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On universal graphs for planar oriented graphs of a given girth

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Abstract

The oriented chromatic number $o(H)$ of an oriented graph H is defined to be the minimum order of an oriented graph H' such that H has a homomorphism to H'. If each graph in a class $\mathscr{K}$ has a homomorphism to the same H', then H' is *$\mathscr{K}$-universal*. Let $\mathscr{P}_k$ denote the class of orientations of planar graphs with girth at least k. Clearly, $\mathscr{P}_3 \supset \mathscr{P}_4 \supset \mathscr{P}_5 \ldots$ We discuss the existence of $\mathscr{P}_k$-universal graphs with special properties. It is known (see Raspaud and Sopena, 1994) that there exists a $\mathscr{P}_3$-universal graph on 80 vertices. We prove here that

(1) there exist no planar $\mathscr{P}_4$-universal graphs;
(2) there exists a planar $\mathscr{P}_{16}$-universal graph on 6 vertices;
(3) for any k, there exist no planar $\mathscr{P}_k$-universal graphs of girth at least 6;
(4) for any k, there exists a $\mathscr{P}_{40k}$-universal graph of girth at least $k+1$. © 1998 Elsevier Science B.V. All rights reserved

1. Introduction

Graphs in this paper can be directed, oriented or unoriented. The difference between *directed* and *oriented* graphs is that in directed graphs opposite arcs are allowed, while in oriented graphs they are not allowed. (Two exceptions: by *directed cycle* we mean

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an oriented cycle without sources and by *directed path* an oriented path with exactly one source and exactly one sink.) In other words, an oriented graph is an orientation of an undirected graph obtained by assigning to every edge one of the two possible orientations. For every graph $G=(V,E)$, V is its set of vertices and E is its set of arcs or edges. Given graphs $G=(V,E)$ and $G'=(V',E')$, a homomorphism from G to G' is any mapping $f: V \to V'$ satisfying

$$xy \in E \;\Rightarrow\; f(x)f(y) \in E'.$$

Here the elements or E and E' either both are edges or both are arcs. The existence of a homomorphism from G to G' will be denoted by $G \to G'$.

Homomorphisms are clearly related to the chromatic number of undirected graphs by the observation that $\chi(G) \leqslant k$ if and only if $G \to K_k$. In other words, an undirected graph G has chromatic number k if and only if G has a homomorphism to K_k but no homomorphism to K_{k-1}. Therefore, the chromatic number $\chi(G)$ of an undirected graph G can equivalently be defined as the minimum number of vertices in an undirected graph H such that G has a homomorphism to H. Homomorphisms of undirected graphs have been extensively studied (see, e.g., [3–7,10]) as a generalization of graph colouring. We can similarly define the *oriented chromatic number* $o(H)$ of an oriented graph H as the minimum number of vertices in an oriented graph H' such that H has a(n oriented) homomorphism to H'. Oriented homomorphisms have been studied in [2,8,9,11,12]. We will often say that a graph G is *H-colourable* if G has a homomorphism to H and the vertices of H will be called *colours*.

A difference between undirected and directed homomorphisms is that every undirected graph G with $\chi(G) \leqslant k$ is K_k-colourable, while the minimum number of vertices in an oriented graph H such that every oriented graph G with $o(G) \leqslant k$ is H-colourable is exponential in k. This difference justifies studying *$\mathscr{K}$-universal oriented graphs*, i.e. the oriented graphs H such that every graph in $\mathscr{K}$ is H-colourable. In this paper we study universal graphs for oriented planar graphs of given girth. By the *girth* (respectively, *length of a path or length of a cycle*) of an oriented graph we mean the girth (respectively, length of a path or length of a cycle) of the underlying undirected graph.

Denote by $\mathscr{P}_k$ the class of planar oriented graphs with girth at least k. In particular, $\mathscr{P}_3$ is the class of all planar oriented graphs. Evidently, $\mathscr{P}_3 \supset \mathscr{P}_4 \supset \mathscr{P}_5 \ldots$, which yields that any $\mathscr{P}_k$-universal graph is also $\mathscr{P}_m$-universal for every $m > k$. The following theorem is a summary of results in [2,9,11,12] related to planar graphs.

Theorem 0. 1. *There is a $\mathscr{P}_3$-universal graph on* 80 *vertices* [11];
2. *there is a $\mathscr{P}_5$-universal graph on* 19 *vertices* [2];
3. *there is a $\mathscr{P}_6$-universal graph on* 11 *vertices* [2];
4. *there is a $\mathscr{P}_8$-universal graph on* 7 *vertices* [2];
5. *there is a $\mathscr{P}_{14}$-universal graph on* 5 *vertices* [2];
6. *for every k, there exists a graph $G \in \mathscr{P}_k$ with $o(G) \geqslant 5$* [9];
7. *there exists a graph $G \in \mathscr{P}_7$ with $o(G) \geqslant 6$* [9];
8. *there exists a planar oriented graph G with $o(G) \geqslant 15$* [12].

