



Note

Transversals in uniform hypergraphs with property (7,2)

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Abstract

Let $f(r, p, t)$ ($p > t \geq 1, r \geq 2$) be the maximum of the cardinality of a minimum transversal over all r -uniform hypergraphs \mathcal{H} possessing the property that every subhypergraph of \mathcal{H} with p edges has a transversal of size t . The values of $f(r, p, 2)$ for $p = 3, 4, 5, 6$ were found in Erdős et al. (Siberian Adv. Math. 2 (1992) 82–88). We give bounds on $f(r, 7, 2)$, partially answering a question in Erdős et al. (1992). © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

A transversal of a family \mathcal{F} of sets is a subset of $\bigcup_{F \in \mathcal{F}} F$ meeting all members of \mathcal{F} . The smallest cardinality $\tau(\mathcal{F})$ of a transversal of \mathcal{F} is called the transversal number of \mathcal{F} . For a hypergraph $\mathcal{H} = (V, \mathcal{E})$, a transversal is a transversal of \mathcal{E} .

Say that \mathcal{B} possesses the property (p, t) if $\tau(\mathcal{F}) \leq t$ for every $\mathcal{F} \subset \mathcal{B}$ with $|\mathcal{F}| = p$. Erdős, Hajnal and Tuza [2] raised the following problem:

For given integers r, p , and t ($p > t \geq 1, r \geq 2$), determine the largest value, $f(r, p, t)$, of $\tau(\mathcal{F})$ taken over the class of r -uniform families \mathcal{F} possessing the property (p, t) .

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Erdős, Fon-Der-Flaass, Kostochka, and Tuza [1] found some bounds for $f(r, p, t)$ in general and determined the exact values of $f(r, p, 2)$ for $3 \leq p \leq 6$. In particular, $f(r, 6, 2) = r$.

In this note we give bounds for $f(r, 7, 2)$. For $3 \leq p \leq 6$, the extremal hypergraphs were the maximal (w.r.t. the number of vertices) complete r -uniform hypergraphs possessing the property (p, t) . It appears that $p = 7$ is the first number such that this is not the case. We will prove that for $k \geq 10$, $f(4k, 7, 2) \geq 3k + 1$ while the worst complete $4k$ -uniform hypergraph possessing property $(7, 2)$ has transversal number $3k$. We will also show that $f(r, 7, 2) \leq \lceil \frac{7r}{8} \rceil$. Note that if the lower bound holds for $k = 2$, then the upper bound is exact for $r = 8$.

2. Lower bound

Clearly, the transversal number of the complete r -uniform hypergraph $K_t(r)$ on t vertices is $t - r + 1$. Observe that $K_{7k-1}(4k)$ possesses property $(7, 2)$. Indeed, let A_1, \dots, A_7 be arbitrary $4k$ -element subsets of the ground set $U = \{1, \dots, 7k - 1\}$. Since $\sum_{i=1}^7 |A_i| = 28k > 4|U|$, there exists $u_1 \in U$ belonging to at least five sets A_i . Since $7k - 1 < 2(4k)$, there exists u_2 meeting the sets A_j not containing u_1 . It follows that $K_{7k-1}(4k)$ possesses property $(7, 2)$ and $f(4k, 7, 2) \geq (7k - 1) - 4k + 1 = 3k$.

On the other hand, $K_{7k}(4k)$ does not possess property $(7, 2)$. Indeed, let F_1, \dots, F_7 be the family of complements of lines in a Fano plane on $V = \{1, \dots, 7\}$. Then no two points meet every F_i . Blowing up every element of V into k elements, we produce a family $\{F'_1, \dots, F'_7\}$ of seven subsets of $V' = \{1, \dots, 7k\}$ with transversal number three. Thus although $K_{7k}(4k)$ has transversal number $7k - 4k + 1 = 3k + 1$, it does not have property $(7, 2)$. In this section, we exhibit a family \mathcal{B} of $4k$ -element sets possessing property $(7, 2)$ with transversal number $3k + 1$.

