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# On the maximum average degree and the oriented chromatic number of a graph

O.V. Borodin[a,1], A.V. Kostochka[b,2], J. Nešetřil[c,3], A. Raspaud[d,*], E. Sopena[d]

[a] *Department of Mathematics, University of Nottingham, Nottingham, NG7 2RD, UK*
[b] *Institute of Mathematics, 630090, Novosibirsk, Russia*
[c] *Department of Appl. Math., Charles University, Prague, Czech Republic*
[d] *LaBRI, Université Bordeaux I, 33405 Talence Cedex, France*

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**Abstract**

The oriented chromatic number o($H$) of an oriented graph $H$ is defined as the minimum order of an oriented graph $H'$ such that $H$ has a homomorphism to $H'$. The oriented chromatic number o($G$) of an undirected graph $G$ is then defined as the maximum oriented chromatic number of its orientations. In this paper we study the links between o($G$) and mad($G$) defined as the maximum average degree of the subgraphs of $G$. 

## 1. Introduction and statement of results

For every graph $G$ we denote by $V(G)$, with $v_G = |V(G)|$, its set of vertices and by $E(G)$, with $e_G = |E(G)|$, its set of arcs or edges. A homomorphism from a graph $G$ to a graph $H$ is a mapping $\varphi$ from $V(G)$ to $V(H)$ which preserves the edges (or the arcs), that is $xy \in E(G) \Rightarrow \varphi(x)\varphi(y) \in E(H)$.

Homomorphisms of graphs have been studied in the literature [4,6–9,11,15–18] as a generalization of graph colouring. It is not difficult to observe that an undirected graph $G$ has chromatic number $k$ if and only if $G$ has a homomorphism to $K_k$ but no homomorphism to $K_{k-1}$ (where $K_n$ denotes the complete graph on $n$ vertices).

* Corresponding author. E-mail: andre.raspaud@labri.u-bordeaux.fr.

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Therefore, the chromatic number of an undirected graph $G$ can equivalently be defined as the minimum number of vertices in an undirected graph $H$ such that $G$ has a homomorphism to $H$. We will often say that a graph $G$ is *$H$-colourable* if $G$ has a homomorphism to $H$ and the vertices of $H$ will be called *colours*.

Oriented graphs are directed graphs without opposite arcs. In other words, an oriented graph is an orientation of an undirected graph obtained by assigning every edge one of the two possible orientations. We can similarly define the *oriented chromatic number* $o(H)$ of an oriented graph $H$ as the minimum number of vertices in an oriented graph $H'$ such that $H$ has a (oriented) homomorphism to $H'$. The oriented chromatic number $o(G)$ of an undirected graph $G$ is then defined as the maximum oriented chromatic number of its orientations. In a sense, this notion measures how 'bad' an orientation of an undirected graph can be. For instance, bipartite graphs may have arbitrarily large oriented chromatic number (see Observation 2 below).

Oriented chromatic number of (undirected) graphs with bounded degree or with bounded treewidth was studied in [18]. In [17] it was proved that every planar graph has oriented chromatic number at most 80. The proof depends on the acyclic 5-colourability of planar graphs proved in [2]. No better upper bound is known up to now. The study of planar graphs is thus particularly challenging. This bound can be significantly decreased under some large girth assumption [15] (recall that the girth $g(G)$ of a graph $G$ is the length of a shortest cycle in $G$). The links between the oriented chromatic number and other parameters of a graph (arboricity, maximum degree, acyclic chromatic number) were studied in [11].

In this, paper, we study the relationship between the oriented chromatic number and the maximum average degree of a graph. The maximum average degree $\mathrm{mad}(G)$ of a graph $G$ is defined as the maximum of the average degrees $\mathrm{ad}(H)=2e_H/v_H$ taken over all the subgraphs $H$ of $G$. Our first result is the following:

**Theorem 1.** (1) *For every graph G with* $\mathrm{mad}(G)<\frac{7}{3}$, $o(G)\leqslant 5$.

(2) *For every graph G with* $\mathrm{mad}(G)<\frac{11}{4}$ *and girth* $g(G)\geqslant 5$, $o(G)\leqslant 7$.

(3) *For every graph G with* $\mathrm{mad}(G)<3$, $o(G)\leqslant 11$.

(4) *For every graph G with* $\mathrm{mad}(G)<\frac{10}{3}$, $o(G)\leqslant 19$.

Concerning the maximum average degree of planar graphs, we have the following:

**Observation 1.** *For every planar graph G with girth at least g*, $\mathrm{mad}(G)<2g/(g-2)$.

To see this, observe that if $G$ has girth at least $g$ then the number of faces in $G$ is at most $2e(G)/g$. By Euler's formula we then get that $2e(G)/v(G)\leqslant 2ge(G)/(2g+(g-2)e(G))$ and thus $\mathrm{mad}(G)<2g/(g-2)$.

In case of planar graphs, Theorem 1 therefore leads to the following corollary, which improves some results given in [15].