In fact, some results in [2] cited above have a stronger form in terms of maximum average degree. *The maximum average degree* $\mathrm{mad}(G)$ of a graph G is defined to be the maximum of the *average degrees* $\mathrm{ad}(H)=2|E(H)|/|V(H)|$ taken over all the subgraphs H of G. Euler formula implies that for each surface, any graph with sufficiently large girth embedded in this surface has small maximum average degree. In particular, for every planar or projective planar graph G with girth at least g, we have (see, e.g. [2])

$$\mathrm{mad}(G)<2g/(g-2). \tag{1}$$

That is, if $\mathcal{MAD}_\alpha$ is the class of all graphs with the maximum average degree strictly less than α, then $\mathcal{MAD}_{2g/(g-2)}\supset\mathcal{P}_g$ for each $g\geqslant 3$. This explains how statements 2,3 and 5 of Theorem 0 are implied by the following facts proved in [2].

2′. *there is a* $\mathcal{MAD}_{10/3}$*-universal graph on* 19 *vertices*;

3′. *there is a* $\mathcal{MAD}_3$*-universal graph on* 11 *vertices*;

5′. *there is a* $\mathcal{MAD}_{7/3}$*-universal graph on* 5 *vertices.*

In the present paper we are looking for $\mathcal{P}_k$-universal graphs which themselves are planar and/or of a given girth. Several existence results are obtained for $\mathcal{MAD}_\alpha$-universal graphs. In contrast with the statement 1 of Theorem 0, we have

Theorem 1. *There are no planar* $\mathcal{P}_3$*- or* $\mathcal{P}_4$*-universal graphs.*

On the other hand, the following is true.

Theorem 2. *There exists a planar graph on* 6 *vertices which is universal for the set of graphs in* $\mathcal{MAD}_{16/7}$ *with girth at least* 11.

Note that, by (1), Theorem 2 yields the following.

Corollary 3. *There exists a planar* $\mathcal{P}_{16}$*-universal graph on* 6 *vertices.*

It can be also proved that an orientation of the planar graph K_5-e is $\mathcal{P}_{31}$-universal. Clearly, each directed cycle is $\mathcal{MAD}_2$-universal, i.e. universal for oriented forests. The situation is similar for graphs G with $\mathrm{mad}(G)=2$:

Proposition 4. *For each* $k\geqslant 3$ *and for any* $\varepsilon>0$, *there exists an outerplanar graph* $F\in\mathcal{MAD}_{2+\varepsilon}$ *of girth* k *which is universal for all graphs with mad at most* 2 *of girth* k.

But as soon as mad is greater than 2, the picture changes. Planar graphs with large girth have, by (1), mad close to 2, and still the following is true.

Theorem 5. *For each* k, *any* $\mathcal{P}_k$*-universal graph has maximum average degree at least* 3.

This together with (1) yields the following.

Corollary 6. *For each k, there exists no planar $\mathscr{P}_k$-universal graph of girth at least* 6.

Note that when mad approaches 4, another jump takes place: we have proved in [2] that for every $\varepsilon > 0$, there exists a $\mathscr{M}\mathscr{A}\mathscr{D}_{4-\varepsilon}$-universal graph, while the oriented chromatic number of graphs in $\mathscr{M}\mathscr{A}\mathscr{D}_4$ can be arbitrarily large.

Finally, we show that there are (non-planar) $\mathscr{P}_k$-universal graphs of large girth if k is sufficiently larger.

Theorem 7. *For every $g \geqslant 2$, there exists a graph H of girth $g+1$ which is $\mathscr{P}_{40g}$-universal and $\mathscr{M}\mathscr{A}\mathscr{D}_{2+1/(12g-2)}$-universal.*

2. Nonexistence of some planar universal graphs

In this section, we prove Theorem 1 by contradiction. If the result is not true, then there exists a minimal by inclusion $\mathscr{P}_4$-universal planar graph H. Below we derive a sequence of properties which are possessed by H. The first two of them are immediately implied by the minimality of H.

