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On kernel-perfect orientations of line graphs

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Abstract

We exploit the technique of Galvin (1995) to prove that an orientation D of a line-graph G (of a multigraph) is kernel-perfect if and only if every oriented odd cycle in D has a chord (or pseudochord) and every clique has a kernel.

Keywords: Line graphs; Orientations; Kernel-perfect digraphs

1. Introduction

A *kernel* in a digraph $D=(V,A)$ is an independent set S of vertices such that S *absorbs* V, i.e. for each $v\in V\backslash S$, there exists an $s\in S$ with $(v,s)\in A$. Kernels originated in the analysis of games on graphs, a kernel representing a set of winning positions. Many digraphs fail to have kernels, the simplest being the directed odd cycles.

A digraph $D=(V,A)$ is called *kernel-perfect* if every induced subgraph of it has a kernel. By the paragraph above, each kernel-perfect digraph D satisfies the property

$$\text{every directed odd cycle in } D \text{ has a chord (or pseudochord).} \qquad (1)$$

(By a pseudochord of a directed cycle $v_1,\dots,v_k$ we mean an arc (v_i,v_{i-1}) for some i.) Another necessary condition for a kernel-perfect digraph D is that D is *normal*, i.e. every clique in D has a kernel (which necessarily has cardinality 1, i.e. is a vertex). (By a clique we mean any subgraph where any two vertices are joined by at least one arc.) A normal digraph D satisfying (1) will be called *odd-chorded*.

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Remark. If D has no opposite arcs then satisfying (1) implies that D is normal (and hence odd-chorded).

For a digraph in general, being odd-chorded is not sufficient for being kernel-perfect. Galeana-Sánchez [2] constructed for each k a triangle-free digraph D_k with no kernel such that every directed odd cycle in D_k has at least k chords. Still, under some restrictions on the structure of the underlying unoriented graph of a digraph D, being odd-chorded is enough for being kernel-perfect.

Berge and Duchet [1] conjectured that every normal orientation of a perfect graph is kernel-perfect. This conjecture is proved only for some special classes of perfect graphs (see [4] for references). In particular, Maffray [4] proved it for perfect graphs which are line-graphs. Since every odd cycle of length at least five in a perfect graph has a chord, Maffray's theorem [4] is equivalent to the statement that an orientation of a perfect line-graph is kernel-perfect if and only if it is odd-chorded.

In his remarkable paper [3] on list edge-colourings of bipartite multigraphs, Galvin presented, among other ideas, a beautiful proof of the fact that normal orientations of line-graphs of bipartite multigraphs are kernel-perfect. (In fact, Galvin considered *line-multigraphs*: if edges f and g have two common ends in a multigraph H, then the corresponding vertices in $L(H)$ are joined by two edges. We shall do the same.) In this note, we elaborate Galvin's argument to prove the following extension of Maffray's theorem.

Theorem 1. *Let a multigraph G be the line-graph $L(H)$ of a multigraph H. Then an orientation D of G is kernel-perfect if and only if it is odd-chorded.*

Observe that an orientation D of the line-graph $L(H)$ of a multigraph H is odd-chorded if and only if it is normal and for each odd cycle C in H, the cycle $L(C)$ in D is not a directed cycle.

2. Proof of the theorem

We prove the theorem by induction on the number of edges in H. For H with $|E(H)| = 1$, the theorem holds.

Let H be a smallest (wrt. the number of edges) counter-example to the theorem, and D be an odd-chorded orientation of $L(H)$. By the minimality of H, we may assume that D has no kernel.

Since D is normal, for each $v \in V(H)$, each edge e incident with v can be labelled by a number $l_v(e)$ so that different edges get different labels and

$$l_v(e') < l_v(e'') \text{ implies that } (e', e'') \text{ is an arc in } D.$$

For every vertex $v \in V(H)$, let $e(v)$ denote the edge incident to v with the maximum label $l_v(e)$.

If for some distinct v and w, $e(v)=e(w)$, then by the choice of H, the restriction of D on $L(H-\{v,w\})$ has a kernel Q. It follows that $Q\cup\{e(v)\}$ is a kernel in D. Thus,

$$\text{all } e(v) \text{ are distinct.} \tag{2}$$

Let $M=\{e(v)\mid v\in V(H)\}$. Due to (2), the ends of any $e\in M$ can be marked as $x(e)$ and $y(e)$, where e is $e(x(e))$ and is not $e(y(e))$.

Case 1. For some $e_1\in M$, there exists $e_2\in E(H)$ incident with $y(e_1)$ such that $l_{y(e_1)}(e_2)<l_{y(e_1)}(e_1)$.