Theorem 1. *Let $k \geq 10$, $|S| = 7k + 1$ and $X \subset S$, $|X| = 4k$. Let \mathcal{B} be the family of $4k$ -element subsets of S whose intersection with X has an odd cardinality. Then \mathcal{B} possesses property $(7, 2)$ and $\tau(\mathcal{B}) = 3k + 1$.*

Remark. By elaborating the arguments of Claims 3 and 4 below, one can prove that the theorem holds already for $k \geq 4$. But for $k = 2$, \mathcal{B} does not possess property $(7, 2)$.

Proof of Theorem 1. Clearly, $S \setminus X$ is a transversal of \mathcal{B} of cardinality $3k + 1$. On the other hand, let T be an arbitrary subset of S with $|T| = 3k$. Then there is $a \in (S \setminus T) \cap X$ and $b \in (S \setminus T) \setminus X$. Thus, either $(S \setminus T) - a \in \mathcal{B}$ or $(S \setminus T) - b \in \mathcal{B}$. It follows that $\tau(\mathcal{B}) = 3k + 1$.

Below in a series of claims we will prove that \mathcal{B} possesses property $(7, 2)$.

Let $\mathcal{A} = \{A_1, \dots, A_7\}$ be an arbitrary family of seven members of \mathcal{B} . Assume that $\tau(\mathcal{A}) > 2$. Below we will derive the properties of such an \mathcal{A} which finally will produce a contradiction.

Claim 1. Every element of S belongs to at most four members of \mathcal{A} .

Proof. Assume that, say, a covers A_1, \dots, A_5 . There exists $b \in A_6 \cap A_7$. Then $\{a, b\}$ is a transversal of \mathcal{A} , a contradiction.

Since $|A_1| + \dots + |A_7| = 28k$, we conclude from Claim 1 that almost every element of S (with at most four exceptions) has degree four in \mathcal{A} . We shall call such elements *standard*; denote the set of standard vertices by St , and $S \setminus \text{St}$ by $\overline{\text{St}}$. The sequence of the degrees of vertices in $\overline{\text{St}}$ must be one of the following: (a) 3,3,3,3; (b) 3,3,2; (c) 3,1; (d) 2,2; (e) 0.

To shorten notation, we set $\overline{Y} = S \setminus Y$, $A_{ij} = A_i \cap A_j$ and $A^{ij} = A_i \cup A_j$.

Claim 2. If there exists $a \in \overline{A^{ij}} \cap \text{St}$ such that $a \notin A_k$ ($k \neq i, j$), then $A_k \cap A_{ij} = \emptyset$.

Proof. If $b \in A_k \cap A_{ij}$, then $\{a, b\}$ is a transversal of \mathcal{A} , a contradiction.

Claim 3. For any $i \neq j$, $|A_{ij}| \leq 2k + 2$. Furthermore,

- (i) if $|A_{ij}| = 2k + 2$, then $A^{ij} \subseteq \text{St}$ and there exists A_k with $A_{ik} = A_i \setminus A_{ij}$ and $|A_{ik}| = 2k - 2$;
- (ii) if $|A_{ij}| = 2k + 1$, then $|\overline{\text{St}} \setminus A^{ij}| \geq 2$ and there exists A_k with $2k - 3 \leq |A_{ik}| \leq 2k - 1$.

Proof. Assume that $m = \max\{|A_{ij}|\}$, $|A_{1,2}| \geq m - 1$ and $m \geq 2k + 2$. Then $|\overline{A^{1,2}}| \geq 7k + 1 - (8k - (m - 1)) = m - k \geq k + 2$ and there exists $a \in \overline{A^{1,2}} \cap \text{St}$. W.l.o.g., suppose that $a \in A_3 \cap A_4 \cap A_5 \cap A_6$. Then, by Claim 2, $A_7 \cap A_{1,2} = \emptyset$.