**Corollary 1.** *Let G be a planar graph, then*
1. *if G has girth at least* 14 *then* $\mathrm{o}(G)\leqslant 5$,
2. *if G has girth at least* 8 *then* $\mathrm{o}(G)\leqslant 7$,
3. *if G has girth at least* 6 *then* $\mathrm{o}(G)\leqslant 11$,
4. *if G has girth at least* 5 *then* $\mathrm{o}(G)\leqslant 19$.

It is easy to see that Observation 1, and hence Corollary 1, holds for the projective plane as well.

Relationships between oriented colourings and acyclic colourings have been discussed in [11,17]. Recall that a $k$-colouring of an undirected graph $G$ is said to be *acyclic* if every cycle in $G$ uses at least three colours. The acyclic chromatic number $\mathrm{acn}(G)$ of $G$ is then defined as the minimum number $k$ of colours such that $G$ has an acyclic $k$-colouring. It has been proved in [17] that $\mathrm{acn}(G)\leqslant k$ implies $\mathrm{o}(G)\leqslant k2^{k-1}$. Considering the relationship between the acn and mad parameters we prove:

**Theorem 2.** *For every* $m>15$ *and every graph G with* $\mathrm{mad}(G)<4(1-1/(m-2))$, $\mathrm{acn}(G)\leqslant m$.

Using the above-mentioned theorem of [17], Theorem 2 leads to the following corollary:

**Corollary 2.** *For every* $k>13$, *if* $\mathrm{mad}(G)<4(1-1/k)$ *then* $\mathrm{o}(G)\leqslant(k+2)2^{k+1}$.

Our next result is the following:

**Theorem 3.** *For every k, there exists a graph G of arbitrarily large girth with* $\mathrm{mad}(G)<4(1-1/k)$ *and* $\mathrm{o}(G)\geqslant k$.

In case of bipartite graphs, we have the following:

**Observation 2.** *For every* $k>0$ *there exists a bipartite graph* $B_k$ *with girth* 6 *such that* $\mathrm{mad}(B_k)=4(1-2/(k+1))$ *and* $\mathrm{o}(B_k)\geqslant k$.

To see this, let $B_k$ be the oriented graph obtained from the complete graph $K_k$ by replacing every edge by a directed 2-path. Clearly, the graph $B_k$ thus obtained has oriented chromatic number at least $k$ (all the 'original' vertices from $K_k$ must have distinct colours) and girth $g=6$. Moreover, no proper subgraph of $B_k$ can have the average degree greater than $B_k$ itself and thus:

$$\mathrm{mad}(B_k)=\frac{2e_{B_k}}{v_{B_k}}=\frac{2k(k-1)}{k(k-1)/2+k}=4\left(1-\frac{2}{k+1}\right).$$

Corollary 2 thus indicates that the mad parameter is a robust parameter. Theorem 3 extends Observation 2 to graphs of arbitrarily large girth.

In what follows, we will simply say that an undirected graph $G$ has a homomorphism to an oriented graph $T$ when $G$ with any orientation $L$ has a homomorphism to $T$. The rest of this paper is devoted to the proofs. For Theorem 1, we will prove that every graph satisfying the corresponding conditions has a homomorphism to some special (oriented) circulant graph. Recall that the circulant graph $G=G(n;\ c_1,\dots,c_d)$ has $V(G)=\{0,1,\dots,n-1\}$ and $xy\in E(G)$ if and only if $y=x+c_i \pmod n$ for some $i$, $1\leqslant i\leqslant d$.

If $n$ is a prime number of the form $4k+3$ and the $c_i$'s are the non-zero quadratic residues of $n$ then $d=\lfloor (n-1)/2\rfloor$ and $G=G(n;\ c_1,\dots,c_d)$ is a tournament. We will extensively use the following property of this kind of tournaments:

**Proposition 1.** *Let $G=G(n;\ c_1,\dots,c_d)$ be a circulant graph such that the $c_i$'s are the non-zero quadratic residues of $n$, where $n$ is a prime number of the form $4k+3$. Then for any two distinct vertices $u,v$ in $G$ and any orientation of a 2-path, there exist at least $(n-3)/4$ 2-paths of given orientation from $u$ to $v$.*

**Proof.** By Lemma 6.24 in [13] we have the following: if $n=3 \pmod 4$ is a prime number then for any $b$ (except 0), the number of solutions of the equation $x^2+y^2=b \pmod n$ is $n+1$ and the number of solutions of the equation $x^2-y^2=b$ is $n-1$. This immediately implies that $G$ has the desired property. □

Let $H$ be any target graph with $n$ vertices called colours. Suppose that we want to construct a homomorphism $f$ of a given graph $G$ to $H$ and let $x,y$ be any two vertices to be coloured in $G$. We will say that *$y$ allows $k$ colours* for $x$ if for a given choice of the colour of $y$ we have in any case at least $k$ choices for colouring $x$. Similarly, we will say that *$y$ forbids $k$ colours* for $x$ if for a given choice of the colour of $y$ we have in any case $n-k$ choices for colouring $x$.