(i) *There is no homomorphism of H to any of its proper subgraphs.*

(ii) *For every arc e in H, there exists a graph $G_e \in \mathscr{P}_4$ such that every homomorphism $f : G_e \to H$ maps some arc of G_e to the arc e.*

(iii) *For every arc e in H, for every planar graph G with girth at least* 4 *and every arc e' in G, there exists a homomorphism from G to H which maps e' to e.*

Proof. We construct an auxiliary graph G' as follows: take a copy of the graph G_e from (ii). To every arc e'' in G_e we 'glue' a copy of G by identifying the arcs e' and e''. The graph G' thus obtained is clearly planar and has girth at least 4. Thus, there exists a homomorphism $f : G' \to H$. Since every homomorphism of G_e to H uses e, there is an arc e'' in G_e which is mapped to e. Let G'' be the corresponding copy of G which is glued to e''. The restricted homomorphism $f|_{G''}$ is obviously a homomorphism from G to H which maps e' to e. □

(iv) *No vertex in H has in-degree or out-degree less than* 3.

Proof. Let $x \in V(H)$. Consider the graph G_0 obtained from the directed 6-cycle (123456) by adding a vertex 7 and three arcs 17, 37 and 57. Clearly, G_0 is planar and has girth 4. By (iii), there exists a homomorphism of G_0 to H which maps vertex 7 to x. Since the vertices 1,3 and 5 must get distinct colors, the in-degree of x is at least 3. Similarly, the out-degree of x also is at least 3. □

Since, by (iv), every vertex in H has degree at least 6, H cannot be planar. This contradiction proves Theorem 1.

3. Existence of a planar $\mathcal{MAD}_{16/7}$-universal graph

The aim of this section is to prove Theorem 2 which immediately yields Corollary 3.

Let T denote the circulant graph $T(6;1,2)$, i.e. the graph with the vertex-set $\{1,2,3,4,5,6\}$ and such that ab is an arc in T if and only if $b-a\equiv 1\,(\mathrm{mod}\,6)$ or $b-a\equiv 2\,(\mathrm{mod}\,6)$. Note that T is planar.

Call a subset I of $\{1,2,3,4,5,6\}$ the *(i,j)-interval* if $I=\{j,j+1,\ldots,j+i-1\}$ (the sums are taken modulo 6). Any (i,j)-interval will be sometimes called an *i-interval* or simply an *interval*. For an oriented path P and $v\in V(T)$, let $N_T^P(v)$ denote the set of vertices $w\in V(T)$ such that T contains a path isomorphic to P connecting v with w. By induction on the number of arcs in P, it is easy to observe the following fact.

Lemma 1. *For any $v\in V(T)$ and any oriented path P with k arcs ($1\leqslant k\leqslant 5$), the set $N_T^P(v)$ is a $(k+1)$-interval. Moreover, if $N_T^P(v)$ is a $(k+1,j)$-interval, then $N_T^P(v+i)$ is a $(k+1,j+i)$-interval.*

Let G be a minimum (with respect to the number of vertices) oriented graph with maximum average degree less than 16/7 which has no homomorphism to T. Clearly, G has no vertices of degree 1. Vertices of degree k will be often referred to as k-vertices; vertices of degree at least three will be also called *senior vertices*. We say that a vertex $w\in V(G)$ is a *quasi-neighbour* of $v\in V(G)$ if it is a neighbour of v or there is a path connecting w and v whose all internal vertices have degree 2. A 3-vertex having exactly i quasi-neighbours of degree 2 will be sometimes called a $(3,i)$-vertex. Similarly, an *i-quasi-neighbour* (respectively, a *$(3,i)$-quasi-neighbour*) of $v\in V(G)$ is a quasi-neighbour of v which is an i-vertex (respectively, a $(3,i)$-vertex). Graph G possesses the following properties.

(G1) *G contains no path of length 5 whose internal vertices have degree 2.*

Proof. Assume that G contains such a path $(v_0,\ldots,v_5)$. By the minimality of G, there exists a homomorphism $f: G\backslash\{(v_1,\ldots,v_4\}\to T$. By Lemma 1, T contains a 5-path from $f(v_0)$ to $f(v_5)$ whose orientation is the same as in $G[\{v_0,\ldots,v_5\}]$. Thus, we can extend f to a homomorphism of G to T. $\square$

Remark. As in the proof of (G1), the main problem with embedding a subgraph of G which is a path with internal 2-vertices into T is to map the internal vertices so that there is a path of given orientation in T connecting the images of the ends of this path. In particular, if $(v_0,\ldots,v_i)$ is a path in G whose internal vertices have degree 2 and we know the image of v_0, then for the image of v_i, by Lemma 1, the path $(v_0,\ldots,v_i)$ forbids exactly $5-i$ colours. Sometimes, we shall say in this situation that v_0 *forbids* $5-i$ colours (or, equivalently, *allows* $i+1$ colours) for v_i.

