $(\alpha_1,\dots,\alpha_s)$, *let* $\mathbf{L}(\alpha_1,\dots,\alpha_s)$ *denote the event that* $|\mathbf{A}(\alpha_1,\dots,\alpha_s)|<\lambda_s(1-k^{-0.1})$, *and let* $\mathbf{L}=\bigcup_{s=1}^{l}\bigcup_{(\alpha_1,\dots,\alpha_s)}\mathbf{L}(\alpha_1,\dots,\alpha_s)$. *Then* $\mathbf{P}\{\mathbf{L}\}<1/3$.

Proof. By (4), $\mathbf{P}\{\mathbf{L}(\alpha_1,\dots,\alpha_s)\} < \exp\{-0.5k^{-0.2}6^5k^{3-s/2l}\} < \exp\{-6^4k^{2.3}\}$. Hence,

$$\mathbf{P}\{\mathbf{L}\} \le \sum_{s=1}^{l}\sum_{(\alpha_1,\dots,\alpha_s)} \exp\{-6^4k^{2.3}\} < 2^{l+1}\exp\{-6^4k^{2.3}\} < 1/3. \qquad \blacksquare$$

For two vectors $\mathbf{a}^{(1)}=(\alpha_1^{(1)},\dots,\alpha_l^{(1)})$ and $\mathbf{a}^{(2)}=(\alpha_1^{(2)},\dots,\alpha_l^{(2)})$, let

$$h(\mathbf{a}^{(1)},\mathbf{a}^{(2)}) = \min\{i \mid \alpha_i^{(1)} \ne \alpha_i^{(2)}\}$$

and

$$\mathbf{C}(\mathbf{a}^{(1)},\mathbf{a}^{(2)}) = \left|\bigcup_{s=h(\mathbf{a}^{(1)},\mathbf{a}^{(2)})}^{l}\Big(\mathbf{A}(\alpha_1^{(1)},\dots,\alpha_s^{(1)})\cap\mathbf{A}(\alpha_1^{(2)},\dots,\alpha_s^{(2)})\Big)\right|.$$

Lemma 2. *Let* $\mathbf{Q}(\mathbf{a}^{(1)},\mathbf{a}^{(2)})$ *be the event that* $|\mathbf{C}(\mathbf{a}^{(1)},\mathbf{a}^{(2)})-\sum_{s=h}^{l}\nu_s| > 0.4\lambda_h$, *where* $h=h(\mathbf{a}^{(1)},\mathbf{a}^{(2)})$. *Let* $\mathbf{Q}=\bigcup_{(\mathbf{a}^{(1)},\mathbf{a}^{(2)})}\mathbf{Q}(\mathbf{a}^{(1)},\mathbf{a}^{(2)})$. *Then* $\mathbf{P}\{\mathbf{Q}\}<1/3$.

Proof. Fix two distinct vectors $\mathbf{a}^{(1)}=(\alpha_1^{(1)},\dots,\alpha_l^{(1)})$ and $\mathbf{a}^{(2)}=(\alpha_1^{(2)},\dots,\alpha_l^{(2)})$. Denote $h=h(\mathbf{a}^{(1)},\mathbf{a}^{(2)})$ and $\mu=\sum_{s=h}^{l}\nu_s$. By (4),

$$\mathbf{P}\{\mathbf{C}(\mathbf{a}^{(1)},\mathbf{a}^{(2)}) < \mu - 0.4\lambda_h\} <$$

$$< \exp\left\{\frac{-(0.4\lambda_h)^2}{2\sum_{s=h}^{l}\nu_s}\right\} \le \exp\left\{\frac{-0.08\cdot 6^{10}k^{6-h/l}}{6^5 q\sum_{s=h}^{l}k^{2-s/l}}\right\} \le \exp\left\{\frac{-0.08\cdot 6^5k^4}{q\sum_{s=h}^{l}k^{(h-s)/l}}\right\} <$$

$$< \exp\left\{\frac{-6^5k^4}{15q}(1-k^{-1/l})\right\} \le \exp\left\{\frac{-6^5k^4}{15q}\left(\frac{\ln k}{l}-\frac{\ln^2 k}{l^2}\right)\right\}$$

$$\le \exp\left\{\frac{-6^5}{18q}k^4\frac{\ln k}{6qk^2}\right\} \le \exp\left\{-\frac{6^3}{3}k^2\frac{\ln k}{q^2}\right\} \le \exp\{-12l\}.$$

