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Colour-critical graphs with few edges

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Abstract

A graph G is called k-critical if G is k-chromatic but every proper subgraph of G has chromatic number at most $k-1$. In this paper the following result is proved. If G is a k-critical graph ($k \geqslant 4$) on n vertices, then $2|E(G)| > (k-1)n + ((k-3)/(k^2-3))n + k - 4$ where $n \geqslant k+2$ and $n \neq 2k-1$. This improves earlier bounds established by Dirac (1957) and Gallai (1963).

Keywords: Chromatic number; Critical graphs

1. Introduction

About 1950, G.A. Dirac introduced the concept of colour criticality. This concept — invented for simplifying graph colouring theory — has given rise to numerous investigations and beautiful theorems.

In this paper a new lower bound for the number of edges possible in a k-critical graph on n vertices is established.

1.1. Terminology

Concepts and notation not defined in this paper will be used as in standard textbooks.

The graphs considered are finite, undirected and without loops and multiple edges. The set of vertices and the set of edges of a graph G are denoted by $V(G)$ and $E(G)$, respectively. An edge of G joining the distinct vertices $x, y \in V(G)$ is denoted by xy or yx, and the vertices x and y are said to be *adjacent* in G. For $x \in V(G)$, let $N_G(x)$ denote the set of all vertices in G that are adjacent to x in G. The *degree* of x with respect to G is $d_G(x) = |N_G(x)|$.

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Let $X \subseteq V(G)$. The subgraph of G induced by X is denoted by $G(X)$, i.e., $V(G(X)) = X$ and $E(G(X)) = \{e \in E(G) \mid e = xy \text{ and } x, y \in X\}$; further, $G - X := G(V(G) - X)$. The set X will be called a *clique* (respectively, an *independent set*) in G if $G(X)$ is a complete graph (respectively, a graph without edges). As usual, K_n denotes the complete graph on n vertices

A (proper) *k-colouring* of a graph G is a mapping φ of $V(G)$ into the (colour-)set $\{1,\dots,k\}$, $k \geqslant 1$, such that $\varphi(x) \neq \varphi(y)$ for any two adjacent vertices $x, y \in V(G)$. A graph G which admits a k-colouring is called *k-colourable*. The *chromatic number* $\chi(G)$ of G is the smallest integer k for which G is k-colourable. If $\chi(G) = k$, then G is called *k-chromatic*.

1.2. Critical graphs

A graph G is called *critical* if $\chi(G') < \chi(G)$ for every proper subgraph G' of G; it is called *k-critical* if it is critical and k-chromatic. K_k is an example of a k-critical graph and for $k = 1, 2$ it is the only one. The 3-critical graphs are the odd circuits. For $k \geqslant 4$ there are k-critical graphs on n vertices for all $n \geqslant k$ except for $n = k + 1$.

Let G be a k-critical graph. If X is a clique in G, then $G - X$ is empty or connected. In particular, G is connected and has no separating vertex. Moreover, $d_G(x) \geqslant k - 1$ for all $x \in V(G)$. The vertices of G whose degrees are equal to $k - 1$ are called the *low vertices* and the others are called the *high vertices*. The subgraph of G induced by the set of low vertices is called the *low-vertex subgraph* of G. By Brooks' theorem [1], a k-critical graph G has no high vertices if and only if G is a K_k or $k = 3$ and G is an odd circuit.

Let $f_k(n)$ denote the minimum number of edges possible in a k-critical graph on n vertices where $k \geqslant 4$ and $n \geqslant k + 2$. Brooks' theorem implies

$$2f_k(n) \geqslant (k - 1)n + 1,$$

and Dirac [2] proved

$$2f_k(n) \geqslant (k - 1)n + k - 3.$$

A graph G is called a *Hajós graph* of order $2k - 1$ if the vertex set of G consists of three non-empty pairwise disjoint sets A, B_1, B_2 with $|B_1| + |B_2| = |A| + 1 = k - 1$ and two additional vertices a, b such that A and $B_1 \cup B_2$ are two cliques in G not joined by any edge, $N_G(a) = A \cup B_1$ and $N_G(b) = A \cup B_2$. Dirac [3] proved that $2|E(G)| = (k - 1)n + k - 3$ for a k-critical graph G on $n \geqslant k + 2$ vertices if and only if G is a Hajós graph of order $2k - 1$. Another generalization of Brooks' theorem due to Gallai [4], is

$$2f_k(n) > (k - 1)n + \frac{k - 3}{k^2 - 3}n.$$

Moreover, Gallai [5] determined the exact value of $f_k(n)$ for $n \leqslant 2k-1$. Recently, Gallai's result has been improved by Krivelevich [8] to

$$2f_k(n) \geqslant (k-1)n + \frac{k-3}{k^2-2k-1}n.$$

Krivelevich [9] also presents improved bounds on the number of edges in k-critical graphs with given clique number or odd girth. In Section 2 we shall prove

$$2f_k(n) > (k-1)n + \frac{k-3}{k^2-3}n + k - 4$$

provided that $n \neq 2k-1$. The exceptional graphs are again the Hajós graphs of order $2k-1$.