Choose e_2 incident with $y(e_1)$ having minimum $l_{y(e_1)}(e_2)$.

As in Galvin's proof [3], delete e_2 and by induction get a kernel Q in $D-e_2$, which turns out to be a kernel in D, too. Indeed, no vertex in D corresponding to an edge in H non-incident with $y(e_1)$ absorbs e_1. Hence, any vertex e_3 in Q absorbing e_1 corresponds to an edge in H incident with $y(e_1)$ (possibly, $e_3=e_1$). But then by the choice of $l_{y(e_1)}(e_2)$, e_3 absorbs e_2, as well.

Case 2. For each $e_1\in M$ and every $e_2\in E(H)$ incident with $y(e_1)$ and distinct from e_1, $l_{y(e_1)}(e_1)<l_{y(e_1)}(e_2)$.

In particular, all $y(e)$ should be distinct. Since $|M|=|V(H)|$, each vertex of H is the $y(e)$ for some $e\in M$. Thus, M forms a 2-factor in H. If at least one of the cycles formed by M is odd, we are done. Let all the cycles formed by M be even. For each $v\in V(H)$, denote by $a(v)$ the label $l_v(e)$ of the edge $e\in M$ with $v=y(e)$, and by $z(v)$ the label $l_v(e')$ of the edge $e'\in M$ with $v=x(e')$. In these terms, the conditions of the case can be rewritten as follows: for each edge $e=(v,w)$ in $H-M$,

$$a(v)<l_v(e)<z(v),\quad a(w)<l_w(e)<z(w). \tag{3}$$

Now we shall run a procedure for finding special subsets of $V(H)$, and according to its results either show a kernel in D or find there an odd directed cycle without chords. In both cases, it will contradict the definition of H.

Step 0. Among cycles formed by the edges in M, choose an arbitrary cycle $C_1=(v(1,1),\dots,v(1,2r_1))$. Put $W_1=\{v(1,2j)\mid 1\leqslant j\leqslant r_1\}$, $B_1=V(C_1)\backslash W_1$. Go to Step 1.

Step k ($k\geqslant 1$). The procedure terminates if either

(i) W_k is not independent in H; or

(ii) no vertex in $V(H)\backslash(B_k\cup W_k)$ is adjacent to W_k (in particular, if $V(H)=B_k\cup W_k$).

Otherwise, choose a vertex $v\in V(H)\backslash(B_k\cup W_k)$ adjacent to W_k. Let $C_{k+1}=(v(k+1,1),\dots,v(k+1,2r_{k+1}))$ be the cycle formed by the edges in M containing v. Renumber the vertices in C_{k+1} so that $v=v(k+1,1)$. Put $W_{k+1}=W_k\cup\{v(k+1,2j)\mid 1\leqslant j\leqslant r_{k+1}\}$, $B_{k+1}=B_k\cup V(C_{k+1})\backslash W_{k+1}$. Go to Step $k+1$.

By the definition, the number of steps is at most the number of the cycles formed by the edges in M. Let the procedure terminate on Step m. Assume first that W_m is independent in H. Then each edge in H incident with W_m is also incident with B_m. By the minimality of H, the subgraph of D induced by the edges of $H'=H-W_m-B_m$ has a kernel Q. By the construction, the set $M'=\{e(v)\mid v\in B_m\}$ is a matching in H and absorbs all the vertices in D corresponding to edges incident with B_m. Thus, $Q \cup M'$ is a kernel in D, a contradiction.

