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# Properties of Descartes' Construction of Triangle-Free Graphs with High Chromatic Number

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We present some nice properties of the classical construction of triangle-free graphs with high chromatic number given by Blanche Descartes and its modifications. In particular, we construct colour-critical graphs and hypergraphs of high girth with moderate average degree.

## 1. Introduction

We cite the description of the famous construction by Blanche Descartes in [8]:

The graph $G_3$ is a circuit of just 7 edges. (Any larger odd number would do.) When $G_i$ is defined, with $m_i$ nodes say, we construct $G_{i+1}$ as follows. We take $\binom{im_i-i+1}{m_i}$ disjoint copies of $G_i$. We adjoin $im_i - i + 1$ extra nodes, which we call 'central'. We set up a 1–1 correspondence between the copies of $G_i$ and the sets of $m_i$ central nodes. We join each copy of $G_i$ to the members of the corresponding set of central nodes by $m_i$ new edges of which no two have a common end. The resulting graph is $G_{i+1}$.

The key properties of this beautiful construction (observed in [8]) are the following.

**Property 1.** *For each $i \geqslant 3$, the chromatic number $\chi(G_i)$ of the graph $G_i$ is at least $i$.* □

**Property 2.** *For each $i \geqslant 3$, the girth $g(G_i)$ of the graph $G_i$ is at least 6.* □

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The aim of this note is to list some other properties of the construction and its modifications. This is mostly done in the next section. We show that this classical construction gives several statements which were proved by an *ad hoc* argument and, in fact, its easy generalization yields a strengthening of these results (to graphs with arbitrary girth). In Section 3, we extend the construction to hypergraphs. In the last section, we use this construction to prove the existence, for each $k \geqslant 3$, of $k$-(chromatically)-critical graphs of arbitrary girth with average degree less than $2(k-2)$ and $k$-(chromatically)-critical $r$-uniform hypergraphs of arbitrary girth with average degree less than $2(k-2)+\epsilon$ for any $\epsilon > 0$. In [11] it is shown that any such graph must have average degree at least $2k - o(k)$.

## 2. Degree properties

A graph is called *$k$-degenerate* if each of its non-empty subgraphs has a vertex of degree at most $k$. Clearly, the chromatic number of any $k$-degenerate graph is at most $k+1$. In a sense, any $k$-degenerate graph can be 'efficiently' coloured with at most $k+1$ colours.

**Property 3.** *For each $i \geqslant 3$, the graph $G_i$ is $(i-1)$-degenerate.*

**Proof.** Trivially, the graph $G_3$ is 2-degenerate. Assume that the property holds for $G_i$, and consider an arbitrary subset $W$ of $V(G_{i+1})$. If $W$ contains only central vertices, then each $w \in W$ has degree 0 in $G_{i+1}[W]$. Otherwise, let $H$ be a copy of $G_i$ with $V(H) \cap W \neq \emptyset$. By the induction assumption, there is $w \in V(H) \cap W$ of degree in $H[W \cap V(H)]$ at most $i-1$. Then the degree of $w$ in $G_{i+1}[W]$ is at most $i$. □

A *star forest* is a forest with each component being a star. I. Křı̌ž [12] constructed for each $n$ and $k$ a $k$-chromatic graph $G(n,k)$ of girth $n$ such that its edge set is the union of $k-1$ star forests.

**Property 4.** *For each $i \geqslant 3$, the graph $G_i$ is the union of a set of vertex-disjoint cycles and $i-3$ star forests.*

**Proof.** For $i = 3$, the statement is trivial. Assume that $G_i$ is the union of a set of vertex-disjoint cycles and $i-3$ star forests, and consider $G_{i+1}$. By the assumption, the subgraph of $G_{i+1}$ induced by all copies of $G_i$ is also the union of a set of vertex-disjoint cycles and $i-3$ star forests, and the edges incident to central vertices form one more star forest. □