(G2) *G contains no $(3,i)$-vertices for any $i\geqslant 7$.*

Proof. Assume that G contains a $(3,i)$-vertex v, and the senior quasi-neighbours of v are u_1, u_2 and u_3. Let the shortest path from v to u_j $(j=1,2,3)$ contain i_j 2-vertices. By the minimality of G, there exists a homomorphism f to T of the graph G' obtained from G by deleting v and all its 2-quasi-neighbours. We claim that f can be extended to a homomorphism of G to T. By the remark above, each u_j forbids for v exactly $4-i_j$ colours. Thus, altogether they forbid for v at most $12-i_1-i_2-i_3=12-i$ colours, and if $i>6$, we have an admissible colour for v. □

(G3) *G contains no $(3,6)$-vertex which is adjacent to a $(3,6)$-, $(3,5)$- or $(3,4)$-vertex.*

Proof. Assume that G contains a $(3,6)$-vertex v which is adjacent to a $(3,j)$-vertex u $(j\geqslant 4)$. Then, by (G1), v is connected with other senior quasi-neighbours by 4-paths. Let v_1 and v_2 be these distinct from u senior quasi-neighbours of v, and u_1 and u_2 be the distinct from v senior quasi-neighbours of u. Let G' be obtained from G by deleting u, v, and their 2-quasi-neighbours. By the minimality of G, there exists a homomorphism f of G' to T. By the remark, v_1 and v_2 forbid for v at most two colours, and u_1 and u_2 forbid for u at most four colours. Let α and β be two colours allowed for u. By the second part of Lemma 1, the quadruple of colours forbidden for v by u if we colour u with β differs from that if we colour u with α. Thus, in at least one case, there is a colour in T allowed for v. □

(G4) *If some $(3,4)$-vertex in G has a $(3,6)$-quasi-neighbour on distance two, then G does not contain another $(3,i)$-quasi-neighbour for $i\geqslant 5$ on distance two.*

Proof. Assume that G contains a $(3,4)$-vertex v which has a $(3,6)$-quasi-neighbour u and a $(3,5)$-quasi-neighbour w, both on distance two from v. Then the third senior quasi-neighbour x is on distance 3 from v. Let u_1 and u_2 (respectively, w_1 and w_2) be the distinct from v senior quasi-neighbours of u (respectively, of w). Since the girth of G is at least 11, none of u_1, u_2, w_1 and w_2 coincides with u, w or x.

Let G' be obtained from G by deleting v, u, w, and their 2-quasi-neighbours. By the minimality of G, there exists a homomorphism f of G' to T. By the remark, w_1 and w_2 forbid for w at most four colours, u_1 and u_2 forbid for u at most three colours and x forbids for v exactly two colours. Let α and β be two colours allowed for w. The size of the union of the set of colours allowed for v by w if we colour w with β and the set of colours allowed for v by w if we colour w with α is at least four. Thus, we can choose a colour for w so that w and x together forbid for v at most four colours.

Recall that we have a choice of three colours for u, each of which allows for v a 3-interval of colours. But the union of three distinct 3-intervals has the size at least five. Consequently, we can extend f on whole G.

If w is a $(3,6)$-vertex, then the proof is only easier. □

The proofs of the following four facts are very similar to that of (G3) and (G4), and we omit them.

(G5) *No* $(3,6)$*-vertex in* G *has a* $(3,6)$*- or* $(3,5)$*-quasi-neighbour on distance two.*

(G6) *No* $(3,6)$*-vertex in* G *has a* $(3,6)$*-quasi-neighbour on distance at most three.*

(G7) *If a* $(3,4)$*-vertex* v *in* G *is adjacent to a* $(3,5)$*-vertex, then it has no other* $(3,5)$*-quasi-neighbour on distance at most two, and no* $(3,6)$*-quasi-neighbour on distance at most three.*

(G8) *If a* $(3,5)$*-vertex* v *in* G *has a* $(3,6)$*-quasi-neighbour on distance three, then* v *has neither another* $(3,6)$*-quasi-neighbour on distance three, nor a* $(3,5)$*-quasi-neighbour on distance at most two.*

Now, let each vertex of G have the *charge* equal to its degree. We define a *discharging procedure* as follows:

(a) each senior vertex gives the amount $1/7$ to each 2-quasi-neighbour;

(b) each senior vertex which is not a $(3,6)$-vertex gives the amount $(4-k)/21$ to each $(3,6)$-quasi-neighbour on distance $k \leqslant 3$;

(c) each senior vertex which is neither a $(3,6)$-vertex nor a $(3,5)$-vertex gives the amount $(3-k)/21$ to each $(3,5)$-quasi-neighbour on distance $k \leqslant 2$.

For each $v \in V(G)$, let $d^\star(v)$ denote the charge of vertex v after this procedure. Since the sum of charges did not change, it is enough to verify that $d^\star(v) \geqslant 16/7$ for each $v \in V(G)$, to prove the theorem.