By (6),

$$Z=\mathbf{P}\{\mathbf{C}(\mathbf{a}^{(1)},\mathbf{a}^{(2)})>\mu+0.4\lambda_h\}<\exp\{0.4\lambda_h-(\mu+0.4\lambda_h)\ln(1+0.4\lambda_h/\mu)\}.$$

Observe that the derivative in μ of the RHS of (6) is positive since $(a-(\mu+a)\ln(1+a/\mu))'_\mu=-\ln(1+a/\mu)+a/\mu>0$. Hence, if $\mu\le\lambda_h$, then

$$Z<\exp\{0.4\lambda_h-1.4\lambda_h\ln 1.4\}=\exp\left\{-0.4\lambda_h\ln(1.4^{3.5}/e)\right\}<\exp\{-6l\}.$$

If $\mu>\lambda_h$, then (6) yields

$$Z<\exp\left\{\frac{2\lambda_h}{5}-(\mu+\frac{2\lambda_h}{5})\left(\frac{2\lambda_h}{5\mu}-\frac{2\lambda_h^2}{25\mu^2}\right)\right\}=\exp\left\{-\frac{2\lambda_h^2}{25\mu}+\frac{4\lambda_h^3}{125\mu^2}\right\}<$$

$$<\exp\left\{\frac{-6\lambda_h^2}{5^3\mu}\right\}\le\exp\left\{\frac{-6(6^5k^{3-h/2l})^2}{5^3\nu_h\sum_{s=h}^{l}k^{(h-s)/l}}\right\}<\exp\left\{-\frac{6^{11}k^{6-h/l}(1-k^{-1/l})}{5^3\cdot 6^5qk^{2-h/l}}\right\}\le$$

$$\le\exp\left\{-\frac{6^6k^4}{5^3q}\left(\frac{\ln k}{l}-\left(\frac{\ln k}{l}\right)^2\right)\right\}\le\exp\left\{-\frac{6^6k^4}{2\cdot 5^3q}\cdot\frac{\ln k}{6qk^2}\right\}\le\exp\{-6l\}.$$

It follows that $\mathbf{P}\{\mathbf{Q}(\mathbf{a}^{(1)},\mathbf{a}^{(2)})\}<e^{-12l}+e^{-6l}$. Hence

$$\mathbf{P}\{\mathbf{Q}\}<\binom{2^l}{2}2e^{-6l}<1/3.$$ ∎

By Lemmas 1 and 2, with probability at least 1/3, neither of the events $\mathbf{L}$ and $\mathbf{Q}$ occurs. This means that for each $i=1,\dots,l$ and each (0-1)-vector $(\alpha_1,\dots,\alpha_i)$ there exist sets $A(\alpha_1,\dots,\alpha_i)$ such that for every vector $(\beta_1,\dots,\beta_s)$,

(10) $$|A(\beta_1,\dots,\beta_s)|\ge\lambda_s(1-k^{-0.1}),$$

and for every two vectors $\mathbf{a}^{(1)}=(\alpha_1^{(1)},\dots,\alpha_l^{(1)})$ and $\mathbf{a}^{(2)}=(\alpha_1^{(2)},\dots,\alpha_l^{(2)})$,

(11) $$\left|\left|\bigcup_{s=h}^{l}\left(A(\alpha_1^{(1)},\dots,\alpha_s^{(1)})\cap A(\alpha_1^{(2)},\dots,\alpha_s^{(2)})\right)\right|-\sum_{s=h}^{l}\nu_s\right|\le 0.4\lambda_h,$$

where $h=h(\mathbf{a}^{(1)},\mathbf{a}^{(2)})$.

