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On Systems of Small Sets with No Large Δ-Subsystems

A. V. KOSTOCHKA[1]†, V. RÖDL[2] and L. A. TALYSHEVA[1]‡

[1] Institute of Mathematics, Siberian Branch of the RAS,
Novosibirsk-90, 630090, Russia
(e-mail: sasha@math.nsc.ru)

[2] Department of Mathematics and Computer Science, Emory University,
Atlanta, GA 30322, USA
(e-mail: rodl@mathcs.emory.edu)

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A family of k sets is called a Δ-*system* if any two sets have the same intersection. Denote by $f(r,k)$ the least integer so that any r-uniform family of $f(r,k)$ sets contains a Δ-system consisting of k sets. We prove that, for every fixed r, $f(r,k) = k^r + o(k^r)$. Using a recent result of Molloy and Reed [5], a bound on the error term is provided for sufficiently large k.

1. Introduction

A family $\mathscr{F}$ of sets is called r-*uniform* if, for every $F \in \mathscr{F}$, $|F| = r$ holds. A family of sets is called a Δ-*system* if any two sets have the same intersection. Define $f(r,k)$ to be the least integer so that any r-uniform family of $f(r,k)$ sets contains a Δ-system consisting of k sets. Erdős and Rado [4] proved that

$$(k-1)^r < f(r,k) < r!(k-1)^r, \tag{1.1}$$

and conjectured that, for each k, there exists a constant C_k so that $f(r,k) < C_k^r$. Erdős (see [3]) offered \$1000 for a proof or disproof of this for $k = 3$. In this note we consider the case when r is fixed and k is large enough.

Abbott, Hanson and Sauer [1] completely solved the case $r = 2$, as follows.

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Theorem A. (Abbott, Hanson and Sauer [1])

$$f(2,k) = \begin{cases} k(k-1), & \text{if } k \text{ is odd,} \\ (k-1)^2 + (k-2)/2, & \text{if } k \text{ is even.} \end{cases} \tag{1.2}$$

Abbott and Hanson [2] obtained an upper bound for $r = 3$, as follows.

Theorem B. (Abbott and Hanson [2]) *For $k \geqslant 7$,*

$$f(3,k) \leqslant 1.8k(k-1)^2 + 8(k-1)^2. \tag{1.3}$$

In this note, we apply the Pippenger–Spencer theorem [6] to show that, for fixed r, the lower bound in (1.1) is asymptotically (in k) tight.

Theorem 1. *Let r be fixed and k be sufficiently large. Then*

$$f(r,k) = k^r + o(k^r). \tag{1.4}$$

Using a recent result of Molloy and Reed [5], we estimate the error term for sufficiently large k as follows.

Theorem 2. *Let r be fixed and k be sufficiently large. Then there exists a constant c_r such that*

$$f(r,k) \leqslant k^r(1 + c_r k^{-2^{-r}}). \tag{1.5}$$

2. Background

Let $H = (V, E)$ be a hypergraph with the vertex-set V and the edge-set E. The *degree* $\deg_H(v)$ of a vertex $v \in V$ is the number of edges in H containing v. Similarly, the *codegree* $\mathrm{codeg}(v, w)$ of a pair of vertices is the number of edges in H containing both v and w. Let $D(H) = \max\{\deg_H(v) \mid v \in V\}$ and $C(H) = \max\{\mathrm{codeg}_H(v, w) \mid v, w \in V\}$. If $C(H) = 1$ then H is called *linear*.

An edge-colouring of a hypergraph H is *proper* if the edges of the same colour are vertex-disjoint, that is, form a *matching*.

Theorem C. (Pippenger and Spencer [6]) *Let r be fixed and D be sufficiently large. Then, for each r-uniform hypergraph H with $D(H) \leqslant D$ and $C(H) = o(D)$, there exists a proper edge-colouring with $D + o(D)$ colours.*

Unfortunately, the theorem does not say which D is 'sufficiently large' for a given r and how small are $o(D)$. The situation with $o(D)$ for $C(H) = 1$ was improved recently by Molloy and Reed, as follows.

Theorem D. (Molloy and Reed [5]) *Let s be fixed and D be sufficiently large. Then there exists c_s such that, for each s-uniform linear hypergraph H with $D(H) \leqslant D$, there exists a proper edge-colouring with $D\left(1 + c_s(\log D)^6 D^{-1/s}\right)$ colours.*

We shall also use the following folklore observation.