Case 1: $d_G(v) = k \geqslant 4$. Note that along any path with internal 2-vertices starting at v, v sends at most $3/7$. Indeed, if it sends something to the senior quasi-neighbour at the end of this path, then this path has less than three 2-vertices. Thus, $d^\star(v) \geqslant k - 3k/7 = 4k/7 \geqslant 16/7$.

Case 2: $d_G(v) = 2$. Then v receives $1/7$ from each of its senior quasi-neighbours. Thus, $d^\star(v) \geqslant 2 + 2/7 = 16/7$.

Case 3: *v is a $(3,i)$-vertex and $i \leqslant 3$.* By the rules, along a path of length $j+1$ with internal 2-vertices starting at v, v sends at most $j/7 + (3-j)/21 = 1/7 + 2j/21$. Hence $d^\star(v) \geqslant 3 - 3/7 - 2i/21 \geqslant 3 - 3/7 - 2/7 = 16/7$.

Case 4: *v is a $(3,4)$-vertex.* If v has no $(3,6)$-quasi-neighbour on distance at most three, then, in view of (G7), it sends to its $(3,5)$-quasi-neighbours at most $3/21$, and $d^\star(v) \geqslant 3 - 4/7 - 3/21 = 16/7$. So, let u_1 be a $(3,6)$-quasi-neighbour of v on minimum distance. By (G3), the distance between v and u_1 is at least two. If this distance is exactly two, then, by (G4) and (G7), at most one $(3,5)$- or $(3,6)$-quasi-neighbour of v distinct from u_1 is on distance at most three, and v gives to that vertex at most $1/21$. Thus, in this case $d^\star(v) \geqslant 3 - 4/7 - 2/21 - 1/21 = 16/7$. Finally, let the distance between v and u_1 be three. By (G7), v is not adjacent to a $(3,5)$-vertex, and hence gives to each of its senior quasi-neighbours at most $1/21$. Again, $d^\star(v) \geqslant 3 - 4/7 - 3/21 = 16/7$.

Case 5: *v is a $(3,5)$-vertex.* If v has no $(3,6)$-quasi-neighbour on distance at most three, then it gives nothing to senior vertices, and $d^\star(v) \geqslant 3 - 5/7 = 16/7$. So, let u_1 be a $(3,6)$-quasi-neighbour of v. By (G3) and (G5), the distance between v and u_1 is exactly three. Moreover, by (G8), none of the remaining senior quasi-neighbours u_2 and u_3 is a $(3,6)$-vertex on distance at most three from v. Since the sum of the distances from v to u_2 and to u_3 is equal to five, one of them, say u_2, is on distance at most two from

v, and the other is on distance at least three from v. By (G8), u_2 is not a $(3,5)$-vertex, and u_3 is not a $(3,6)$-vertex on distance at most three from v. Thus, v gives nothing to u_3 and receives exactly $1/21$ from u_2. In total, $d^\star(v) \geqslant 3 - 5/7 - 1/21 + 1/21 = 16/7$.

Case 6: *v is a $(3,6)$-vertex.* If v has no senior quasi-neighbour on distance at least four, then, by (G6) and (b), it receives from each senior quasi-neighbour at least $1/21$, and $d^\star(v) \geqslant 3 - 6/7 + 3/21 = 16/7$. So, let the distance between v and one of its senior quasi-neighbour u_1 be at least four. If at least one of the remaining senior quasi-neighbours u_2 and u_3 is adjacent to v, then v receives $1/7$ from this vertex and has $d^\star(v)$ at least $16/7$. If this is not the case, then one of u_2 and u_3 is on distance at most two and the other at most three from v. Again, v receives at least $2/21 + 1/21 = 1/7$ from u_2 and u_3.

Therefore, each vertex v in G has $d^\star(v) \geqslant 16/7$ which contradicts the fact that $\mathrm{mad}(G) < 16/7$. This proves Theorem 2.

In fact, the proof of Theorem 2 can be rewritten as an algorithm which for each graph $G \in \mathcal{MAD}_{16/7}$ with girth at least 11, constructs a homomorphism of G into $T(6;1,2)$ in polynomial time (actually, in time $O(|V(G)|^2)$).

4. On the girth of planar universal graphs

In this section, we prove Proposition 4 and Theorem 5.

Let $\mathcal{M}(k)$ denote the set of oriented graphs with girth at least k and maximum average degree at most two which have no homomorphism to other graphs with girth at least k and maximum average degree at most two. Since any $\mathcal{M}(k)$-universal graph admits a homomorphism from each oriented graph with maximum average degree at most two, we first describe $\mathcal{M}(k)$.