More information concerning the number of edges of critical graphs can be found in [6, Ch. 5].

1.3. The basic theorems of Gallai

A maximal connected subgraph B of G such that any two edges of B are contained in a circuit of G is called a *block* of G. Obviously, a vertex of G is a separating vertex of G iff it is contained in more than one block of G. An *endblock* of G is a block which contains at most one separating vertex of G.

A connected graph G all of whose blocks are complete graphs and/or odd circuits is called a *Gallai tree*; a *Gallai forest* is a graph all of whose components are Gallai trees.

The next result due to Gallai [4] is fundamental in the theory of critical graphs. For a short proof of this result the reader is referred to [7].

Theorem 1.1. *If G is a k-critical graph $(k \geqslant 4)$, then the low-vertex subgraph of G is a Gallai forest (possibly empty).*

An ε_k*-graph* is a graph defined recursively as follows:
(i) a K_{k-1} is an ε_k-graph;
(ii) if G_1 and G_2 are two disjoint ε_k-graphs and, for $i=1,2$, x_i is a vertex of degree $k-2$ in G_i, then the graph obtained from G_1 and G_2 by adding the edge x_1x_2 is an ε_k-graph.

Clearly, every ε_k-graph is a Gallai tree. Gallai [4] proved the following.

Theorem 1.2. *If G is a k-critical graph $(k \geqslant 5)$ with exactly one high vertex x, then $G-x$ is an ε_k-graph and $N_G(x)$ is the set of vertices of degree $k-2$ in $G-x$.*

1.4. Extension of colourings

Let G be a graph and $H \subseteq V(G)$. For $x \in V(G)$, let $N(x:H) = N_G(x) \cap H$. For a subgraph C of G, let $N(C:H) = \bigcup_{x \in V(C)} N(x:H)$. H is said to be a *k-set* of G if $d_G(x) = k-1$ for all $x \in V(G) - H$. Clearly, if the graph G is k-critical and H contains all high vertices of G, then H is a k-set of G. The next theorem is crucial for the proof of our main result.

Theorem 1.3. *Let G be a graph, $H \subseteq V(G)$ a k-set of G ($k \geqslant 4$), and C a component of $G-H$. Assume that there is a $(k-1)$-colouring φ of $G(H)$ satisfying one of the following three conditions.*
(a) $|\varphi(N(x:H)| < |N(x:H)|$ *for some* $x \in V(C)$, *or*
(b) $\varphi(N(x:H)) \neq \varphi(N(y:H))$ *for two non-separating vertices x, y contained in the same endblock B of C, or*
(c) *C is an ε_k-graph and* $|\varphi(N(C:H))| \geqslant 2$.
Then φ can be extended to a $(k-1)$-colouring of $G(H \cup V(C))$.

Proof. (By induction on $|C|$). First, assume that (a) holds. For $|C|=1$, the statement is evident. If $|C| \geqslant 2$, then there is a non-separating vertex $y \neq x$ in C. Since y has at most $k-2$ neighbours in H, φ can be extended to a $(k-1)$-colouring φ' of $G(H \cup \{y\})$. Obviously, $H' = H \cup \{y\}$ is a k-set of G, $C' = C - y$ is a component of $G - H'$, and $|\varphi'(N(x:H'))| < |N(x:H')|$. Therefore, by the induction hypothesis, φ' can be extended to a $(k-1)$-colouring of $G(H' \cup V(C')) = G(H \cup V(C))$.

Next, assume that (b) holds. Since B is an endblock, (b) implies that there are two adjacent non-separating vertices u, v of C contained in B such that $\varphi(N(u:H)) \neq \varphi(N(v:H))$. Let $i \in \varphi(N(u:H)) - \varphi(N(v:H))$. Clearly, φ can be extended to a $(k-1)$-colouring φ' of $G(H \cup \{v\})$ where $\varphi'(v) = i$. Then $H' = H \cup \{v\}$ is a k-set of G, $C' = C - v$ is a component of $G - H'$, and $|\varphi'(N(u:H'))| < |N(u:H')|$. Therefore, because of (a), φ' can be extended to a $(k-1)$-colouring of $G(H' \cup V(C')) = G(H \cup V(C))$.