All the proofs will be based on the so-called method of *reducible configurations*, usually attributed to Heesch [5], and used in particular by Appel and Haken [1] in their proof of the four-colour theorem. But we must note that Franklin [3] had already used it in the 1920s. We first provide a (small) set of *forbidden configurations*, that is a set of graphs that a minimal counter-example $G$ to our theorem cannot contain as subgraphs. We will then assume that every vertex $v$ in $G$ is valued by its degree $\deg(v)$ and define a *discharging procedure* which specifies some transfer of values among the vertices in $G$ keeping the sum of all the values constant. We will then get a contradiction by considering the *modernized degree* $\deg^*(v)$ of every vertex $v$, that is the value obtained by $v$ owing to the discharging procedure.

*Drawing conventions*: In all the figures depicting forbidden configurations, we will draw vertices with prescribed degrees as 'white vertices' and vertices with unbounded degree as 'black vertices'. All the neighbours of white vertices are drawn. Unless otherwise specified, two or more black vertices may coincide in a single vertex, provided that they do not share a common white neighbour.

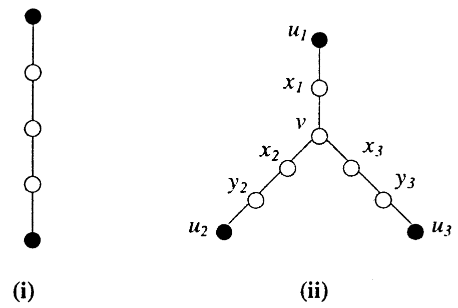


Fig. 1. Forbidden configurations for Theorem 1(1).

## 2. Proof of Theorem 1(1)

We will use here the circulant graph $T_5 = G(5;\ 1,2)$ as target graph. The following can easily be checked:

**Observation 3.** *The graph $T_5$ is such that for each $v \in V(T_5)$, each $i$, $1 \leqslant i \leqslant 4$ and each orientation of an i-path, the number of vertices in $T_5$ reachable from $v$ by an i-path of this orientation is $i+1$.*

We will now prove that every graph with maximum average degree less than $\frac{7}{3}$ has a homomorphism to $T_5$. Assume that $G$ with an orientation $L$ is a minimum counter-example to the theorem. Then $G$ must avoid all the configurations depicted in Fig. 1:

(i) This follows directly from Observation 3.

(ii) Let $f$ be any $T_5$-colouring of $G \backslash \{v, x_1, x_2, y_2, x_3, y_3\}$. By Observation 3 the vertex $u_1$ forbids two colours for $v$, while each of $u_2$ and $u_3$ forbids only one colour. So we can extend $f$ to a $T_5$-colouring of $G$, a contradiction.

We now use the following discharging procedure: each vertex of degree at least 3 gives $\frac{1}{3}$ to each of its 2-neighbours having itself a 2-neighbour and $\frac{1}{6}$ to each of its other 2-neighbours.

Let us check that the modernized degree $\deg^*$ of each vertex is at least $\frac{7}{3}$ which contradicts the assumption $\text{mad}(G) < \frac{7}{3}$. We consider the possible cases for the old degree $\deg(v)$:

1. $\deg(v) = 2$: by (i), $v$ has at least one neighbour of degree at least 3. If $v$ has a 2-neighbour then it receives $\frac{1}{3}$ so that $\deg^*(v) = \frac{7}{3}$. If $v$ has no 2-neighbour then it receives $\frac{1}{6}$ from each of its neighbours and thus $\deg^*(v) = \frac{7}{3}$,
2. $\deg(v) = 3$: by (ii) $v$ gives at most $\max\{2 \times \frac{1}{3}, \frac{1}{3} + 2 \times \frac{1}{6}, 3 \times \frac{1}{6}\} = \frac{2}{3}$ and thus $\deg^*(v) \geqslant \frac{7}{3}$,
3. $\deg(v) = k \geqslant 4$: $v$ gives at most $k\frac{1}{3}$ and thus $\deg^*(v) \geqslant k - k/3 > \frac{7}{3}$.

Therefore, every vertex in $G$ gets a modernized degree at least $\frac{7}{3}$ and the theorem is proved.

## 3. Proof of Theorem 1(2)

We will use here the circulant graph $T_7 = G(7;\ 1,2,4)$ as target graph. The following observation is a direct consequence of Proposition 1:

**Observation 4.** *The graph $T_7$ is such that for each $u,v \in V(T)$ with $u \neq v$, there exist directed 2-paths of all four possible orientations connecting $u$ with $v$. It follows that there is a directed 3-path of every possible orientation between any two vertices (not necessarily distinct).*

Observe that if the arcs of a directed 2-path have opposite directions, then in $T_7$ there is a directed 2-path from any vertex $v$ to itself. All that essentially means that if we have a path $vxu$ in $G$ with $x$ of degree 2 then $v$ 'forbids' for $u$ at most one colour (actually its own colour, if the two arcs have the same direction). More precisely, we have the following:

**Observation 5.** *Let $G$ be an oriented graph and $u,v$ be two neighbour vertices in $G$. Let $f$ be some $T_7$-colouring of $G\backslash\{v\}$. If for a fixed colouring of the neighbours of $v$ distinct from $u$ we can use $i$ distinct colours for $v$, then $v$ forbids for $u$ at most two colours for $i=2$, one colour for $i=3,4$ and no colours for $i>4$.*

We will now prove that every graph with maximum average degree less than $\frac{11}{4}$ and girth at least 5 has a homomorphism to $T_7$. Assume that $G$ with an orientation $L$ is a minimum counter-example to the theorem. Then $G$ must avoid all the configurations depicted in Fig 2:

(i) This follows directly from Observation 4.
Observe that since $G$ has girth at least 5 and (by (i)) no adjacent vertices of degree 2, no white vertices can coincide and no black vertex can coincide with a white one. Some black vertices can coincide but it does not matter.

(ii) Let $f$ be any $T_7$-colouring of $G\backslash\{v,x_1,x_2\}$. Then $u_3$ allows exactly three colours for $v$ while $u_1,u_2$ forbid at most two of them. Thus $f$ can be extended to a $T_7$-colouring of $G$, a contradiction.

(iii) Let $f$ be any $T_7$-colouring of $G\backslash\{v,x_1,x_2,x_3,x_4\}$. All the $u_i$'s forbid at most four colours for $v$, and thus $f$ can be extended to a $T_7$-colouring of $G$, a contradiction.

(iv) Let $f$ be any $T_7$-colouring of $G\backslash\{v,x_1,x_2,x_3,x_4,y_4\}$. Then $u_4$ and $u_5$ allow at least two colours for $x_4$ and thus, by Observation 5, at least five colours for $v$ while $u_1,u_2,u_3$ forbid at most three of them. Thus $f$ can be extended to a $T_7$-colouring of $G$, a contradiction.

(v) Let $f$ be any $T_7$-colouring of $G\backslash\{v_1,v_2,v_3,x_1,x_2,x_3\}$. The vertices $u_1$ and $u_4$ (respectively, $u_3$ and $u_5$) allow at least two colours for $v_1$ (resp. for $v_3$). Then, by Observation 5, $v_1$ and $v_3$ allow at least three colours for $v_2$ while $u_2$ forbids at most one of them. Thus $f$ can be extended to a $T_7$-colouring of $G$, a contradiction.

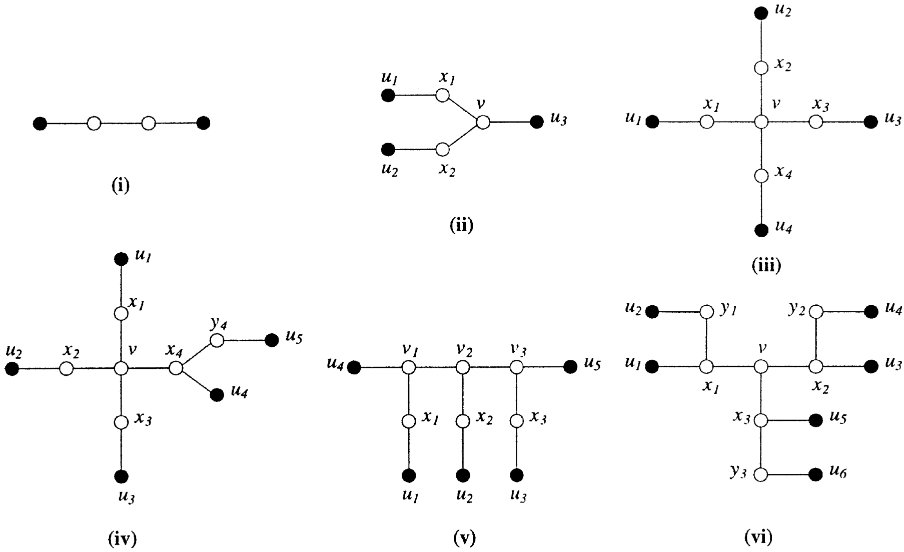


Fig. 2. Forbidden configurations for Theorem 1(2).

(vi) Let $f$ be any $T_7$-colouring of $G\backslash\{v,x_1,x_2,x_3,y_1,y_2,y_3\}$. All the $u_i$'s allow at least two colours for each $x_i$ and thus at least one colour for $v$. Thus $f$ can be extended to a $T_7$-colouring of $G$, a contradiction.

We now use the following discharging procedure: each vertex of degree at least 3 gives $\frac{3}{8}$ to each of its 2-neighbours and $\frac{1}{8}$ to each of its 3-neighbours having a 2-neighbour.