Property 4 is more restrictive than the property of being the union of $i-1$ star forests. On the other hand, Křı̌ž's construction gives graphs of arbitrary girth. To avoid this disadvantage, observe (as in [16, 10, 14]) the following. We can view Descartes' construction of $G_{i+1}$ as replacing each hyperedge in the complete $m_i$-uniform hypergraph on $im_i - i + 1$ vertices by a copy of $G_i$ joined by a matching with vertices of this hyperedge. If, instead of the complete $m_i$-uniform hypergraph on $im_i - i + 1$ vertices, we take any $m_i$-uniform $(i+1)$-chromatic hypergraph, then the chromatic number of the resulting $G_{i+1}$

can also not be less than $i+1$ (for the same reasons). Moreover, Properties 3 and 4 hold as well (the proofs are the same). In addition, if we choose as $G_3$ an odd cycle of length at least $g$, and at every step the auxiliary $m_i$-uniform $(i+1)$-chromatic hypergraph is also of girth $g$ (such a hypergraph exists by the classical result of P. Erdős and A. Hajnal [9]), then the girth of the resulting $G_{i+1}$ is also at least $g$.

**Remark.** Actually, the auxiliary $m_i$-uniform $(i+1)$-chromatic hypergraph only needs girth $\lceil g/3 \rceil$ in order for $G_{i+1}$ to have girth at least $g$.

For a (hyper)graph $G$, let the *density* of $G$, denoted $\mathrm{den}(G)$, be the maximum of the ratio $\frac{|E(H)|}{|V(H)|}$ over all sub(hyper)graphs $H$ of $G$.

**Property 5.** *For each* $i \geqslant 3$, $\mathrm{den}(G_{i+1}) < 1 + \mathrm{den}(G_i)$.

**Proof.** Consider an arbitrary (induced) subgraph $H$ of $G_{i+1}$. Let $V'$ be the set of central vertices of $G_{i+1}$ which belong to $V(H)$ and $V'' = V(H) \setminus V'$. Let $H' = H[V']$ and $H'' = H[V'']$. Clearly,

$$\frac{|E(H'')|}{|V(H'')|} \leqslant \mathrm{den}(G_i).$$

By the construction, $|E(H)| \leqslant |V(H'')| + |E(H'')|$ and, for the equality, we need $V' \neq \emptyset$. It follows that

$$\frac{|E(H)|}{|V(H)|} \leqslant \frac{|V(H'')|}{|V(H)|} + \frac{|E(H'')|}{|V(H'')|} < 1 + \mathrm{den}(G_i).$$

This proves the proposition. □

Since $k$-degenerate graphs can be coloured so easily with $k+1$ colours and have average degree less than $2k$, it might be expected that the vertex set of any such graph (or any such graph with large girth) can be partitioned into $k$ subsets each of which induces a subgraph with maximum degree of, say, at most 100. But this is not true. O. V. Borodin [4] gave for each integer $d$ an example of a 2-degenerate graph $G(d)$ such that, in each bipartition $\{V_1, V_2\}$ of $V(G(d))$, either $V_1$ or $V_2$ induces a subgraph with maximum degree at least $d$. The idea of Descartes' construction lets us extend this result as follows.

**Property 6.** *For any positive integers $k, g$, and $d$, there exists a $k$-degenerate graph $G(k, g, d)$ of girth at least $g$ such that, in each partition $\{V_1, \ldots, V_k\}$ of $V(G(k, g, d))$, at least one of the $V_i$ induces a subgraph with maximum degree at least $d$.*

**Proof.** We fix positive integers $g$ and $d$, and construct the graph $G_k = G(k, g, d)$ inductively. The graph $G_1$ is the star $K_{1,d}$. Suppose that $G_k$ is constructed and has $m_k$ vertices. By [9], there exists an $m_k$-uniform $(d \cdot m_k)$-chromatic hypergraph of girth $g$. Such a hypergraph contains a $(d \cdot m_k)$-critical subhypergraph $H(m_k)$ whose minimum degree must be at least $d \cdot m_k - 1$. Let $H(m_k)$ have $s_k$ vertices. Again, by [9], there exists an $s_k$-uniform $(k+2)$-chromatic hypergraph $F$ of girth $g$. Replacing each hyperedge in $F$ by a copy of $H(m_k)$, we obtain an $m_k$-uniform hypergraph $F'$ with the property that in each partition of $V(F')$ into $k+1$ subsets, at least one of them induces a subhypergraph containing a copy of $H(m_k)$. The vertices of $F'$ will be 'central' in $G_{k+1}$. Then we adjoin $|E(F')|$ disjoint copies of $G_k$ to $F'$ and set up a 1–1 correspondence between the copies of $G_k$ and the hyperedges of $F'$. Finally, we join the vertices of every hyperedge in $F'$ by a matching with the corresponding copy of $G_k$ and delete the hyperedges. This is $G_{k+1}$.