If there exists $b \in \overline{A^{1,2}} \cap \text{St} \cap A_7$, then b misses A_i for some $i \in \{3, 4, 5, 6\}$, say A_6 . Again by Claim 2, $A_6 \cap A_{1,2} = \emptyset$, and $|A_{6,7}| = |A_6| + |A_7| - |A^{6,7}| \geq 8k - (7k + 1 - (m - 1)) = k - 2 + m$, a contradiction to the choice of m . It follows that

$$A_7 \subseteq (A^{1,2} \setminus A_{1,2}) \cup \overline{\text{St}}. \tag{1}$$

Observe that

$$|(A^{1,2} \setminus A_{1,2}) \cup \overline{\text{St}}| = 8k - 2|A_{1,2}| + |\overline{\text{St}} \setminus A^{1,2}|.$$

If $|A_{1,2}| \geq 2k + 3$ then the last expression is at most $4k - 2$. This contradicts the fact that $|A_7| = 4k$. If $|A_{1,2}| = 2k + 2$ then to satisfy (1) we need $|\overline{\text{St}} \setminus A^{1,2}| = 4$ and $A_7 = (A^{1,2} \setminus A_{1,2}) \cup \overline{\text{St}} = (A_1 \setminus A_{1,2}) \cup (A_2 \setminus A_{1,2}) \cup \overline{\text{St}}$, so that $A^{1,2} \subseteq \text{St}$, $|A_1 \setminus A_{1,2}| = |A_2 \setminus A_{1,2}| = 2k - 2$ and $A_{1,7} = A_1 \setminus A_{1,2}$. This proves (i).

Finally, if $|A_{1,2}| = 2k + 1$ then to satisfy (1) we need $|\overline{\text{St}} \setminus A^{1,2}| \geq 2$ and $|(A^{1,2} \setminus A_{1,2}) \setminus A_7| \leq 2$. Since $|A_{1,7}| \leq |A_1| - |A_{1,2}| = 2k - 1$, this proves (ii).

Claim 4. For any $i \neq j$, $|A_{ij}| \geq 2k - 6$.

Proof. Let $i = 1, j = 2$.

Case 1. There exists A_3 with $|A_{1,3}| \geq 2k + 2$. Then by Claim 3, $|A_{1,3}| = 2k + 2$, $A_1 \subseteq \text{St}$, and for some A_4 , $|A_{1,4}| = 2k - 2$. Then $\sum_{l=2}^7 |A_l \cap A_1| = 3|A_1| = 12k$, and again by Claim 3,

$$\sum_{l=3}^7 |A_l \cap A_1| = |A_3 \cap A_1| + |A_4 \cap A_1| + \sum_{l=5}^7 |A_l \cap A_1| \leq 4k + 3(2k + 2) = 10k + 6.$$

Thus $|A_{1,2}| = \sum_{l=2}^7 |A_l \cap A_1| - \sum_{l=3}^7 |A_l \cap A_1| \geq 12k - (10k + 6) = 2k - 6$.

Case 2. $\max\{|A_l \cap A_1| : 3 \leq l \leq 7\} = 2k + 1$. Let $|A_{1,3}| = 2k + 1$. Then by part (ii) of Claim 3, $\sum_{l=2}^7 |A_l \cap A_1| \geq 3|A_1| - 2 = 12k - 2$ and for some A_s , $2k - 3 \leq |A_{1,s}| \leq 2k - 1$. If $s = 2$, then we are done. Thus we may assume $s = 4$. It follows that

$$\sum_{l=3}^7 |A_l \cap A_1| = |A_3 \cap A_1| + |A_4 \cap A_1| + \sum_{l=5}^7 |A_l \cap A_1| \leq 4k + 3(2k + 1) = 10k + 3,$$

and $|A_{1,2}| \geq (12k - 2) - (10k + 3) = 2k - 5$.

Case 3. $\max\{|A_l \cap A_1| : 3 \leq l \leq 7\} \leq 2k$. Then $\sum_{l=2}^7 |A_l \cap A_1| \geq 12k - 4$ and $\sum_{l=3}^7 |A_l \cap A_1| \leq 10k$. Consequently, $|A_{1,2}| \geq 2k - 4$.

For $a \in S$ let the spectrum $s(a)$ be the set of indices i such that $a \in A_i$.

Claim 5. For any $i \neq j$ and any $a, b \in \overline{A^{ij}} \cap \text{St}$, $s(a) = s(b)$.