Eventually, assume that (c) holds. Let B be an endblock of C. Since C is an ε_k-graph, B is a K_{k-1}. If $C = B$, then every vertex of C has exactly one neighbour in H. From $|\varphi(N(C:H))| \geqslant 2$ it then follows that $\varphi(N(x:H)) \neq \varphi(N(y:H))$ for two vertices x, y of $C = B$. Then, because of (b), φ can be extended to a $(k-1)$-colouring of $G(H \cup V(C))$. If $C \neq B$, then exactly one separating vertex y of C is contained in B. For $x \in V(B-y)$, let $h(x)$ denote the only neighbour of x in G belonging to H. If $\varphi(h(x)) \neq \varphi(h(x'))$ for some vertices $x, x' \in V(B-y)$, then, because of (b), φ can be extended to a $(k-1)$-colouring of $G(H \cup V(C))$. Otherwise, $\varphi(h(x)) = 1$ for all $x \in V(B-y)$ and, since B is a K_{k-1} and y has no neighbour in H, φ can be extended to a $(k-1)$-colouring φ' of $G(H \cup V(B))$ where $\varphi'(y) = 1$. Then $H' = H \cup V(B)$ is a k-set and $C' = C - B$ is an ε_k-graph. Since y has exactly one neighbour in C', $|\varphi'(N(C:H'))| \geqslant 2$. Therefore, by the induction hypothesis, φ' can be extended to a $(k-1)$-colouring of $G(H' \cup V(C')) = G(H \cup V(C))$.

This proves Theorem 1.3. □

2. On the number of edges in critical graphs

In this section the following result is proved.

Theorem 2.1. *Let G be a k-critical graph where $k \geqslant 4$. If G is neither a complete graph of order k nor a Hajós graph of order $2k-1$, then*

$$2|E(G)| > |V(G)| \left(k - 1 + \frac{k-3}{k^2-3}\right) + k - 4.$$

In what follows, G denotes a k-critical graph ($k \geqslant 4$) satisfying the assumption of Theorem 2.1. Let $V = V(G)$, $E = E(G)$, $H = \{x \in V \mid d_G(x) \geqslant k\}$ and $L = \{x \in V \mid d_G(x) = k-1\}$. Clearly, $|V| = |H| + |L|$ and $H \neq \emptyset$. Theorem 2.1 is obviously true if $L = \emptyset$. We may therefore assume that $L \neq \emptyset$. The *excess* Δ of G is defined as

$$\Delta = 2\,|E| - |V| \left(k - 1 + \frac{k-3}{k^2-3}\right).$$

We have to show that $\Delta > k-4$. We start with some preliminaries. Let

(a) $\beta = |E(G(H))|$, $\gamma = \sum_{v \in H}(d_G(v) - k)$, and

(b) $\delta = |L|\,r_k - 2|E(G(L))|$ where $r_k = k - 2 + 2/(k-1)$.

For the number of edges of G, on the one hand, we have

$$2|E| = 2\beta + 2(k-1)|L| - 2|E(G(L))| = 2\beta + \delta + |L| \left(k - \frac{2}{k-1}\right)$$

and, on the other, we have

$$2|E| = (k-1)|V| + |H| + \gamma.$$

Adding the last equation multiplied by $(k - 2/(k-1))$ to the previous one, by an easy calculation we obtain

$$\Delta = \gamma \frac{k^2-k-2}{k^2-3} + (2\beta + \delta)\frac{k-1}{k^2-3} > \frac{k-1}{k^2}(k\gamma + 2\beta + \delta). \tag{1}$$

2.1. Lower bounds for δ

Next, we shall prove some auxiliary propositions. For an arbitrary graph F, let $\delta(F) = |V(F)|\,r_k - 2|E(F)|$. Let $\mathscr{C}$ denote the set of all components of $G(L)$. Then

$$\delta = \sum_{C \in \mathscr{C}} \delta(C). \tag{2}$$

Let $\mathscr{T}_k$ denote the set of all Gallai trees distinct from K_k and with maximum degree at most $k-1$. By Thoerem 1.1, $\mathscr{C} \subseteq \mathscr{T}_k$. For $C \in \mathscr{T}_k$ and some endblock B of C, let $C_B = C - (B - x)$ where x is the only separating vertex of C contained in B (if there is no such vertex, then $C = B$ and an arbitrary vertex of B may be taken). The proof of the next result is left to the reader.