Let $G \in \mathcal{M}(k)$. If two arcs wv and uv enter the same vertex in G then the graph G' obtained from G by identifying w with u must be not in $\mathcal{M}(k)$. The only reason for it can be that the path (wvu) is a part of a cycle of length k or $k+1$ in G. Similar observation holds if two arcs leave the same vertex in G. These observations imply the following lemma.

Lemma 2. *Let $\mathcal{M}(k)_1$ denote the set of all directed cycles of length at least k, and $\mathcal{M}(k)_2$ denote the set of unicyclic oriented graphs whose cycle has length k or $k+1$ and such that each source in this cycle is entered by exactly one directed path, and from each sink in the cycle starts exactly one directed path. Then $\mathcal{M}(k) = \mathcal{M}(k)_1 \cup \mathcal{M}(k)_2$.*

Let $m = \max\{k, \lceil 1/\varepsilon \rceil\}$. We construct the universal graph F in question as the disjoint union of graphs F_1 and F_2. Each component of F_1 consists of two directed cycles with exactly one common vertex. One of these cycles has length $2m$ and another has one of the lengths $k, k+1, \ldots, k+2m$. Each component of F_2 consists of some oriented cycle C of length k or $k+1$ with exactly one directed cycle of length $2m$ attached to every source or sink in C.

Observe that every component of F is outerplanar of girth at least k and has maximum average degree less than $2 + 2/(2m) \leqslant 2 + \varepsilon$. It follows that F also possesses these properties. On the other hand, each graph in $\mathscr{M}(k)_1$ has a homomorphism to some component of F_1, and each graph in $\mathscr{M}(k)_2$ has a homomorphism to some component of F_2. This proves the proposition.

To prove Theorem 5, consider an arbitrary minimal by inclusion $\mathscr{P}_k$-universal graph H. If $\mathrm{mad}(H) \geqslant 3$, we are done. Suppose that $\mathrm{mad}(H) < 3$. Repeating the argument of Section 2, we obtain that the properties (i)–(iii) in Section 2 hold also for our H (with replacing 4 by k in (iii)). Instead of (iv) we can prove only the following weaker statement.

(iv′) *No vertex in H has in-degree or out-degree equal to* 0.

Proof. Let $x \in V(H)$. Consider the directed cycle C_k as the graph G_0. Since, by (iii), we can map any of its vertices to x, x has positive in- and out-degrees. □

Call an arc e incident with a vertex v *exceptional for* v, if e is the only arc which leaves v or the only arc which enters v. Denote by EA the set of all exceptional arcs in H, and by n_m the number of vertices of degree m in H. The following observation is obvious in view of (iv′).

(v) *If $d(v) = 2$, then both arcs incident with v are exceptional for v. If $d(v) = 3$, then exactly one arc incident with v is exceptional for v.*

(vi) *An arc uv cannot be exceptional for both u and v.*

Proof. Assume it is. Let G be the cycle $(x_1, \ldots, x_{2k+1})$ whose arcs are $x_{2i-1}x_{2i}$ and $x_{2i+1}x_{2i}$ $(i = 1, \ldots, k)$ and x_1x_{2k+1}. By (iii), there exists a homomorphism of G into H mapping x_1x_2 to uv. Since uv is exceptional for v, x_3x_2 also must be mapped to uv. Similarly, x_3x_4 must be mapped to uv and so on. Finally, x_{2k+1} must be mapped to u which is a contradiction. □

From (v) and (vi) we conclude that

(vii) $|EA| \geqslant 2n_2 + n_3$.

(viii) *A vertex v of degree m cannot be adjacent to $m-1$ arcs which are exceptional for other vertices.*

Proof. Assume it is. Because of symmetry, we may assume that all arcs entering v are exceptional and these arcs are $y_1v, \ldots, y_tv$. Denote $Y = \{y_1, \ldots, y_t\}$.

Let G be as in the proof of (vi). By (iii), there exists a homomorphism of G to H mapping x_1x_2 to y_1v. By the definition of Y, the arc x_3x_2 must be mapped to an arc of the kind y_iv. Since y_iv is exceptional for y_i, x_3x_4 also must be mapped to y_iv, and so on. Finally, x_{2k+1} must be mapped to y_j for some j which yields existence of the arc y_1y_j. Similarly, we obtain that the out-degree of every y_i in $H[Y]$ is at least 1; in particular, $|Y| \geqslant 3$. But then the average degree of $H[Y \cup \{v\}]$ is at least 3, which contradicts the assumption $\mathrm{mad}(H) < 3$. □

By (viii), we have $|EA| \leqslant \sum_{m=2}^{\infty} n_m(m-2)$. Comparing this with (vii), we get

$$2n_2 + n_3 \leqslant \sum_{m=2}^{\infty} n_m(m-2),$$

which is equivalent to

$$n_2 \leqslant \sum_{m=4}^{\infty} n_m(m-2)/2.$$

Since $(m-2)/2 \leqslant m-3$ for $m \geqslant 4$, this implies that

$$n_2 \leqslant \sum_{m=4}^{\infty} n_m(m-3),$$

that is,

$$\sum_{v \in V(H)} d_H(v) = \sum_{m=2}^{\infty} n_m m \geqslant \sum_{m=2}^{\infty} n_m 3 = 3|V(H)|.$$

This contradicts the fact that $\text{mad}(H) < 3$.