Proof. If $s(a) \neq s(b)$, let $k \in s(b) \setminus s(a)$ and $l \in s(a) \setminus s(b)$. By Claim 2, $A^{kl} \cap A_{1,2} = \emptyset$. Then, by Claim 4,

$$|A_{kl}| = |A_k| + |A_l| - |A^{kl}| \geq 8k - (7k + 1 - (2k - 6)) = 3k - 7 > 2k + 2,$$

a contradiction to Claim 3.

For any $i \neq j$, let $c(i, j)$ denote the number k such that $k \neq i, j$ and $A_k \cap \overline{A^{ij}} \cap \text{St} = \emptyset$. By Claim 5, this number is unique. In particular, $A_{c(i,j)} \cap \text{St} \subseteq A^{ij} \setminus A_{ij}$. It follows that $c(i, c(i, j)) = j$ and $c(j, c(i, j)) = i$. In other words, St is the disjoint union of seven sets D_1, \dots, D_7 with equal spectra inside each set. And the A_i -s form the complements of lines of the Fano plane on these D_j -s.

Claim 6. No element of $\overline{\text{St}}$ has a spectrum $\{i, j, c(i, j)\}$ for some i, j .

Proof. Suppose that $s(a) = \{i, j, c(i, j)\}$. By the above, each $b \in \overline{A^{ij}}$ has the spectrum $\{1, \dots, 7\} \setminus \{i, j, c(i, j)\}$. Then a and b cover \mathcal{A} .

Let A_i, A_j and A_k be such that $k = c(i, j)$. Since each of them has an odd intersection with X , the number of elements in X belonging to an odd number of members of $\{A_i, A_j, A_k\}$ is odd. But by the above, each $a \in \text{St}$ belongs to an even number of members of $\{A_i, A_j, A_k\}$. Thus,

$$|A_i \cap \overline{\text{St}} \cap X| + |A_j \cap \overline{\text{St}} \cap X| + |A_k \cap \overline{\text{St}} \cap X|$$

is odd. In particular, $\overline{St} \cap X \neq \emptyset$. Moreover, if $\overline{St} \cap X = \{a\}$, then by Claim 6, there are some $i, j, c(i, j)$ such that none of these contains a . Thus, $|\overline{St} \cap X| \geq 2$.

On the other hand, since the cardinality of each A_i is $4k$,

$$|A_i \cap \overline{St}| + |A_j \cap \overline{St}| + |A_k \cap \overline{St}|$$

is even for every $i \neq j$. By the reasons similar to above, $|\overline{St} \setminus X| \geq 2$.

The only possibility we are left with is that $|\overline{St}| = 4$ and the degree sequence is 3,3,3,3. Furthermore, $|\overline{St} \cap X| = 2$.

However, since $\sum_{i=1}^7 |A_i \cap X|$ is odd, the sum of degrees of the nonstandard vertices in X must be odd. This is a final contradiction.

3. Upper bound

It will be easier to prove the upper bound in the following form.

Theorem 2. *Let \mathcal{B} be an r -uniform family. If $\tau(\mathcal{B}) > \lceil 7r/8 \rceil$, then there exists $\mathcal{F} \subset \mathcal{B}$ with $|\mathcal{F}| \leq 7$ such that $\tau(\mathcal{F}) > 2$.*

Proof. Let $r = 8k + s$, where $k \geq 1$ and $0 \leq s \leq 7$.

Suppose that there exists an r -uniform family \mathcal{B} possessing the property (7,2) with $\tau(\mathcal{B}) > \lceil 7r/8 \rceil = 7k + s$. For each set A , we set

$$\mathcal{B}_A = \{B \in \mathcal{B} \mid B \cap A = \emptyset\}.$$

Below, any triple (A_1, A_2, A_3) of members of \mathcal{B} with $A_1 \cap A_2 \cap A_3 = \emptyset$ will be called a *good triple*. To shorten notation, below we set $A_{ij} = A_i \cap A_j$ and $a_{ij} = |A_{ij}|$. Our main tool will be the following fact.