Let us check that the modernized degree $\deg^*$ of each vertex is at least $\frac{11}{4}$ which contradicts the assumption $\mathrm{mad}(G)<\frac{11}{4}$. We consider the possible cases for the old degree $\deg(v)$:

1. $\deg(v)=2$: by (i), $v$ receives exactly $\frac{3}{4}$, and thus $\deg^*(v)=\frac{11}{4}$.
2. $\deg(v)=3$: if $v$ has a 2-neighbour then by (ii) and (v), $v$ gives at most $\frac{3}{8}+\frac{1}{8}=\frac{1}{2}$ but then it also receives $\frac{1}{8}+\frac{1}{8}$ so that $\deg^*(v)\geqslant\frac{11}{4}$; if $v$ has no 2-neighbour then by (vi), it gives at most $\frac{2}{8}$ to its 3-neighbours having a 2-neighbour and thus $\deg^*(v)\geqslant\frac{11}{4}$.
3. $\deg(v)=4$: by (iii) and (iv), $v$ gives at most $\frac{9}{8}$ and thus $\deg^*(v)\geqslant\frac{23}{8}>\frac{11}{4}$.
4. $\deg(v)=k>4$: $v$ gives at most $3k/8$ and thus $\deg^*(v)\geqslant 5k/8\geqslant\frac{25}{8}>\frac{11}{4}$.

Therefore, every vertex in $G$ gets a modernized degree at least $\frac{11}{4}$ and the theorem is proved.

## 4. Proof of Theorem 1(3)

Here we use as target graph the circulant $T_{11}=G(11;\ 1,3,4,5,9)$. From Proposition 1 we have:

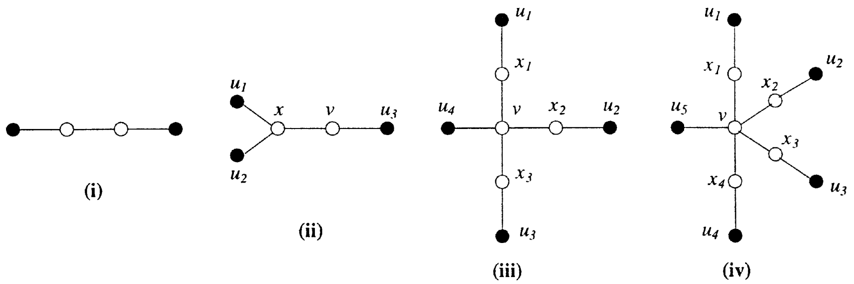


Fig. 3. Forbidden configurations for Theorem 1(3).

**Observation 6.** *The graph $T_{11}$ is such that for every two distinct vertices $u,v$ in $T_{11}$, there are at least two distinct 2-paths linking $u$ and $v$ for every possible orientation of this 2-path and there is a 3-path of every possible orientation linking any two (not necessarily distinct) vertices in $T_{11}$.*

We will now prove that every graph with maximum average degree less than 3 has a homomorphism to $T_{11}$. Assume that $G$ with an orientation $L$ is a minimum counter-example to the theorem. Then $G$ must avoid all the configurations depicted in Fig. 3:

(i) This directly follows from Observation 6.

(ii) Let $f$ be some $T_{11}$-colouring of $G\backslash\{v\}$. By Observation 6, we can choose $f$ such that $f(x)\neq f(u_3)$ and thus $f$ can be extended to a $T_{11}$-colouring of $G$.

(iii) Let $f$ be some $T_{11}$-colouring of $G\backslash\{v,x_1,x_2,x_3\}$. Then $u_4$ allows five colours for $v$ and $u_1,u_2,u_3$ forbid at most three colours. Thus $f$ can be extended to a $T_{11}$-colouring of $G$.

(iv) As before, let $f$ be some $T_{11}$-colouring of $G\backslash\{v,x_1,x_2,x_3,x_4\}$. Then $u_5$ allows five colours for $v$ and $u_1,u_2,u_3,u_4$ forbid at most four colours. Thus $f$ can be extended to a $T_{11}$-colouring of $G$.

We now use the following discharging procedure: each vertex of degree at least 4 gives $\frac{1}{2}$ to each of its 2-neighbours.

Let us check that the modernized degree $\deg^*$ of each vertex is at least 3, which contradicts the assumption $\mathrm{mad}(G)<3$. We consider all possible cases for the old degree $\deg(v)$:

1. $\deg(v)=2$: by (i) and (ii), $v$ receives exactly 1 and thus $\deg^*(v)=3$.
2. $\deg(v)=3$: no changes.
3. $\deg(v)=4$: by (iii), $v$ gives at most 1 and thus $\deg^*(v)\geqslant 3$.
4. $\deg(v)=5$: by (iv), $v$ gives at most $\frac{3}{2}$ and thus $\deg^*(v)\geqslant\frac{7}{2}>3$.
5. $\deg(v)=k\geqslant 6$: $v$ gives at most $k/2$ and thus $\deg^*(v)\geqslant k/2\geqslant 3$.

Therefore, every vertex in $G$ gets a modernized degree at least 3 and the theorem is proved.