Lemma 1. (1.1) *If B is a complete graph of order $b \leqslant k-1$, then*

$$\delta(B) = b(r_k - b + 1) \geqslant \begin{cases} r_k & \text{if } 1 \leqslant b \leqslant k-2, \\ 2 & \text{if } b = k-1. \end{cases}$$

(1.2) *If B is an odd circuit of order $b\geqslant 5$, then $\delta(B)=b(r_k-2)\geqslant r_k$.*
(1.3) *If B is a complete graph of order b with $2\leqslant b\leqslant k-3$ or an odd circuit of order $b\geqslant 5$ and $k\geqslant 5$, then $\delta(B)\geqslant 2(r_k-1)$.*
(1.4) *If B is an endblock of $C\in\mathcal{T}_k$, then $\delta(C)=\delta(C_B)+\delta(B)-r_k$.*

Lemma 2. *For $C\in\mathcal{T}_k$, the following statements hold:*
(2.1) *$\delta(C)\geqslant 2$.*
(2.2) *If C is not an ε_k-graph, then $\delta(C)\geqslant r_k$.*
(2.3) *If no endblock of C is a K_{k-1} and C is neither a K_1 nor a K_{k-2} and $k\geqslant 5$, then $\delta(C)\geqslant 2(r_k-1)$.*

Proof. We prove Lemma 2 by induction on the number m of blocks of C. For $m=1$, Lemma 2 is an immediate consequence of Lemma 1. Note that $r_k\geqslant k-2\geqslant 2$. Now, assume $m>1$.

If C is an ε_k-graph, then C_B is not an ε_k-graph for any end-block B of C and, by the induction hypothesis and Lemma 1, $\delta(C)\geqslant\delta(C_B)+\delta(B)-r_k\geqslant\delta(B)\geqslant 2$.

If C is not an ε_k-graph, then we argue as follows. First, consider the case that some endblock B of C is a K_{k-1}. Since the maximum degree of C is at most $k-1$, there is exactly one block $B'=K_2$ having a vertex in common with B. Then $C'=(C_B)_{B'}$ is not an ε_k-graph and, by the induction hypothesis and Lemma 1, we obtain

$$\delta(C)=\delta(C')+\delta(B)+\delta(B')-2r_k\geqslant r_k+2+2(r_k-1)-2r_k=r_k.$$

Now, consider the case that no endblock of C is a K_{k-1}. Let B be an endblock of C. Then C_B is not an ε_k-graph and, by the induction hypothesis and Lemma 1, $\delta(C)\geqslant r_k$.

If C satisfies the assumption of (2.3), then we choose an endblock B of C where $|V(B)|$ is maximum. Let $C'=C_B$. Clearly, C' is not a K_1. If some endblock of C' is a K_{k-1}, then B is a K_2 and C' is not an ε_k-graph. Hence, by Lemma 1 and (2.2), $\delta(C)\geqslant\delta(C')+\delta(B)-r_k\geqslant\delta(B)\geqslant 2(r_k-1)$. If C' is a K_{k-2}, then, since the maximum degree of C is at most $k-1$ and $k\geqslant 5$, B is a K_{k-2} and $k=5$, implying $\delta(C)=5r_k-12\geqslant 2(r_k-1)$. If C' is not a K_{k-2} and no endblock of C is a K_{k-1}, then, by the induction hypothesis and Lemma 1, $\delta(C)\geqslant\delta(C')+\delta(B)-r_k\geqslant\delta(C')\geqslant 2(r_k-1)$.

This proves Lemma 2. □

As a consequence of Lemma 2 and (2), we obtain $\delta\geqslant 2$. Clearly, both β and γ are non-negative. Hence, by (1), $\Delta>0$. In particular, Theorem 2.1 is true for $k=4$. In what follows, we may therefore assume that $k\geqslant 5$.

The fact that $\delta\geqslant 2$ and, therefore, $\Delta>0$ was first proved by Gallai [4] in 1963.

Next, we shall prove Theorem 2.1 for the case that $|H|=1$, say $H=\{x\}$. Then Theorem 1.2 implies that $G-x=G(L)$ is an ε_k-graph. Let g denote the number of blocks of $G-x$ isomorphic to K_{k-1}. Since G is neither a K_k nor a Hajós graph of order $2k-1$, we have $g\geqslant 3$. Moreover, $G-x$ has $g(k-2)$ vertices of degree $k-2$. Hence, $d_G(x)=g(k-2)\geqslant 3(k-2)$ and, therefore, $\gamma=d_G(x)-k\geqslant 2(k-3)$. From (1) we then conclude that

$$\Delta > \frac{k-1}{k}(2(k-3)) > k-4.$$

This proves Theorem 2.1 in case of $|H| = 1$. In what follows, we may therefore assume that $|H| \geq 2$.

2.2. Good colourings

Next, we partition $\mathscr{C}$ into three classes. For $x \in C \in \mathscr{C}$, let $N(x:H) = N_G(x) \cap H$ and $N(C:H) = \bigcup_{y \in V(C)} N(y:H)$. Let $\mathscr{C}_1$ denote the set of all ε_k-graphs of C. Since $|H| \geq 2$ and G has no separating vertex, $|N(C:H)| \geq 2$ for every $C \in \mathscr{C}_1$. Denote by G^* the graph obtained from the high-vertex subgraph $G(H)$ by adding an edge uv for each $C \in \mathscr{C}_1$ where u, v are two distinct vertices of $N(C:H)$.