Lemma 1. *Let \mathcal{B} be an r -uniform family containing a good triple (A_1, A_2, A_3) satisfying the following inequalities:*

$$a_{12} \leq 4k + \lceil s/2 \rceil; \tag{2}$$

$$\max\{a_{13}, a_{23}\} \leq 3k + s/2; \tag{3}$$

$$a_{13} + a_{23} \leq 5k + s. \tag{4}$$

Then the theorem holds for \mathcal{B} .

Proof. Suppose that $\tau(\mathcal{B}) > 7k + s$. For the proof, we may clearly assume

$$a_{12} \geq a_{13} \geq a_{23}, \tag{5}$$

since reordering to assure (5) will not violate (2), (3) or (4).

Case 1. $a_{13} \leq (k + a_{12})/2$. Then, by (5), $|A_3 - A_1 - A_2| \geq r - 2a_{13} \geq 7k + s - a_{12}$, and so there exists $B_0 \subseteq A_3 - A_1 - A_2$ with $|B_0| = 7k + s - a_{12}$ such that $A_3 - B_0$ can be

partitioned into two parts B_1 and B_2 so that $|B_1| = \lfloor (k + a_{12})/2 \rfloor$, $|B_2| = \lceil (k + a_{12})/2 \rceil$ and $A_{i3} \subseteq B_i$ for $i = 1, 2$. Furthermore, for $i = 1, 2$, there exists $B'_i \subseteq A_i - A_{i3} - A_{12}$ with $|B'_i| = b_i = 7k + s - a_{12} - |B_{3-i}|$, provided that $b_i \geq 0$. But this is always the case since by (2),

$$a_{12} + |B_{3-i}| \leq \lfloor \frac{3}{2}a_{12} + \frac{1}{2}k + \frac{1}{2} \rfloor \leq \lfloor 6k + \frac{3}{2}\lceil s/2 \rceil + \frac{1}{2}k + \frac{1}{2} \rfloor \leq 7k + s.$$

Since $\tau(\mathcal{B}) > 7k + s$, there exist $A_4 \in \mathcal{B}_{A_{12} \cup B_0}$, $A_5 \in \mathcal{B}_{A_{12} \cup B_1 \cup B'_2}$ and $A_6 \in \mathcal{B}_{A_{12} \cup B_2 \cup B'_1}$. For $B_4 = (A_1 \cup A_2) - A_{12} - B'_1 - B'_2$, we have

$$\begin{aligned} |B_4| &= 2(8k + s) - 2a_{12} - (7k + s - a_{12} - |B_2|) - (7k + s - a_{12} - |B_1|) \\ &= 2k + |B_1| + |B_2| = 3k + a_{12} \leq 7k + s. \end{aligned}$$

Hence there exists $A_7 \in \mathcal{B}_{B_4}$.

Suppose that there are two elements x and y covering $\mathcal{F} = \{A_1, \dots, A_7\}$. In order to cover A_1, A_2 and A_3 , at least one of them, say x , belongs to some A_{ij} , where $1 \leq i < j \leq 3$. Assume first that $x \in A_{12}$. Then $y \in A_3 = B_0 \cup B_1 \cup B_2$. It follows that one of the edges A_4, A_5 and A_6 is not covered. If $x \in A_{i3}$ ($i = 1, 2$) and $y \notin A_{12}$, then to meet both A_{3-i} and A_7 , $y \in A_{3-i} - A_1 - B_4 = B'_{3-i}$. But in this case, y misses A_{i+4} . This completes Case 1.

Case 2. $a_{13} > (k + a_{12})/2$. Let $B_1 \subseteq A_2 - A_1 - A_3$ with $|B_1| = 7k + s - a_{12} - a_{13}$. Let $B_2 \subseteq A_3 - A_1 - A_2$ with $|B_2| = \min\{7k + s - a_{12}, 8k + s - a_{13} - a_{23}\}$. Let $B_3 = A_3 - A_1 - B_2$. Observe that $B_3 \supset A_{23}$ and

$$\begin{aligned} |B_3| &= 8k + s - a_{13} - \min\{7k + s - a_{12}, 8k + s - a_{13} - a_{23}\} \\ &= \max\{a_{12} - a_{13} + k, a_{23}\}. \end{aligned}$$

Let $B_4 \subseteq A_1 - A_3 - A_2$ with $|B_4| = \min\{7k + s - a_{12} - |B_3|, 8k + s - a_{12} - a_{13}\}$. By the cardinality constraints on B_1, B_2, B_3 and B_4 , there exist $A_4 \in \mathcal{B}_{A_{12} \cup A_{13} \cup B_1}$, $A_5 \in \mathcal{B}_{A_{12} \cup B_2}$, and $A_6 \in \mathcal{B}_{A_{12} \cup B_3 \cup B_4}$.