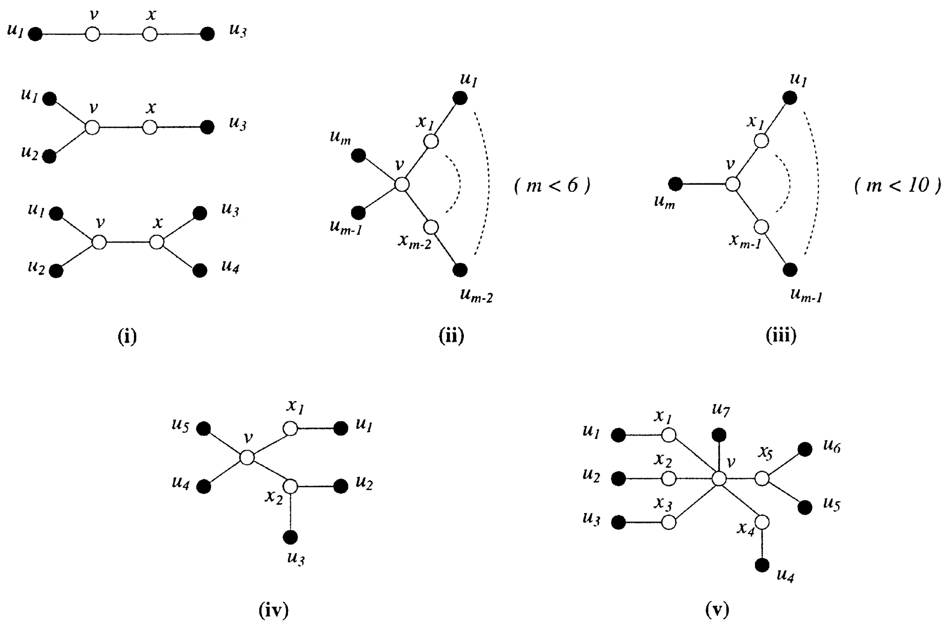


Fig. 4. Forbidden configurations for Theorem 1(4).

## 5. Proof of Theorem 1(4)

Here we use as target graph the circulant $T_{19} = G(19;\ 1,4,5,6,7,9,11,16,17)$ on 19 vertices. From Proposition 1 we have:

**Observation 7.** *For each distinct $u,v \in V(T_{19})$, and for each of all four possible orientations of a 2-path, there exist at least four distinct 2-paths connecting u with v with given orientation.*

By a tedious case analysis we also get the following:

**Observation 8.** *For each distinct $u,v,w \in V(T_{19})$, and for each of all eight possible orientations of edges connecting u, v and w with a vertex, there exists a vertex $x \in V(T_{19})$ such that the edges connecting x with u, v and w have given orientation.*

We will now prove that every graph with maximum average degree less than $\frac{10}{3}$ has a homomorphism to $T_{19}$. Assume that $G$ with an orientation $L$ is a minimum counter-example to the theorem. Then $G$ must avoid all the configurations depicted in Fig. 4:

(i) Let $f$ be some $T_{19}$-colouring of $G \backslash \{vx\}$ (that is we delete the edge $vx$). By Observation 7, we have at least four choices for colouring $x$, so that we can choose for $x$ a colour distinct from the colours of $u_1$ and $u_2$ (if exists). By Observation 8, we can then find a colour for $v$, which is a contradiction.

(ii) Let $f$ be some $T_{19}$-colouring of $G\backslash\{x_1,\dots,x_{m-2}\}$. By Observation 7, we have at least four choices for colouring $v$ while $u_1,\dots,u_{m-2}$ forbid at most three of them. Thus we can extend $f$ to a $T_{19}$-colouring of $G$, a contradiction.

(iii) Let $f$ be some $T_{19}$-colouring of $G\backslash\{x_1,\dots,x_{m-1}\}$. We have nine choices for colouring $v$ while $u_1,\dots,u_{m-1}$ forbid at most eight of them. Thus we can extend $f$ to a $T_{19}$-colouring of $G$, a contradiction.

(iv) Let $f$ be some $T_{19}$-colouring of $G\backslash\{x_1\}$. By Observation 7, $u_4$ and $u_5$ allow at least four colours for $v$ while $u_2,u_3$ forbid at most two of them. Hence, we can choose for $v$ a colour distinct from the colour of $u_1$ and extend $f$ to a $T_{19}$-colouring of $G$, a contradiction.

(v) Let $f$ be some $T_{19}$-colouring of $G\backslash\{v,x_1,x_2,x_3,x_4\}$. Then $u_7$ allows nine colours for $v$ while $u_1,\dots,u_4$ forbid at most four of them and $u_5$, $u_6$ at most two of them. Hence we can colour $v$ with a colour distinct from the colours of $u_1,\dots,u_7$. By Observation 8, we can colour $x_5$. We have then extended $f$ to a $T_{19}$-colouring of $G$, a contradiction.

We now use the following discharging procedure: each vertex of degree at least 4 gives $\frac{2}{3}$ to each of its 2-neighbours and $\frac{1}{9}$ to each of its 3-neighbours.