Now, consider a component $C \in \mathscr{C} - \mathscr{C}_1$. Let x, y be a pair of non-separating vertices of C contained in the same endblock of C. Since x and y have the same degree in G as well as in C,

$$|N(x:H)| = |N(y:H)|.$$

We call (x, y) a *light pair* of C if

$$N(x:H) \neq N(y:H).$$

Let $\mathscr{C}_2$ denote the set of all components $C \in \mathscr{C} - \mathscr{C}_1$ for which there exists a light pair. Moreover, let $\mathscr{C}_3 = \mathscr{C} - \mathscr{C}_1 - \mathscr{C}_2$ and, for $1 \leq i \leq 3$, let $c_i = |\mathscr{C}_i|$.

For each $C \in \mathscr{C}_2$, choose a light pair (x, y) of C and denote by M the set of all pairs chosen. A t-colouring φ of G^* is said to be *good* if, for any pair $(x, y) \in M$, $|\varphi(N(z:H)| < |N(z:H)|$ for $z = x$ or $z = y$, or $\varphi(N(x:H)) \neq \varphi(N(y:H))$. If φ is a good t-colouring of G^*, then φ is a t-colouring of $G(H)$ and $|\varphi(N(C:H))| \geq 2$ for all components $C \in \mathscr{C}_1$. In case of $t \leq k-1$, it follows from Theorem 1.3 that φ can be extended to a $(k-1)$-colouring of $G(H \cup V(C))$ for all components $C \in \mathscr{C}_1 \cup \mathscr{C}_2$.

In what follows, let t^* denote the smallest integer for which there exists a good t^*-colouring of G^*. As we shall see later, if t^* is large then Δ must be large, too. First, we prove that

$$2(\beta + c_1 + c_2) \geq t^*(t^* - 1). \tag{3}$$

Clearly, $\beta + c_1 = |E(G(H))| + |\mathscr{C}_1| \geq |E(G^*)|$ and $c_2 = |M|$. Hence, (3) is a consequence of the following result.

Lemma 3. *Let F be an arbitrary graph, and let M be a set of pairs (X, Y) of subsets of $V(F)$ such that $X \neq Y$ and $|X| = |Y|$. Then, for some t satisfying $|E(F)| + |M| \geq \binom{t}{2}$, there is a t-colouring φ of F which is good for every pair $(X, Y) \in M$; that is $|\varphi(X)| < |X|$, or $|\varphi(Y)| < |Y|$, or $\varphi(X) \neq \varphi(Y)$.*

Proof (*By induction on* $|M|$). For $|M|=0$, Lemma 3 is evidently true. Assume $|M|\geqslant 1$. Consider an arbitrary vertex v of F. Let $M^*=\{(X,Y)\in M \mid v\in X\cap Y \text{ or } v\notin X\cup Y\}$ and $M'=\{(X-v,Y-v)\mid (X,Y)\in M^*\}$.

By induction, there is a t'-colouring φ' of $F-v$ with $|E(F-v)|+|M'|\geqslant\binom{t'}{2}$ such that φ' is good for every pair of M'. Now, for a colour $i\in\{1,\dots,t',t'+1\}$, define a mapping $\varphi^{(i)}$ by $\varphi^{(i)}(x)=\varphi'(x)$ for $x\in V(F-v)$ and $\varphi^{(i)}(v)=i$. Clearly, $\varphi^{(i)}$ is good for every pair of M^* and $\varphi^{(t'+1)}$ is a $(t'+1)$-colouring of F which is good for every pair of M. If, for some $i\in\{1,\dots,t'\}$, $\varphi^{(i)}$ is a t'-colouring of F that is good for every pair of M, then there is nothing to prove. Otherwise, for every $i\in\{1,\dots,t'\}$, there is an edge $e=vx$ such that $\varphi'(x)=i$ or there is a pair in $M-M^*$, for which $\varphi^{(i)}$ is not good (in this case we say that i is forbidden for this pair). Since $v\in(X-Y)\cup(Y-X)$ for every pair $(X,Y)\in M-M^*$, we easily conclude that for each pair of $M-M^*$ at most one colour is forbidden. Consequently, $|E(F)|-|E(F-v)|+|M|-|M^*|\geqslant t'$. Since $|M'|=|M^*|$, we then conclude that $|E(F)|+|M|\geqslant\binom{t'+1}{2}$. This proves Lemma 3. □

2.3. *Proof of Theorem 2.1 — the main part*

To prove Theorem 2.1, let us now suppose that

(1) $\Delta\leqslant k-4$, and

(2) $|V|$ is a minimum subject to (i).

To arrive at a contradiction, we shall show that these assumptions lead to a $(k-1)$-colouring of G. Note that $k\geqslant 5$ and $|H|\geqslant 2$. First, we prove that

(P1) G is 3-connected.