Let $B_5 = (A_1 \cup A_2) - A_{12} - B_1 - B_4$. If we prove that

$$|B_5| \leq 7k + s, \tag{6}$$

then the lemma would follow. Indeed, in this case there exists $A_7 \in \mathcal{B}_{B_5}$. Thus, if two elements x and y cover $\mathcal{F} = \{A_1, \dots, A_7\}$ and $x \in A_{12}$, then $y \in A_3 \subseteq A_{13} \cup B_2 \cup B_3$ and hence at least one of the edges A_4, A_5 and A_6 is not covered. If $x \in A_{13}$ and $y \notin A_{12}$, then $y \in A_2 - A_1 \subseteq B_1 \cup B_5$ and hence A_4 or A_7 is not covered. Finally, if $x \in A_{23}$ and $y \notin A_{12} \cup A_{13}$, then $y \in A_1 - A_2 \subseteq B_4 \cup B_5$ and hence A_6 or A_7 is not covered.

By the definition,

$$\begin{aligned} |B_5| &= 2(8k + s) - 2a_{12} - (7k + s - a_{12} - a_{13}) \\ &\quad - \min\{7k + s - a_{12} - |B_3|, 8k + s - a_{12} - a_{13}\} \\ &= 9k + s - a_{12} + a_{13} - (7k + s - a_{12}) - \min\{-|B_3|, k - a_{13}\} \\ &= 2k + a_{13} + \max\{|B_3|, -k + a_{13}\}. \end{aligned}$$

Hence if $|B_3| \leq -k + a_{13}$, then by (2),

$$|B_5| = 2k + a_{13} - k + a_{13} \leq k + 2(3k + s/2) = 7k + s.$$

Otherwise,

$$\begin{aligned} |B_5| &= 2k + a_{13} + |B_3| = 2k + a_{13} + \max\{a_{12} - a_{13} + k, a_{23}\} \\ &= \max\{3k + a_{12}, 2k + a_{13} + a_{23}\}. \end{aligned}$$

By (2) and (4), in both cases $|B_5| \leq 7k + s$. This proves the lemma.

Proof of the theorem. *Case 1.* There are $A_1, A_3 \in \mathcal{B}$ with $2k \leq |A_1 \cap A_3| \leq 3k + s/2$. Let $A_{13} \subset B_1 \subset A_1$ with $|B_1| = 4k + \lfloor s/2 \rfloor$ and $A_{13} \subset B_3 \subset A_3$ with $|B_3| = 3k + a_{13} + \lfloor s/2 \rfloor$. Since $|B_1 \cup B_3| = 7k + s$, there exists $A_2 \in \mathcal{B}_{B_1 \cup B_3}$. By the definition of B_1 and B_3 , (A_1, A_2, A_3) is a good triple, $|A_2 \cap A_1| \leq 4k + \lfloor s/2 \rfloor$ and $|A_2 \cap A_3| \leq 5k + \lfloor s/2 \rfloor - a_{13}$. Hence, all conditions (2)–(4) of Lemma 1 are satisfied and it can be applied.

Case 2. There are $A_1, A_2 \in \mathcal{B}$ with $3k + s/2 < |A_1 \cap A_2| \leq 4k + s/2$. For $i = 1, 2$, let $A_{12} \subset B_i \subset A_i$ with $|B_i| = 5k + \lfloor s/2 \rfloor$. Since

$$|B_1 \cup B_2| = 10k + 2\lfloor s/2 \rfloor - a_{12} \leq 7k + \lfloor s/2 \rfloor,$$

there exists $A_3 \in \mathcal{B}_{B_1 \cup B_2}$. By the definition of B_1 and B_2 , (A_1, A_2, A_3) is a good triple and $|A_3 \cap A_i| \leq 3k$ for $i = 1, 2$. Moreover, since Case 1 does not hold, $|A_3 \cap A_i| < 2k$ for $i = 1, 2$. Again, Lemma 1 can be applied.