Exactly as above, $G_{k+1}$ is $(k+1)$-degenerate and has girth at least $g$. Consider an arbitrary partition $\{V_1, \ldots, V_{k+1}\}$ of $V(G_{k+1})$. From above, at least one of $V_i$ induces on $V(F')$ a subhypergraph containing a copy of $H(m_k)$. Hence, we may assume that some $W \subseteq V_1 \cap V(F')$ with $|W| = s_k$ induces a copy of $H(m_k)$. By construction, $W$ is joined in $G_{k+1}$ by matchings with at least $d \cdot s_k$ copies of $G_k$. If each of these copies contains a vertex in $V_1$, then $W$ is joined in $G_{k+1}$ to at least $d \cdot s_k$ vertices in $V_1$, and some vertex in $W$ has degree at least $d$ in $G_{k+1}[V_1]$. Otherwise, some copy $C$ of $G_k$ is disjoint from $V_1$ and, by the induction assumption, at least one of the sets $V_2 \cap V(C), \ldots, V_{k+1} \cap V(C)$ induces a subgraph with maximum degree at least $d$. □

## 3. A hypergraph construction

A version of Descartes' construction for $r$-uniform hypergraphs can be stated as follows:

The hypergraph $G_2 = G_2(r, g)$ is a hyperedge on just $r$ vertices. When $G_i = G_i(r, g)$ is defined, with $m_i$ nodes, say, we construct $G_{i+1}$ as follows. Let $F = F(r, g, i)$ be an $(r-1)m_i$-uniform $(i+1)$-chromatic hypergraph of girth $g$ (which exists by [9]) with, say, $l_i$ hyperedges. We take $l_i$ disjoint copies of $G_i$. We adjoin a ('central') copy of $F$. We set up a 1–1 correspondence between the copies of $G_i$ and the hyperedges of $F$. For each copy $C$ of $G_i$, we delete the corresponding hyperedge $e_C$ of $F$ and join vertices of $C$ to the set $e_C$ by $m_i$ new edges of size $r$ of which no two have a common end and each has exactly one vertex in common with $C$. The resulting hypergraph is $G_{i+1}$.

By the definition, $G_i(r, g)$ is an $r$-uniform hypergraph. Analogously to the previous construction, the following properties hold for $G_i$.

**Property 1′.** *For each $i \geqslant 2$, the chromatic number $\chi(G_i)$ of the hypergraph $G_i$ is at least $i$.*

**Property 2′.** *For each $i \geqslant 2$, $g \geqslant 3$, the girth $g(G_i(r, g))$ of the hypergraph $G_i(r, g)$ is at least $g$.*

**Property 3′.** *For each $i \geqslant 2$, the hypergraph $G_i$ is $(i-1)$-degenerate.*

Define a *hypergraph star forest* to be a hypergraph in which every edge contains a vertex of degree 1. Then, in a similar way to Property 4, the following holds.

**Property 4′.** *For each $i \geqslant 2$, the hypergraph $G_i$ is the union of a matching and $i-2$ star forests.*

**Property 5′.** *For each $i \geqslant 2$, $\operatorname{den}(G_{i+1}) < 1 + \operatorname{den}(G_i)$.*

M. I. Burstein [5], L. Lovász [13], P. D. Seymour [15] and D. R. Woodall [17] proved independently that the density of any 3-chromatic hypergraph is at least 1. M. I. Burstein [5] has constructed for every $r \geqslant 2$ an example of a 3-chromatic (in fact, 3-critical) $r$-uniform hypergraph with density 1. It seems that the examples of 3-critical $r$-uniform hypergraphs of girth 4, 5 and 6 given by H. L. Abbott, D. R. Hare, and B. Zhou [2, 3] can have density arbitrarily close to 1. Descartes' construction provides such examples for arbitrary girth.