Suppose that this is not true. Since G is connected and has no separating vertex, we infer that G contains a separating set $\{x,y\}$. By a result of Dirac (for a proof the reader is referred to [4, Theorem (2.7)]) it then follows that x and y are not joined by an edge in G and $G-\{x,y\}$ has precisely two components G_1' and G_2'. Moreover, the notation may be chosen so that the graph G_1 obtained from $G-V(G_2')$ by adding the edge xy to it is k-critical, and the graph G_2 obtained from $G-V(G_1')$ by identifying x and y to a single new vertex z is k-critical. Then $|V(G)|=|V(G_1)|+|V(G_2)|-1$ and $|E(G)|\geqslant|E(G_1)|+|E(G_2)|-1$. For $i=1,2$, let

$$\Delta_i=2|E(G_i)|-|V(G_i)|\left(k-1+\frac{k-3}{k^2-3}\right)$$

be the excess of G_i. Then

$$\Delta\geqslant\Delta_1+\Delta_2-2+\left(k-1+\frac{k-3}{k^2-3}\right)=\Delta_1+\Delta_2+\left(k-3+\frac{k-3}{k^2-3}\right).$$

If G_i is a K_k, then $\Delta_i\geqslant-1$. If G_i is a Hajós graph of order $2k-1$, then $2|E(G_i)|=(k-1)|V(G_i)|+(k-3)=(k-1)(2k-1)+(k-3)$ implying $\Delta_i=(k-3)-(2k-1)((k-3)/(k^2-3))\geqslant 0$. If G_i is neither a complete graph of order k nor a Hajós graph of order $2k-1$, then, by (ii), $\Delta_i > k-4 \geqslant 1$. Therefore, if G_1 or G_2 is not a K_k, then $\Delta > k-4$, contradicting (i). If both G_1 and G_2 are complete graphs of order k, then G is a Hajós graph of order $2k-1$, contradicting the assumption of Theorem 2.1. This proves (P1).

Now, let us consider a good t^*-colouring φ^* of G^*. If $t^* \geqslant k-1$, then (3) implies $2(\beta + c_1 + c_2) \geqslant (k-1)(k-2)$. Because of (2) and Lemma 2, $\delta \geqslant 2(c_1+c_2)$. From (1) we conclude that

$$\Delta > \frac{k-1}{k^2}(2\beta + \delta) \geqslant \frac{k-1}{k^2}(k-1)(k-2) > k-4,$$

a contradiction to (i). Therefore,

(P2) $t^* \leqslant k-2$

and φ^* is a good $(k-2)$-colouring of G^*. Obviously, φ^* is a $(k-2)$-colouring of $G(H)$ and, using Theorem 1.3, we conclude that φ^* can be extended to a $(k-1)$-colouring of $G(H \cup V(C))$ for all components $C \in \mathscr{C}_1 \cup \mathscr{C}_2$. If this is also the case for all components $C \in \mathscr{C}_3$, then G has a $(k-1)$-colouring, contradicting $\chi(G)=k$.

Therefore, for the rest of the proof, we assume that, for some component $D \in \mathscr{C}_3$, φ^* cannot be extended to a $(k-1)$-colouring of $G(H \cup V(D))$. Then, by (2) and Lemma 2,

$$\begin{aligned} \delta &\geqslant \delta(D) + 2c_1 + (k-2)(c_2+c_3-1) \\ &= \delta(D) + 2(c_1+c_2) + (k-4)(c_2+c_3-1), \end{aligned}$$

and therefore, by (1) and (3),

$$\Delta \geq \frac{k^2-k-2}{k^2-3}\gamma + \frac{k-1}{k^2-3}(\delta(D) + t^*(t^*-1) + (k-4)(c_2+c_3-1)) \qquad (4)$$

implying, in particular,

$$\Delta > \frac{k-1}{k^2}(k\gamma + \delta(D) + t^*(t^*-1)). \qquad (5)$$

Since D contains no light pair, for every endblock B of C, there is a subset $P(B) \subseteq H$ such that $N(x:H) = P(B)$ for each non-separating vertex x of D contained in B. Since φ^* cannot be extended to a $(k-1)$-colouring of $G(H \cup V(D))$, we conclude from Theorem 1.3 that $|\varphi^*(P(B))| = |P(B)|$. This implies that

(P3) $t^* \geq |P(B)|$ for every endblock B of D.

If some endblock B of D is a K_{k-1}, then $P(B) = \{y\}$. Moreover, since D is not an ε_k-graph, $B \neq D$ and, therefore, B contains a separating vertex x of D. This implies that $\{x,y\}$ is a separating set of G, contradicting (P1). If D is a K_{k-2}, then $|P(D)| = 2$ and, since G is not a K_k, $P(D)$ is a separating set of G, contradicting (P1), too. Therefore,

(P4) D is not a K_{k-2} and no endblock of D is a K_{k-1}.