The members of our family $\mathcal{F}$ will be the 2^l sets $B(\alpha_1,\dots,\alpha_l)=\bigcup_{i=1}^{l}A(\alpha_1,\dots,\alpha_i)$.

4. Properties of the family $\mathcal{F}$

Consider three arbitrary vectors $\mathbf{a}^{(1)} = (\alpha_1^{(1)}, \dots, \alpha_l^{(1)})$, $\mathbf{a}^{(2)} = (\alpha_1^{(2)}, \dots, \alpha_l^{(2)})$, and $\mathbf{a}^{(3)} = (\alpha_1^{(3)}, \dots, \alpha_l^{(3)})$. Let $i = h(\mathbf{a}^{(1)}, \mathbf{a}^{(3)})$, $j = h(\mathbf{a}^{(1)}, \mathbf{a}^{(2)})$. We can reorder $\mathbf{a}^{(1)}$, $\mathbf{a}^{(2)}$ and $\mathbf{a}^{(3)}$ so that $i < j$. We shall prove that

$$|B(\mathbf{a}^{(1)}) \cap B(\mathbf{a}^{(2)})| > |B(\mathbf{a}^{(1)}) \cap B(\mathbf{a}^{(3)})|.$$

Let $C_t = B(\mathbf{a}^{(t)}) \setminus \bigcup_{s=1}^{i-1} N_s$, $t = 1,2,3$. Since the sets $B(\mathbf{a}^{(1)}), B(\mathbf{a}^{(2)})$ and $B(\mathbf{a}^{(3)})$ coincide on N_s for $s < i$,

$$|B(\mathbf{a}^{(1)}) \cap B(\mathbf{a}^{(2)})| - |B(\mathbf{a}^{(1)}) \cap B(\mathbf{a}^{(3)})| = |C_1 \cap C_2| - |C_1 \cap C_3|.$$

By (10) and (11),

$$|C_1 \cap C_2| - |C_1 \cap C_3| \geq \sum_{s=i}^{j-1} (1 - k^{-0.1})\lambda_s + \sum_{s=j}^{l} \nu_s - 0.4\lambda_j - \sum_{s=i}^{l} \nu_s - 0.4\lambda_i =$$

$$= \sum_{s=i}^{j-1} ((1 - k^{-0.1})\lambda_s - \nu_s) - 0.4(\lambda_i + \lambda_j) > 0.1\lambda_i.$$

This proves Theorem 1. Moreover, $\mathcal{F}$ is "far" from containing weak Δ-systems in the sense that any three its members F_1, F_2 and F_3 can be ordered so that

$$|F_1 \cap F_2| - |F_1 \cap F_3| > 0.1\lambda_l = 0.1 \cdot 6^5 k^{2.5} > n^{5/12}. \tag{12}$$

It is not hard to prove that the size of any family with property (12) is at most $2^{O(n^{7/12} \log n)}$.

Remark. Observe that for any r sets $B(\mathbf{a}^{(1)}), \dots, B(\mathbf{a}^{(r)})$ in $\mathcal{F}$, the vectors $\mathbf{a}^{(1)}, \dots, \mathbf{a}^{(r)}$ can be reordered in such a way that

$$h(\mathbf{a}^{(1)}, \mathbf{a}^{(2)}) - h(\mathbf{a}^{(1)}, \mathbf{a}^{(3)}) \geq \log_2 r \; - 1.$$

This implies that the same construction with l and m replaced by $l' = \lfloor qk^2 \log_2 r \rfloor$ and $m' = \lfloor n/l' \rfloor$ yields

$$F(n, r) \geq r^{c(n \ln n)^{1/3}}.$$

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