Observation E. *Let $\mathscr{F}$ be an r-uniform family not containing a Δ-system of size k, and let W be a subset of the set $\bigcup_{F\in\mathscr{F}} F$. Further, let $\mathscr{F}_W = \{F \setminus W \mid F \in \mathscr{F},\ W \subset F\}$. Then $\mathscr{F}_W$ is an $(r - |W|)$-uniform family not containing a Δ-system of size k. In particular,*

$$|\{F \in \mathscr{F} \mid W \subset F\}| = |\mathscr{F}_W| < f(r - |W|, k).$$

3. Proofs

Proof of Theorem 1. First we prove Theorem 1 by induction on r. For $r = 1$, it is trivial. For $r = 2$, it follows from Theorem A. Suppose that Theorem 1 is proved for all $r < r_0$. Let $\mathscr{F}$ be an r_0-uniform family not containing a Δ-system consisting of k sets, where k is sufficiently large. Let $H = (V, E)$ be the hypergraph with $E = \mathscr{F}$ and v be an arbitrary vertex in H. By Observation E and then by the induction assumption, we have

$$|\mathscr{F}_{\{v\}}| < f(r_0 - 1, k) = k^{r_0-1} + o(k^{r_0-1}).$$

In other words, $D(H) \leqslant k^{r_0-1} + o(k^{r_0-1})$. Similarly, $C(H) \leqslant k^{r_0-2} + o(k^{r_0-2})$. Thus, for some $D = k^{r_0-1} + o(k^{r_0-1})$, we have $D(H) \leqslant D$ and $C(H) = o(D)$. By Theorem C, the edges of H can be partitioned into $t = D + o(D) = k^{r_0-1} + o(k^{r_0-1})$ matchings $\mathscr{M}_1, \mathscr{M}_2, \ldots, \mathscr{M}_t$. Any matching is a Δ-system. Thus, $|\mathscr{M}_j| < k$ for each j. It follows that $|\mathscr{F}| = |E| < kt = k^{r_0} + o(k^{r_0})$. □

To prove Theorem 2, we need the following fact.

Lemma 3. *Let $H_i = (V_i, E_i)$ be an r-uniform hypergraph such that*

(a) *the family E_i does not contain a Δ-system of size k,*
(b) *$|e \cap e'| \leqslant r - i$ for each $e, e' \in E_i$.*

Then E_i can be partitioned into $k\big(1 + c_{\binom{r}{i}}(\log k)^6 D^{1-1/\binom{r}{i}}\big)$ subsets $E_{i+1,j}$ such that $|e \cap e'| \leqslant r - i - 1$ for each $e, e' \in E_{i+1,j}$.

Proof. Consider the auxiliary hypergraph G_i whose vertices are $(r - i)$-subsets of V_i, and $\binom{r}{r-i}$ vertices of G_i form an edge if and only if the union of the corresponding $(r - i)$-subsets of V_i is an edge in H_i. Thus there is one-to-one correspondence between edge sets of H_i and G_i.

First observe that G_i is simple. Indeed, if two distinct vertices of G_i belong to at least two common edges, then the two corresponding distinct $(r-i)$-subsets of V_i are contained in two common edges of H_i, a contradiction to (b).

Also, due to (b), the set of edges in H_i containing a given $(r - i)$-subset of V_i is a Δ-system. Thus, $D(G_i) < k$, and applying Theorem D to G_i, we partition $E(G_i)$ into $k\big(1 + c_{\binom{r}{i}}(\log k)^6 k^{-1/\binom{r}{i}}\big)$ matchings. Note that the corresponding partition of E_i satisfies the statement of the lemma. □

Proof of Theorem 2. We are ready to prove Theorem 2. Let $\mathscr{F}$ be an r-uniform family not containing a Δ-system consisting of k sets, and let $H = (V, E)$ be the hypergraph with $E = \mathscr{F}$. Clearly, $H_1 = H$ satisfies the conditions of Lemma 3 for $i = 1$. Applying the lemma, we get that every hypergraph $H_{2,j} = (V(H_{2,j}), E_{2,j})$ satisfies the conditions of Lemma 3 for $i = 2$ and so on. Note that, after $r-1$ steps, 'matchings' in $H_{r-,j}$ correspond to real matchings in $H_1 = H$. Thus, we have split E in at most

$$t = k^{r-1} \prod_{i=1}^{r-1} \left(1 + c_{\binom{r}{i}} (\log k)^6 k^{-1/\binom{r}{i}}\right)$$

matchings. Let $c'_r = \max\{c_{\binom{r}{i}} \mid 1 \leqslant i \leqslant r-1\}$. Then

$$\begin{aligned} t &\leqslant k^{r-1} \prod_{i=1}^{r-1} \exp\left\{c'_r (\log k)^6 k^{-1/\binom{r}{i}}\right\} \\ &\leqslant k^{r-1} \exp\left\{r\, c'_r (\log k)^6 k^{-1/\binom{r}{\lceil r/2 \rceil}}\right\}. \end{aligned}$$

For large k, the last expression is at most $k^{r-1}(1 + k^{-2^{-r}})$. As above, no matching can contain k or more edges. Thus, $|\mathscr{F}| < k \cdot k^{r-1}(1 + k^{-2^{-r}})$, as required. □

4. A remark

M. Molloy and B. Reed have informed the authors that they are able to prove that the error term in the Pippenger–Spencer theorem (Theorem C) is at most

$$c_r (\log D)^{\text{const}} D \left(\frac{C(H)}{D}\right)^{1/r}.$$

In the proof of Theorem 1, we have $C(H)/D \leqslant 1/(k + o(k))$. Then, repeating the proof of Theorem 1, we get the bound $f(r,k) \leqslant k^r(1 + k^{(-1+\epsilon)/r})$.

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