A triple (x,y,z) of vertices of D is called *soft* if xy, $xz \in E(D)$, $yz \notin E(D)$, and $D-\{y,z\}$ is connected. Assume that D contains a soft triple (x,y,z). Then the $(k-2)$-colouring φ^* of $G(H)$ can be extended to a $(k-1)$-colouring φ of $G(H \cup \{y.z\})$ where $\varphi(y)=\varphi(z)=k-1$. Since $D-\{x,y\}$ is connected, it then follows from Theorem 1.3 that φ can be extended to a $(k-1)$-colouring of $G(H \cup V(D))$. This is a contradiction to the choice of D. Therefore,

(P5) D contains no soft triple.

To establish the existence of a $(k-1)$-colouring of G, let us now consider an arbitrary endblock B of D with $b=|V(B)|$. Because of (P4), neither B nor D_B is an ε_k-graph. From Lemmas 1 and 2 we conclude that $\delta(D) \geqslant \delta(C_B)+\delta(B)-r_k \geqslant \delta(B)$.

If B is not a complete graph, then B is an odd circuit with $b \geqslant 5$ and, by Lemma 1, $\delta(D) \geq \delta(B)=b(r_k-2) \geqslant 5(k-4)$. Since $|P(B)|=k-3$, this yields, by (5) and (P3),

$$\Delta > \frac{k-1}{k^2}(\delta(D)+t^*(t^*-1)) \geqslant \frac{k-1}{k^2}(5(k-4)+(k-3)(k-4)) > k-4,$$

a contradiction to (i).

Therefore, B is a K_b where $b \leqslant k-2$ and, by Lemma 1, $\delta(D) \geqslant \delta(B) \geqslant b(k-b-1)$. Let $P=P(B)$. Then $|P|=k-b$ and, because of (P2) and (P3), $t^* \geqslant k-b$ and $b \geqslant 2$. Consequently, $\delta(D)+t^*(t^*-1) \geqslant k(k-b-1)$.

If $b=2$, then (5) implies $\Delta > ((k-1)/k^2)(k(k-3)) > k-4$, contradicting (i). Hence, $b \geqslant 3$.

Let $m=1$ if $c_2+c_3-1>0$ and $m=0$ if $c_2+c_3-1=0$. If $\gamma+m \geqslant b-2$, then, using (4), we obtain by an easy calculation that

$$\Delta \geqslant \frac{k^2-k-2}{k^2-3}(b-2-m)+\frac{k-1}{k^2-3}(k(k-b-1)+(k-4)m)$$

$$=\frac{k^3-4k^2+(3-4m)k-2b+(4+6m)}{k^2-3} > k-4,$$

contradicting (i). Hence,

$$\gamma+m \leqslant b-3. \tag{6}$$

This, in particular, implies that $d_G(v) \leqslant k+b-3$ for each $v \in H$. Now, let x denote the only separating vertex of D contained in B in case of $D \neq B$ or, an arbitrary vertex of B in case of $B=D$. Then, since P and $B-x$ are completely joined in G,

$$d_{G-(B-x)}(v) \leqslant k-2 \quad \text{for each } v \in P. \tag{7}$$

Let us first consider the case that $m=0$ and there are two non-adjacent vertices in $G(P)$. Then $\mathscr{C}=\mathscr{C}_1 \cup \{D\}$ and we argue as follows. First, in the good $(k-2)$-colouring φ^* of G^* we recolour two independent vertices of $G(P)$ by the new colour $k-1$. Since

$|\varphi^*(N(C:H)| \geqslant 2$ for all $C \in \mathscr{C}_1$ and, by (P1), $|N(C:H)| \geqslant 3$ for all $C \in \mathscr{C}_1$, this results in a $(k-1)$-colouring φ of $G(H)$ where $|\varphi(N(C:H))| \geqslant 2$ for all $C \in \mathscr{C}_1$. Moreover, $|\varphi(N(y:H)| < |N(y:H)|$ for all $y \in V(B-x)$. From Theorem 1.3 we conclude that φ can be extended to some $(k-1)$-colouring of G, contradicting $\chi(G) = k-1$.

Therefore, in what follows, we assume that $m=1$, or that $m=0$ and $G(P)$ is a complete graph. We distinguish two cases.

