ELSEVIER

Discrete Mathematics 188 (1998) 261–266

DISCRETE MATHEMATICS

Note

On the independent domination number of graphs with given minimum degree

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Received 23 October 1996; revised 15 October 1997; accepted 27 October 1997

Abstract

We prove a new upper bound on the independent domination number of graphs in terms of the number of vertices and the minimum degree. This bound is slightly better than that of Haviland (1991) and settles the case $\delta = 2$ of the corresponding conjecture by Favaron (1988). © 1998 Elsevier Science B.V. All rights reserved

1. Introduction

The independent domination number $i(G)$ of a graph G is defined to be the minimum cardinality among all maximal (by inclusion) independent sets of G. Let $i(n,\delta)$ denote the maximum of the independent domination numbers over all graphs with n vertices and minimum degree δ.

By a theorem of Bollobás and Cockayne [1], $i(n,1) \leqslant n + 2 - 2\sqrt{n}$. Favaron [2] proved the bound $i(n,\delta) \leqslant n + 3\delta - 2\sqrt{\delta(n+2\delta)}$ and conjectured that

$$i(n,\delta) \leqslant n + 2\delta - 2\sqrt{\delta n}. \tag{1}$$

This last bound (if true), for every fixed positive integer δ, is attained on infinitely many graphs. Haviland [3] improved the bound of Favaron as follows: if $0 \leqslant \delta \leqslant (n-2)/7$, then

$$i(n,\delta) \leqslant n + 3\delta - \min\{1 + 2\sqrt{\delta(n+2\delta-2)}, 2\sqrt{\delta(n+9\delta/4)}\},$$

and if $(n-2)/7 \leqslant \delta \leqslant n/4$, then $i(n,\delta) \leqslant 2(n-\delta)/3$.

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[1] This work was partially supported by the grant 96-01-01591 of the Russian Foundation for Fundamental Research.

[2] This work was partially supported by the grants 96-01-01614 and 97-01-01075 of the Russian Foundation for Fundamental Research and by the Network DIMANET of the European Union.

PII S0012-365X(97)00267-7

The aim of this note is to prove the following:

Theorem 1. *For each integer* $\delta \geqslant 2$,

$$i(n,\delta) \leqslant n + 3\delta - 2\sqrt{\delta(n+2\delta-4)} - 2. \tag{2}$$

The bound in Theorem 1 is better than that of Haviland for $n > 7\delta+2$ and for $\delta = 2$ is equivalent to (1). This means that for $\delta = 2$ the bound is sharp for infinitely many n.

2. Proof of Theorem 1

Let n and δ be positive integers with $2 \leqslant \delta < n$. Choose a graph G on n vertices with the minimum degree at least δ and $i(G) = i(n,\delta)$. In the sequel, I is an independent dominating set in G with $i = |I| = i(n,\delta)$, $K = V(G)\setminus I$, $k = |K| = n - i$. For $X, Y \subseteq V(G)$ and $v \in V(G)$, we denote $[X,Y] = \{(v,w) \in E(G) \mid v \in X, w \in Y\}$, $N(v) = \{w \in V(G) \mid (v,w) \in E(G)\}$, $N_I(v) = N(v) \cap I$, $d_I(v) = |N_I(v)|$, $N(X) = \bigcup_{v \in X} N(v)$.

In these terms, we need to prove that

$$k \geqslant 2\sqrt{\delta(n+2\delta-4)} - 3\delta + 2.$$

Assume that the opposite holds, i.e., that

$$k < 2\sqrt{\delta(n+2\delta-4)} - 3\delta + 2. \tag{3}$$

We shall show that this assumption leads to a contradiction. First, we observe that (3) yields that for each positive x,

$$k(x-1+\delta) < x^2 - 1 + \delta(n-x-1). \tag{4}$$

Indeed, for each positive x,

$$2\sqrt{\delta(n+2\delta-4)} - 3\delta + 2 \leqslant (x-1+\delta) + \frac{\delta(n+2\delta-4)}{x-1+\delta} - 3\delta + 2 = \frac{x^2-1+\delta(n-x-1)}{x-1+\delta},$$

and so, (3) implies (4).

Inequality (4) can be rewritten also in the form

$$(k-x-1)(x-1) < \delta(n-k-x-1). \tag{5}$$

Since $E(G) \neq \emptyset$, $K \neq \emptyset$. For each $v \in K$, we define $S(v) = \{w \in K \mid N_I(w) \subseteq N_I(v)\}$ and $\widetilde{S}(v) = S(v)\setminus N(v)$. By definition, $v \in \widetilde{S}(v)$.