Now assume that W_m is not independent in H. Let $e=(a,b)$ be such that $a,b \in W_m$. We may assume that $a \in V(C_q)$ and $b \in V(C_p)$ for some $1 \leqslant p,q \leqslant m$ (in fact, at least one of q and p is m). If $p=q$, then our odd cycle is formed by (a,b) and the part of C_p connecting b with a and such that for the first edge e of this path, $x(e)=b$. Let $p \neq q$. Note that all vertices in B_m have odd indices in cycles $C_1,\dots,C_m$ and all vertices in W_m have even indices in these cycles. Moreover, for each k, $2 \leqslant k \leqslant m$, there is a number $f(k)$, $1 \leqslant f(k) \leqslant k-1$ such that $v(k,1)$ is adjacent to a vertex $w(k) \in V(C_{f(k)}) \cap W_{f(k)}$. Thus, there exist sequences $1=j_{1,1}<j_{1,2}<\cdots<j_{1,s_1}=q$ and $1=j_{2,1}<j_{2,2}<\cdots<j_{2,s_2}=p$ such that $j_{1,i}=f(j_{1,i+1})$ for each $1 \leqslant i \leqslant s_1-1$ and $j_{2,i}=f(j_{2,i+1})$ for each $1 \leqslant i \leqslant s_2-1$. Let $h=j_{1,t(1)}=j_{2,t(2)}$ be the largest common number in sequences $j_{1,1},j_{1,2},\dots,j_{1,s_1}$ and $j_{2,1},j_{2,2},\dots,j_{2,s_2}$. Vertices $w(j_{1,t(1)+1})$ and $w(j_{2,t(2)+1})$ lying on C_h may coincide. In this case we may assume that in D the arc connecting vertices corresponding to the edges $(v(j_{1,t(1)+1},1),w(j_{1,t(1)+1}))$ and $(v(j_{2,t(2)+1},1),w(j_{2,t(2)+1}))$ leads towards $(v(j_{1,t(1)+1},1),w(j_{1,t(1)+1}))$. Now we define some pieces of a future odd directed cycle. For each i, $t(1)+1 \leqslant i \leqslant s_1-1$, let P_i be the part of $C_{j_{1,i}}$ connecting $v(j_{1,i},1)$ with $w(j_{1,i+1})$ and such that for the first edge e of this path, $x(e)=v(j_{1,i},1)$. By P_{s_1} denote the part of C_q connecting $v(j_{1,s_1},1)=v(q,1)$ with a and such that for the first edge e of this path, $x(e)=v(q,1)$. Furthermore, let P'_{s_2} denote the part of C_p connecting b with $v(j_{2,s_2},1)=v(p,1)$ and such that for the first edge e of this path, $x(e)=b$. Then for each i, $t(2)+1 \leqslant i \leqslant s_2-1$, let P'_i be the part of $C_{j_{2,i}}$ connecting $w(j_{2,i+1})$ with $v(j_{2,i},1)$ and such that for the first edge e of this path, $x(e)=w(j_{2,i+1})$. If $w(j_{1,t(1)+1})=w(j_{2,t(2)+1})$ then we define $P_0=\emptyset$, otherwise let P_0 be the part of C_h connecting $w(j_{2,t(2)+1})$ with $w(j_{1,t(1)+1})$ and such that for the first edge e of this path, $x(e)=w(j_{2,t(2)+1})$. Our odd cycle C is as follows:

$$\begin{aligned} C = {} & w(j_{1,t(1)+1}),\ v(j_{1,t(1)+1},1)P_{t(1)+1}w(j_{1,t(1)+2}),\ \dots, \\ & v(j_{1,s_1},1)P_{s_1}a,\ bP'_{s_2}v(j_{2,s_2},1), w(j_{2,s_2})P'_{s_2-1}\cdots \\ & v(j_{2,t(2)+1},1),\ w(j_{2,t(2)+1})P_0. \end{aligned}$$

By (3), the cycle in D whose vertices correspond to the edges of C is a directed odd cycle (without chords).

3. Concluding remarks

A number of results on line-graphs can be extended to graphs without induced $K_{1,3}$ or without induced $K_{1,3}$ and K_5-e. It looks as though this is not the case here. Indeed, consider the digraph $D_7=(V,E)$ with $V=\{1,\dots,7\}$ and $E=\{(i,i+1),(i,i+2)\,|\,i=1,\dots,7\}$ (indices are taken mod 7). Since D_7 is an orientation of the complement of the 7-cycle C_7, it contains neither K_4 nor $K_{1,3}$. It is clear that any chordless directed cycle C in D_7 makes exactly one round around $\{1,\dots,7\}$. To make this round, any such C needs at least four arcs. If it has at least five arcs then among them there are two consecutive arcs of length one, and C has a chord. Thus, D_7 is odd-chorded. Assume that M is a kernel in D_7. Since any vertex in D_7 absorbs 3 vertices (counting itself), $|M| \geqslant 3$. But C_7 has no 3-cliques, a contradiction.

In fact, the proof contains a polynomial algorithm to find a kernel in an arbitrary odd-chorded orientation of a line-graph. It could be noted that Galvin's proof gives a polynomial algorithm for finding list-colourings of edges if their lists are as big as the maximum degree.

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References

[1] C. Berge, P. Duchet, Seminaire MSH, Paris, January 1983.
[2] H. Galeana-Sánchez, A counterexample to a conjecture of Meyniel on kernel-perfect graphs, Discrete Math. 41 (1982) 105–107.
[3] F. Galvin, The list chromatic number of a bipartite multigraph, J. Combin. Theory Ser. B 63 (1995) 153–158.
[4] F. Maffray, Kernels in perfect line-graphs, J. Combin. Theory Ser. B 55 (1992) 1–8.