Case 1: $B=D$. Then $N(x:H)=P$ and, since G contains no K_k, $G(P)$ is not a complete graph implying $m=1$. Let Z be a maximal independent set of $G(P)$, $z=|Z|$ and $G' = G-B-P$. Clearly, $z \geqslant 2$. Since G is k-critical, there is a $(k-1)$-colouring φ of G'. If there are two distinct vertices $a, b \in Z$ such that φ' can be extended to a $(k-1)$-colouring of $G(V(G') \cup \{a,b\})$ such that a and b receive the same colour, then, by (7), this colouring can be first extended to a $(k-1)$-colouring of $G(V(G') \cup P)$ and then, by Theorem 1.3, to a $(k-1)$-colouring of G, contradicting $\chi(G)=k$. Otherwise, for each colour $i \in \{1, \dots, k-1\}$, the number of edges of G joining a vertex of colour i with a vertex of Z is at least $z-1$. For $y \in Z$, let $d'(y) = |N_G(y) \cap V(G')|$ and $d''(y) = d_{G(P)}(y)$. Then

$$\sum_{y \in Z} d'(y) \geqslant (k-1)(z-1),$$

and, since Z is a maximal independent set in $G(P)$,

$$\sum_{y \in Z} d''(y) \geqslant |P-Z| = k-b-z.$$

Because of $Z \subseteq H$ and $d_G(y) = d'(y) + d''(y) + b$ for all $y \in Z$, the last two inequalities imply

$$\gamma \geqslant \sum_{y \in Z} (d_G(y) - k) \geqslant (k-1)(z-1) + (k-b-z) + (b-k)z$$

$$= z(b-2) - b + 1.$$

Since $b \geqslant 3$ and $z \geqslant 2$, this yields, on the one hand, $\gamma \geqslant b-3$. On the other hand, because of (6) and $m=1$, we have $\gamma \leqslant b-4$, a contradiction.

Case 2: $B \neq D$. Let $A = N(x:H) \cap P$. Clearly, $A \neq P$. Let Z be a maximal independent set in $G(P-A)$, $Z' = Z \cup \{x\}$, and $z' = |Z'|$. Then Z' is a maximal independent set in $G'' = G(P \cup \{x\})$ and $z' \geqslant 2$. Let $G' = G-B-P$. Since G is k-critical, there is a $(k-1)$-colouring φ of G'. As in case 1, we conclude that, for each colour $i \in \{1, \dots, k-1\}$, the number of edges of G joining a vertex of colour i with a vertex of Z' is at least $z'-1 = z$. For $y \in Z'$, let $d'(y) = |N_G(y) \cap V(G')|$ and $d''(y) = d_{G''}(y)$. Then

$$\sum_{y \in Z'} d'(y) \geqslant (k-1)(z'-1) = (k-1)z,$$

and, since Z' is a maximal independent set in G'',

$$\sum_{y \in Z'} d''(y) \geqslant |(P \cup \{x\}) - Z'| = k-b-z.$$

Since $Z \subseteq H$ and $d_G(y) = d'(y) + d''(y) + b - 1$ for all $y \in Z \cup \{x\}$ and $d_G(x) = k-1$, it then follows that

$$\begin{aligned}\gamma &\geqslant \sum_{y\in Z} (d_G(y) - k) \geqslant (k-1)z + (k-b-z) + (b-k-1)z - (k-b)\\ &= z(b-3).\end{aligned}$$

By (6), $\gamma \leqslant b-3-m$. Since $b \geqslant 3$ and $z \geqslant 1$, this is possible only if $m=0$ and $\gamma = b-3$.

Now, let B' be an endblock of D distinct from B. Clearly, B' is also a complete graph of order b' with $3 \leqslant b' \leqslant k-2$. Let $D' = (D_B)_{B'}$. Since $k \geqslant 5$, we conclude from (P4) and (P5) that no endblock of D' is a K_{k-1} and D' is neither a K_1 nor a K_{k-2}. Consequently, by Lemmas 1 and 2, $\delta(D') \geqslant 2(r_k - 1)$ and, therefore,

$$\begin{aligned}\delta(D) &\geqslant \delta(D') + \delta(B) + \delta(B') - 2r_k\\ &\geqslant b(r_k - b + 1) + b'(r_k - b' + 1) - 2.\end{aligned}$$

Because of (P3), $t^* \geqslant |P| = k-b$. Hence

$$\begin{aligned}&k\gamma + \delta(D) + t^*(t^*-1)\\ &\geqslant (k(b-3) + b(r_k - b + 1) + b'(r_k - b' + 1) - 2 + (k-b)(k-b-1))\\ &= \left(k^2 - 4k + \frac{2b}{k-1} - 2 + b'(r_k - b' + 1)\right)\\ &\geqslant \left(k^2 - 4k + \frac{2b}{k-1} - 2 + (k-2)\left(1 + \frac{2}{k-1}\right)\right)\\ &\geqslant k^2 - 3k - 2\end{aligned}$$

implying, by (5), that

$$\Delta > \frac{k-1}{k^2}(k^2 - 3k - 2) > k - 4,$$

a contradiction to (i).

Thus in both cases 1 and 2 we arrive at a contradiction. This proves Theorem 2.1.

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