Let us check that the modernized degree deg$^*$ of each vertex is at least $\frac{10}{3}$ which contradicts the assumption $\mathrm{mad}(G)<\frac{10}{3}$. We consider the possible cases for the old degree $\deg(v)$:

1. $\deg(v)=2$: by (i), $v$ receives exactly $\frac{4}{3}$ and thus $\deg^*(v)\geqslant\frac{10}{3}$.
2. $\deg(v)=3$: by (i), $v$ receives exactly $3\times\frac{1}{9}=\frac{1}{3}$ and thus $\deg^*(v)=\frac{10}{3}$.
3. $\deg(v)=4$: if $v$ has no 2-neighbour then it gives at most $4\times\frac{1}{9}$ so that $\deg^*(v)\geqslant\frac{32}{9}>\frac{10}{3}$. Otherwise, by (ii) and (iv) it gives exactly $\frac{2}{3}$ and thus $\deg^*(v)=\frac{10}{3}$.
4. $\deg(v)=5$: by (ii), $v$ gives at most $2\times\frac{2}{3}+3\times\frac{1}{9}$ to its 2- and 3-neighbours so that $\deg^*(v)\geqslant\frac{10}{3}$.
5. $\deg(v)=6$: by (iii) and (v), $\deg^*(v)\geqslant 6-\max\{3\times\frac{2}{3}+3\times\frac{1}{9},4\times\frac{2}{3}\}=\frac{10}{3}$.
6. $7\leqslant\deg(v)=n\leqslant 9$: by (iii), $\deg^*(v)\geqslant n-(n-2)\frac{2}{3}-2\times\frac{1}{9}=\frac{n}{3}+\frac{10}{9}>\frac{10}{3}$.
7. $\deg(v)=n\geqslant 10$: in that case $\deg^*(v)\geqslant n/3\geqslant\frac{10}{3}$.

Therefore, every vertex in $G$ gets a modernized degree at least $\frac{10}{3}$ and the theorem is proved.

## 6. Proof of Theorem 2

We will prove in this section that every graph $G$ with $\mathrm{mad}(G)<4(1-1/(m-2))$, $m>15$, has acyclic chromatic number at most $m$. Suppose $G$ is a minimum counter-example to the theorem. Then $G$ satisfies the following properties:

(i) *No 2-vertex in G is adjacent to some j-vertex for j<m.* Assume that $\deg(v)=2$ and let $x_1,x_2$ be its neighbours, where $x_1$ has degree at most $m-1$. Let $f$ be some acyclic $m$-colouring of $G\backslash\{v\}$. If $f(x_1)\neq f(x_2)$ we can colour $v$ with any

colour distinct from $f(x_1)$ and $f(x_2)$. Otherwise, we can colour $v$ with a colour distinct from $f(N(x_1))$ and $f(x_2)$. In both cases, $f$ can be extended to an acyclic $m$-colouring of $G$, a contradiction.

(ii) *For every* 3*-vertex* $v$ *in* $G$ *with* $N(v)=\{x_1,x_2,x_3\}$, *we have* $\deg(x_i)+\deg(x_j)\geqslant m+1$ *for every* $1\leqslant i<j\leqslant 3$. Assume this is not the case and let $f$ be any acyclic $m$-colouring of $G\backslash\{v\}$. We have three cases to consider:

1. $f(x_1)\neq f(x_2)\neq f(x_3)\neq f(x_1)$. We can obviously colour $v$ with any colour distinct from the colours of its neighbours.
2. $1=f(x_1)=f(x_2)\neq f(x_3)=2$. If some colour $c\in\{3,\dots,m\}$ is not used on $N(x_1)$ or $N(x_2)$ then we can colour $v$ with $c$. Otherwise, $\deg(x_1)\geqslant m-1$ and $\deg(x_2)\geqslant m-1$. Since, by (i), $\deg(x_3)\geqslant 3$, the result follows.
3. $f(x_1)=f(x_2)=f(x_3)=1$. Suppose $\deg(x_1)+\deg(x_2)\leqslant m$. We can then colour $v$ with a colour $c\neq 1$ which is used neither on $N(x_1)$ nor on $N(x_2)$.

Hence, in all cases $f$ can be extended to an acyclic $m$-colouring of $G$, a contradiction.

(iii) *Let* $v$ *be a vertex in* $G$ *with* $N(v)=\{u,y_1,y_2,y_3,x_1,\dots,x_t\}$ *such that* $x_1,\dots,x_t$ *have degree two and* $y_1,y_2,y_3$ *have degree at most three. Then* $t\geqslant(m-10)(m-5)+1$. Assume on the contrary that $t\leqslant(m-10)(m-5)$. For every $i$, $1\leqslant i\leqslant t$, denote by $z_i$ the neighbour of $x_i$ distinct from $v$ and let $f$ be some acyclic $m$-colouring of $G\backslash\{v,x_1,\dots,x_t\}$. Denote $A=f(\{u,y_1,y_2,y_3\}\cup N(y_1)\cup N(y_2)\cup N(y_3))$. Clearly, $|A|\leqslant 10$. There exists a colour $c\in\{1,\dots,m\}\backslash A$ such that at most $t/(m-10)$ $z_j$'s (counted with multiplicity) are coloured with $c$. We colour $v$ with $c$ and then colour every $x_j$ such that $f(z_j)\neq c$ with any colour distinct from $c$ and $f(z_j)$. Finally, we colour $x_j$'s such that $f(z_j)=c$ with distinct colours in $\{1,\dots,m\}\backslash(\{c\}\cup f(\{u,y_1,y_2,y_3\}))$. Since $t/(m-10)\leqslant m-5$, we can do that and thus extend $f$ to an acyclic $m$-colouring of $G$, a contradiction.