Case 3. There are $A_1, A_2 \in \mathcal{B}$ with $k \leq |A_1 \cap A_2| < 2k$. For $i = 1, 2$, let $A_{12} \subset B_i \subset A_i$ with $|B_i| = 4k + \lfloor s/2 \rfloor$. Since $|B_1 \cup B_2| = 8k + 2\lfloor s/2 \rfloor - a_{12} \leq 7k + s$, there exists $A_3 \in \mathcal{B}_{B_1 \cup B_2}$. By the definition of B_1 and B_2 , (A_1, A_2, A_3) is a good triple and $|A_3 \cap A_i| \leq 4k + \lfloor s/2 \rfloor$ for $i = 1, 2$. Moreover, since Cases 1 and 2 do not hold, $|A_3 \cap A_i| < 2k$ for $i = 1, 2$. Once more, Lemma 1 can be applied.

Case 4. For any $A, B \in \mathcal{B}$, either $|A \cap B| > 4k + \lfloor s/2 \rfloor$ or $|A \cap B| < k$. Let $x = \max\{|A \cap B| : A, B \in \mathcal{B}, |A \cap B| \leq 4k + s/2\}$ and $A_1, A_3 \in \mathcal{B}$ with $|A_1 \cap A_3| = x$. For $i = 1, 3$, let $A_{13} \subset B_i \subset A_i$ with $|B_1| = \lceil (7k + s + x)/2 \rceil$ and $|B_3| = \lfloor (7k + s + x)/2 \rfloor$. Since $|B_1 \cup B_3| = 7k + s$, there exists $A_2 \in \mathcal{B}_{B_1 \cup B_3}$. By the construction,

$$\max\{|A_1 \cap A_2|, |A_3 \cap A_2|\} \leq 8k + s - \lfloor (7k + s + x)/2 \rfloor.$$

Since (A_1, A_2, A_3) is a good triple, either $|A_1 \cap A_2| \leq 4k + s/2$ or $|A_3 \cap A_2| \leq 4k + s/2$. We will assume that $|A_3 \cap A_2| \leq 4k + s/2$. Then, under conditions of the case, $|A_3 \cap A_2| \leq x$.

Since $a_{13} + a_{23} \leq 2x \leq 2k - 2 < \lfloor (5k + s + x)/2 \rfloor \leq 7k + s - a_{12}$, we can partition A_3 into four parts B_1, \dots, B_4 so that

$$\begin{aligned} |B_1| &= 7k + s - a_{12}, & |B_2| &= 7k - a_{12}, \\ |B_3| &= |B_4| = a_{12} - 3k & \text{and } B_1 &\supseteq A_{13} \cup A_{23}. \end{aligned}$$

For $i = 1, 2$, there exists $A_{i+3} \in \mathcal{B}_{A_{12} \cup B_i}$.

Let $i \in \{1, 2\}$ and let $C_i = A_4 \cap A_i$. Since $A_4 \cap A_{12} = \emptyset$, we have $|C_i| \leq x$. Let $D_i = A_{12} \cup B_{i+2} \cup A_{i3} \cup C_{3-i}$. Since

$$|D_i| \leq a_{12} + (a_{12} - 3k) + x + x \leq 5k + 2\lceil s/2 \rceil + 2(k - 1) < 7k + s,$$

there exists $A_{i+5} \in \mathcal{B}_{D_i}$.

Assume that some two elements x and y cover $\mathcal{F} = \{A_1, \dots, A_7\}$. If $x \in A_{12}$ then $y \in B_j$ for some $j \in \{1, 2, 3, 4\}$. But in this case A_{j+3} is not covered. So, let $x \in A_{i3}$ for some $i \in \{1, 2\}$. In order to cover A_4 and A_{3-i} , we need $y \in C_{3-i}$. Then A_{i+5} is missed. This is the final contradiction.

References

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