**Property 7.** *For each positive integers $g \geqslant 3$, $r \geqslant 2$ and $m$, there exists a 3-chromatic $r$-uniform hypergraph $G_3(r,g,m)$ of girth at least $g$ such that*

$$\mathrm{den}(G_3(r,g,m)) < 1 + 1/m.$$

**Proof.** We prove the property by induction on $m$. Since $\mathrm{den}(G_2(r,g)) = 1/r$, we conclude from Property 5′ that $\mathrm{den}(G_3(r,g)) \leqslant 1 + 1/r$ and so we can take $G_3(r,g,2) = G_3(r,g)$.

Now, let the property be true for all $m \leqslant m_0$ and any $r$ and $g$. In order to construct $G_3(r,g,2m_0)$, we apply the construction to the hyperedge $G_2 = G_2(r,g)$ with $F = G_3(r(r-1), g, m_0)$. We claim that the resulting hypergraph $G$ possesses the required property.

Indeed, assume that $\mathrm{den}(G) > 1 + 1/2m_0$. Let a subhypergraph $G'$ of $G$ be the smallest (with respect to the number of vertices) subhypergraph with $\frac{|E(G')|}{|V(G')|} > 1 + 1/2m_0$. Then it cannot have vertices of degree 0 and 1. It follows that together with any non-central vertex $v$ of $G$, $G'$ contains the 'new' edge including $v$ and all edges connecting this edge with central vertices. In other words, the edges of $G'$ can be partitioned into groups of cardinality $r+1$ such that each group was created in the course of constructing $G$ by replacing an edge of $F$ by the edges of this group. Thus, if we have $x$ such groups and $G'$ contains $y$ central vertices, then

$$\frac{|E(G')|}{|V(G')|} = \frac{x(r+1)}{rx+y} = \frac{r+1}{r+y/x}.$$

Since $\mathrm{den}(F) \leqslant 1 + 1/m_0$, we have $x/y \leqslant 1 + 1/m_0$, and so

$$\frac{|E(G')|}{|V(G')|} \leqslant \frac{r+1}{r+\frac{m_0}{1+m_0}} = \frac{r+1}{r+1-\frac{1}{1+m_0}} = 1 + \frac{1/(1+m_0)}{r+1-\frac{1}{1+m_0}} < 1 + \frac{1}{rm_0}.$$

This contradicts the choice of $G'$. □

## 4. Sparse colour-critical graphs and hypergraphs

In [11] it is shown that any $k$-critical hypergraph without graph edges on $n$ vertices has at least $kn(1 - 3k^{-1/3})$ edges. Constructions by H. L. Abbott, D. R. Hare and B. Zhou [1, 2, 3] show that this bound is asymptotically (in $k$) tight even for $k$-critical $r$-uniform hypergraphs of girth $g$ for $g \leqslant 5$ and arbitrary $r$. Namely, they construct for every $\epsilon > 0$, $k \geqslant 3$ and $r$ an example of a $k$-critical $r$-uniform hypergraph $G$ of girth 5 with $|E(G)|/|V(G)| \leqslant k - 2 + \epsilon$. Descartes' construction allows us to extend the upper bound to any girth.

**Theorem 4.1.** *For each $\epsilon > 0$, $k \geqslant 3$, $g \geqslant 3$ and $r \geqslant 2$, there exists a $k$-critical $r$-uniform hypergraph $G = G(k, g, r, \epsilon)$ of girth $g$ with $|E(G)|/|V(G)| \leqslant k - 2 + \epsilon$.*

**Proof.** By Properties 7 and 5′, there exists a $k$-chromatic $r$-uniform hypergraph $H = H(k, g, r, \epsilon)$ of girth $g$ with $\text{den}(H) \leqslant k - 2 + \epsilon$. This $H$ contains a $k$-critical subhypergraph $G = G(k, g, r, \epsilon)$. By the definition of density, this $G$ will do. □

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