5. On the girth of $\mathscr{P}_k$-universal graphs

In this section, we prove Theorem 7. To do it, we need the following lemma (see [1, pp. 238–239]):

Lemma 3. *Let Y be the sum of n mutually independent indicator variables, $\mu = \boldsymbol{E}(Y)$. For all $\varepsilon > 0$,*

$$\boldsymbol{P}[Y < (1-\varepsilon)\mu] < \mathrm{e}^{-\varepsilon^2 \mu/2}.$$

Let g be fixed and $\alpha = 1/3g$. We choose any n such that

$$n^{\alpha} > 200 \ln n \tag{2}$$

and construct a random directed graph $\mathscr{G}$ (with loops) as follows. Let $U = \{u_1, \ldots, u_n\}$, $W = \{w_1, \ldots, w_n\}$. For each w_i and u_j, the arc $w_i u_j$ exists with probability $p = n^{\alpha - 1}$ independently of any other arcs. The graph $\mathscr{G}$ is obtained from this bipartite oriented graph by identifying w_i with u_i into the vertex v_i for each $i \in \{1, \ldots, n\}$. Denote $V = \{v_1, \ldots, v_n\}$.

So defined $\mathscr{G}$ with high probability has short cycles and even loops, but not many. Let S_1 be the event that there exists k, $2 \leqslant k \leqslant 2g-1$, and $M \subset V$ with $|M| = k$ such that $|E(\mathscr{G}(M))| \geqslant k+1$. Note that S_1 includes the event that a vertex with a loop belongs to some cycle of length at most $2g-1$.

Lemma 4. $\boldsymbol{P}[S_1] < 1/3$.

Proof.

$$P[S_1] \leqslant \sum_{k=2}^{2g-1} \binom{n}{k}\binom{k^2}{k+1} p^{k+1} \leqslant \sum_{k=2}^{2g-1} \frac{1}{2\pi k}\left(\frac{ne}{k}\right)^k \left(\frac{k^2 e}{k+1}\right)^{k+1} n^{(\alpha-1)(k+1)}$$

$$\leqslant \sum_{k=2}^{2g-1} \frac{1}{2\pi} e^{2k+1} n^{\alpha(k+1)-1} \leqslant e^{2(2g-1)} n^{2g\alpha-1} = e^{2(2g-1)} n^{-1/3}.$$

By (2), the last expression is less than $(e^4/200)^g < 1/3$. □

For each $A \subset V$, let $N^+(A) = \{v \in V \mid \exists x \in A : xv \in E(\mathcal{G})\}$ and $N^-(A) = \{v \in V \mid \exists x \in A : vx \in E(\mathcal{G})\}$. Let S_2 be the event that for some $A \subset V$ with $|A| < 1/p$, the inequality $\min\{|N^+(A|, |N^-(A|\} < pn|A|/4$ holds.

Lemma 5. $P[S_2] \leqslant 1/10$.

Proof. Let $A \subset V$ with $|A| = a < 1/p$ and $v \in V$. Clearly, $P[v \in N^+(A)] = 1 - (1-p)^a$. Since $a < 1/p$, we have

$$1 - (1-p)^a \geqslant 1 - 1 + pa - p^2\binom{a}{2} = pa\left(1 - \frac{1}{2}p(a-1)\right) > \frac{pa}{2}.$$

It follows that $E[|N^+(A)|] > npa/2$. By Lemma 3, we have

$$P\left[|N^+(A)| < \frac{npa}{4}\right] < \exp\left\{-\frac{pan}{16}\right\}.$$

Similarly, $P[|N^-(A)| < npa/4] < \exp\{-pan/16\}$. Thus,

$$P[S_2] < \sum_{a=1}^{\lceil 1/p \rceil} 2\binom{n}{a} \exp\left\{-\frac{pan}{16}\right\}$$

$$\leqslant \sum_{a=1}^{\lceil 1/p \rceil} \left(\frac{ne}{a}\right)^a \exp\{-an^\alpha/16\} < \sum_{a=1}^{\infty} \left(ne^{1-n^\alpha/16}\right)^a.$$

By (2), $n^\alpha/16 > 10 \ln n$, and so,

$$P[S_2] < \sum_{a=1}^{\infty} n^{-3a} < \frac{1}{10}. \qquad \square$$

Let S_3 be the event that for some $A \subset V$ with $|A| = \lceil 1/p \rceil$, the inequality $\min\{|N^+(A|, |N^-(A|\} < [4(e-1)/5e]n$ holds.