Proposition 1. *Let $v \in K$ and M be an arbitrary independent subset of $\widetilde{S}(v)$, containing v and dominating $\widetilde{S}(v)$. Then*

$$|M| \geqslant d_I(v).$$

Proof. Observe that $(I \setminus N_I(v)) \cup M$ is an independent dominating set in G. This yields the proposition. $\square$

It follows from Proposition 1 that

$$|\widetilde{S}(v)| \geqslant d_I(v). \tag{6}$$

Choose $v_1 \in K$ with $|N_I(v_1)| = \max\{|N_I(v)| : v \in K\}$, and denote $x = |N_I(v_1)|$.

Proposition 2. $x \geqslant \delta + 1$.

Proof. If $x \leqslant \delta$, then

$$\delta k \geqslant xk \geqslant |[K, I]| \geqslant \delta |I| = \delta(n-k),$$

which gives $k \geqslant n/2$. But $n/2 \geqslant 2\sqrt{\delta(n+2\delta-4)} - 3\delta + 2$, since $(\sqrt{n+2\delta-4} - 2\sqrt{\delta})^2 \geqslant 0$. This contradicts (3). $\square$

Proposition 3. *If $v \in K$ and $d_I(v) = x$ then $N_I(v) \cap N_I(v_1) \neq \emptyset$.*

Proof. Let m be the maximum integer for which there exist vertices $v_1, \ldots, v_m$ in K such that $d_I(v_j) = x$ for each $1 \leqslant j \leqslant m$ and $N_I(v_j) \cap N_I(v_h) = \emptyset$ for each $1 \leqslant j < h \leqslant m$. Assume that $m > 1$. Denote $N = \bigcup_{j=1}^{m} N_I(v_j)$ and $S = \bigcup_{j=1}^{m} S(v_j)$. Then $|(I \setminus N) \cap N(v)| \leqslant x - 1$ for each $v \in K \setminus S$, and by (6) we get

$$(k - mx)(x-1) \geqslant |[K \setminus S, I \setminus N]| \geqslant \delta(n - k - mx).$$

Rewriting this and applying Proposition 2, we obtain

$$\begin{aligned} k(x - 1 + \delta) &\geqslant mx(x-1) + \delta(n - mx) \\ &= mx(x - \delta - 1) + \delta n \\ &\geqslant 2x(x - \delta - 1) + \delta n \\ &= x^2 - 1 + \delta(n - x - 1) + (x-1)(x - \delta - 1) \\ &\geqslant x^2 - 1 + \delta(n - x - 1), \end{aligned}$$

which contradicts (4). $\square$

Proposition 4. *For each $v \in K$, $\widetilde{S}(v) \cap \widetilde{S}(v_1) \neq \emptyset$. In particular, $N_I(v) \cap N_I(v_1) \neq \emptyset$.*

Proof. Assume that for some $v \in K$, $\widetilde{S}(v) \cap \widetilde{S}(v_1) = \emptyset$. Denote $y = d_I(v)$. Then, by Proposition 3, $(k-x-y)(x-1) \geqslant \delta(n-k-x-y)$, and hence

$$\begin{aligned} k(x-1+\delta) &\geqslant x^2 - 1 + \delta(n-x-1) + (y-1)(x-1-\delta) \\ &\geqslant x^2 - 1 + \delta(n-x-1), \end{aligned}$$

which again contradicts (4). □

Proposition 5. *There exists $v_2 \in K$ such that $d_I(v_2) = x$ and $|N_I(v_2) \cap N_I(v_1)| = 1$.*

Proof. Otherwise,

$$(k-x)(x-2) \geqslant |[K \setminus S(v_1), I \setminus N_I(v_1)]| \geqslant \delta(n-k-x).$$

It follows that

$$k(x-2+\delta) \geqslant x(x-2) + \delta(n-x) = (x-2\delta)(x-2+\delta) + \delta(n+2\delta-4),$$

i.e.

$$k \geqslant (x-2+\delta) + \frac{\delta(n+2\delta-4)}{x-2+\delta} - 3\delta + 2 \geqslant 2\sqrt{\delta(n+2\delta-4)} - 3\delta + 2,$$

which contradicts (3). □

In view of Proposition 5, we can consider a vertex $z \in I$ and $Z \subseteq K$ such that $\{z\} = N_I(v_1) \cap N_I(v_2)$ and $Z = \{v \in K \mid N_I(v) = \{z\}\} = S(v_1) \cap S(v_2)$.