We now use the following two-step discharging procedure:

*Step* 1: each vertex of degree at least 8 gives $\frac{1}{2}$ to each of its 3-neighbours,

*Step* 2: each vertex of degree at least $m$ (before discharging) leaves 4 for itself and uniformly distributes the rest among its 2-neighbours.

Let us check that the modernized degree $\deg^*$ of each vertex is at least $4(1-1/(m-2))$ which contradicts the assumption of Theorem 2. We consider the possible cases for old degree $\deg(v)$:

1. $\deg(v)\geqslant 8$: after the first step of discharging, $v$ has at least $0.5\deg(v)$, so it can afford to save 4 for itself and thus $\deg^*(v)\geqslant 4$.
2. $4\leqslant\deg(v)\leqslant 7$: $\deg^*(v)=\deg(v)$.
3. $\deg(v)=3$: by (ii), $v$ cannot have two neighbours with degree at most 7. Thus $v$ receives at least 1 and $\deg^*(v)\geqslant 4$.
4. $\deg(v)=2$: let us check that $v$ receives at least $(m-4)/(m-2)$ from each of its neighbours. Indeed, let $(v,u)$ be an edge in $E(G)$. By (i), the degree $d$ of $u$ is at least $m$. If $u$ has $a$ neighbours of degree at least 4 and $b$ neighbours of degree 3

then $v$ receives from $u$

$$h(d,a,b) = \frac{d-4-b/2}{d-a-b} = 1 - \frac{4-b/2-a}{d-a-b}.$$

Hence if $a+b/2 \geqslant 4$, we are done. Suppose on the contrary that $a+b/2 \leqslant 3.5$. Under these conditions, $h(d,a,b)$ increases if any of $d,a,b$ increases and others are fixed or if $a+b$ is fixed and $a$ increases. We have three cases to consider:

(a) $a \geqslant 2$. By the above, $h(d,a,b) \geqslant (m,2,0) = (m-4)/(m-2)$.

(b) $a \leqslant 1, a+b \geqslant 5$. By our previous observation concerning the behaviour of the function $h$ and the fact that if $b=4$ then $a=1$, we have $h(d,a,b) \geqslant h(m,0,5) = (m-6.5)/(m-5) > (m-4)/(m-2)$.

(c) $a \leqslant 1, a+b \leqslant 4$. Then by (iii), $\deg(u) \geqslant (m-10)(m-5)+5$ and

$$h(d,a,b) \geqslant h(m^2-15m+55,0,0) = \frac{m^2-15m+51}{m^2-15m+55} > \frac{m-4}{m-2}.$$

Thus we get $\deg^*(v) \geqslant 2 + 2(m-4)/(m-2) = 4(1-1/(m-2))$.

Therefore, every vertex in $G$ gets a modernized degree at least $4(1-1/(m-2))$ and the theorem is proved. □

## 7. Proof of Theorem 3

In [12] Kříž constructed for every $k,g$ a graph $H_{k,g}$ having girth $g$ and chromatic number $k$. Moreover, this graph $H_{k,g}$ is the union of $(k-1)$ forests whose every component is a star.

By the easy part of the Nash-Williams theorem [14] this means that for every subgraph $H'$ of $H_{k,g}$ we have $e_{H'} \leqslant (k-1)(v_{H'}-1)$. Consider the oriented graph $H^*_{k,g}$ obtained from $H_{k,g}$ by replacing every edge by a directed 2-path. Clearly the graph $H^*_{k,g}$ has girth $2g$ and oriented chromatic number at least $k$.

We claim that $m = \mathrm{mad}(H^*_{k,g})$ is at most $4(1-1/k)$. To see that, let $G$ be any subgraph of $H^*_{k,g}$ whose average degree is exactly $m$. It is not difficult to see that $G = (H')^*$ for some subgraph $H'$ of $H_{k,g}$ and thus $e_G = 2e_{H'}$ and $v_G = v_{H'} + e_{H'}$. Hence,

$$m = \frac{2e_G}{v_G} = \frac{4e_{H'}}{v_{H'}+e_{H'}} = 4 \cdot \left(1 - \frac{v_{H'}}{v_{H'}+e_{H'}}\right).$$

Since $e_{H'} \leqslant (k-1)(v_{H'}-1)$ we get $v_{H'}+e_{H'} \leqslant kv_{H'}-(k-1)$ and thus $m \leqslant 4(1-1/k)$. The theorem is proved. □

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