Lemma 6. $P[S_3] \leqslant 1/n$.

Proof. Let $A \subset V$ with $|A| = a = \lceil 1/p \rceil$ and $v \in V$. Since $pa \geqslant 1$,

$$P[v \in N^+(A)] = 1 - (1-p)^a > 1 - e^{-pa} \geqslant 1 - e^{-1}.$$

It follows that $\boldsymbol{E}[|N^+(A)|] > n(\mathrm{e}-1)/\mathrm{e}$. By Lemma 3, we have

$$\boldsymbol{P}[|N^+(A)| < 0.8n(\mathrm{e}-1)/\mathrm{e}] < \exp\{-0.02n(\mathrm{e}-1)/\mathrm{e}\}.$$

Similarly, $\boldsymbol{P}[|N^-(A)| < 0.8n(\mathrm{e}-1)/\mathrm{e}] < \exp\{-0.02n(\mathrm{e}-1)/\mathrm{e}\}$. Thus,

$$\begin{aligned} \boldsymbol{P}[S_3] &< 2\binom{n}{\lceil 1/p \rceil} \exp\{-0.02n(\mathrm{e}-1)/\mathrm{e}\} \\ &\leqslant n^{1/p} \exp\{-0.01n\} < \exp\{n^{1-\alpha} \ln n - 0.01n\}. \end{aligned}$$

By (2), $n^{1-\alpha} \ln n < n/200$, and so,

$$\boldsymbol{P}[S_3] < \exp\{-\ln n\} = 1/n. \qquad \square$$

By Lemmas 4–6, with probality at least $\frac{1}{3}$, $\mathscr{G}$ possesses the following properties:

(i) No two cycles of length at most g have a common vertex (in particular, no vertex with a loop belongs to a cycle of length at most g);

(ii) For each $A \subset V$ with $|A| < 1/p$, the inequality $\min\{|N^+(A)|, |N^-(A)|\} \geqslant pn|A|/4$ holds;

(iii) For each $A \subset V$ with $|A| = \lceil 1/p \rceil$, the inequality $\min\{|N^+(A)|, |N^-(A)|\} \geqslant [4(\mathrm{e}-1)/5\mathrm{e}]n$ holds.

It follows that there exists a digraph $G=(V,E)$ possessing all properties (i)–(iii). Denote by $H=(V,E')$ the oriented graph obtained from G by deleting one arc from each cycle of length at most g in G (in particular, every loop and an arc in every 2-cycle must be deleted). By (i), it can be done and the resulting H has girth at least $g+1$. By (ii), for each $A \subset V$ with $|A| < 1/p$, we have

$$\min\{|N_H^+(A)|, |N_H^-(A)|\} \geqslant pn|A|/4 - |A| > pn|A|/5 = n^\alpha |A|/5. \tag{3}$$

Similarly, under conditions (2), for each $A \subset V$ with $|A| = \lceil 1/p \rceil$, we have

$$\min\{|N_H^+(A)|, |N_H^-(A)|\} \geqslant \frac{4(\mathrm{e}-1)}{5\mathrm{e}} n - \left\lceil \frac{1}{p} \right\rceil > 0.505n - n^\alpha > \frac{n}{2}. \tag{4}$$

For an oriented path P and $v \in V$, let $N^P(v)$ denote the set of vertices $w \in V$ such that H contains a path isomorphic to P connecting v with w. Now, we prove that for each $v \in V$ and for each oriented path P of length $4g$,

$$|N^P(v)| > \frac{n}{2}. \tag{5}$$

Indeed, if $|N^{P_k}(v)| \geqslant 1/p$ for at least one initial subpath P_k with k edges of P, $(1 \leqslant k \leqslant 4g-1)$, then this follows from (4). Otherwise, by (3) and (2),

$$|N^P(v)| \geqslant (n^\alpha/5)^{4g} = n^{4/3}/5^{4g} > n,$$

which is impossible.

Inequality (5) implies that for each $v, x \in V$ and for each orientation of a $8g$-path, x can be reached from v by a path of this orientation. Now we are ready to prove Theorem 7.

A smallest counterexample to any of the statements of the theorem must have no 1-vertices. By above, it has no subpath on $8g-1$ vertices of degree 2. Then the first statement follows from the fact that any planar graph without 1- and 2-vertices has girth at most five, and the second follows from the discharging procedure when each vertex v of degree at least three gives $1/(24g-2)$ to each of its 2-quasi-neighbours.

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