Proposition 6. *For each $v \in K$, $z \in N(v)$. In particular, $Z \subseteq S(v)$.*

Proof. *Case* 1: $v \in S(v_1)$. By definition, $N_I(v) \subseteq N_I(v_1)$. By Proposition 4, $N_I(v) \cap N_I(v_2) \neq \emptyset$. Since $N_I(v_1) \cap N_I(v_2) = \{z\}$, we get $N_I(v) \cap N_I(v_2) = \{z\}$.

Case 2: $v \in K \setminus S(v_1)$. By Proposition 4, there exists $w \in S(v) \cap S(v_1)$. According to Case 1, $z \in N(w)$. Since $w \in S(v)$, $N_I(w) \subseteq N_I(v)$. □

Proposition 7. $|Z| = x - \delta$.

Proof. If $|Z| \geqslant x - \delta + 1$, then

$$(k-x+\delta-1)(x-1) \geqslant |[K \setminus Z, I \setminus \{z\}]| \geqslant \delta(n-k-1).$$

Consequently, $(k-x-1)(x-1) \geqslant \delta(n-k-x)$, which contradicts (5).

If $|Z| \leqslant x - \delta - 1$, then

$$\begin{aligned} (k-x-\delta-1)(x-1) &\geqslant |[K \setminus (S(v_1) \cup S(v_2)), I \setminus (N_I(v_1) \cup N_I(v_2))]| \\ &\geqslant \delta(n-k-2x+1). \end{aligned}$$

Consequently, $(k-x-1)(x-1) \geqslant \delta(n-k-x)$, which again contradicts (5). □

Proposition 8. *For each* $v \in K$, $|S(v) \setminus Z| \leqslant \delta$.

Proof. Assume that $|S(v) \setminus Z| \geqslant \delta + 1$. Then, due to Propositions 6,7 and 4, $|S(v)| \geqslant |Z| + \delta + 1 \geqslant x + 1$, and

$$(k - x - 1)(x - 1) \geqslant |[K \setminus S(v), I \setminus N_I(v)]| \geqslant \delta(n - k - |N_I(v)|).$$

Thus, $(k - x - 1)(x - 1) \geqslant \delta(n - k - x)$ which contradicts (5). □

Proposition 9. *For each* $v \in Z$, $|N(v) \cap (K \setminus Z)| \geqslant \delta - 1$.

Proof. If the statement is not fulfilled, then Z is not independent and $S(v_1)$ is bigger than any of its independent subsets. In this case, by Proposition 1, $|S(v_1)| \geqslant x + 1$. On the other hand, by Propositions 7 and 8,

$$|S(v_1)| \leqslant |Z| + |S(v_1) \setminus Z| \leqslant (x - \delta) + \delta = x. \quad \square$$

Proposition 10. *For each* $v \in K$, $|(Z \cup I) \cap N(v)| \leqslant x$.

Proof. By (6) and Propositions 7 and 8, we have

$$\begin{aligned}|(Z \cup I) \cap N(v)| &= |Z \cap N(v)| + |N_I(v)| \leqslant |Z \cap N(v)| + |\widetilde{S}(v)| \\ &= |Z \cap N(v)| + |Z \cap \widetilde{S}(v)| + |\widetilde{S}(v) \setminus Z| \leqslant (x - \delta) + \delta = x. \quad \square\end{aligned}$$

Now, we come to the final part of the proof of Theorem 1. By Propositions 10, 7 and 6,

$$|[Z \cup (I \setminus \{z\}), K \setminus Z]| \leqslant |K \setminus Z|(x - 1) = (k - x + \delta)(x - 1).$$

On the other hand, by Propositions 7 and 9,

$$\begin{aligned}|[Z \cup (I \setminus \{z\}), K \setminus Z]| &= |[Z, K \setminus Z]| + |[I \setminus \{z\}, K \setminus Z]| \\ &\geqslant (\delta - 1)|Z| + \delta|I \setminus \{z\}| = (\delta - 1)(x - \delta) \\ &\quad + \delta(n - k - 1).\end{aligned}$$

Taking into account that $\delta \geqslant 2$ and $x \geqslant \delta + 1$, we obtain

$$\begin{aligned}k(x - 1 + \delta) &\geqslant (x - \delta)(x - 1) + (\delta - 1)(x - \delta) + \delta(n - 1) \\ &= x^2 - 1 + (\delta - 2)(x - \delta) + \delta(n - x - 1) + 1 > x^2 - 1 \\ &\quad + \delta(n - x - 1),\end{aligned}$$

which contradicts (4). □

References

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