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Variable degeneracy: extensions of Brooks' and Gallai's theorems

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Abstract

We introduce the concept of variable degeneracy of a graph extending that of k-degeneracy. This makes it possible to give a common generalization of the point partition numbers and the list chromatic number. In particular, the list point arboricity of a graph is considered. We extend Brooks' and Gallai's theorems in terms of covering the vertices of a graph by disjoint induced subgraphs $G_1,\dots,G_s$ such that G_i is strictly $\boldsymbol{f}_i$-degenerate, given nonnegative-integer-valued functions $\boldsymbol{f}_1,\dots,\boldsymbol{f}_s$ whose sum is bounded below at each vertex by the degree of that vertex.
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1. Variable degeneracy

The notion of a k-degenerate graph proved to be useful in a number of graph colouring problems. We introduce here the more flexible concept of variable degeneracy of G which is expressed in terms of a function from $V(G)$ to the positive integers. This makes it possible to unify the seemingly remote problems of determining the point partition numbers and the list chromatic number and to absorb a number of known results in these directions. In particular, the natural concept of the list point arboricity of a graph is introduced and studied (Corollary 5, $t=2$).

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Our main results are extensions of Brooks' and Gallai's theorems (Theorems 4 and 8, respectively) in terms of covering the vertices of a graph by disjoint induced subgraphs $G_1,\dots,G_s$ such that G_i is strictly $\boldsymbol{f}_i$-degenerate, given nonnegative-integer-valued functions $\boldsymbol{f}_1,\dots,\boldsymbol{f}_s$ whose sum is bounded below at each vertex by the degree of that vertex.

In Section 4, we use a simple argument to strengthen Theorems 4 and 8 by showing that, in addition, each vertex $v \in G_i$ may have degree (in G_i) at most $\boldsymbol{f}_i(v)$. We also consider an application of these results to graphs embedded on surfaces (Section 5).

We now proceed to the definitions. For a graph G, we denote by $\boldsymbol{d}_G$ the degree function on $V(G)$. Let k be a positive integer. A graph G is said to be *strictly k-degenerate* if in every subgraph G' of G there is a vertex v such that $\boldsymbol{d}_{G'}(v) < k$. By the definition, the strictly 1-degenerate graphs are precisely those without edges, and the strictly 2-degenerate graphs are precisely the forests. Lick and White [11] defined a graph to be *k-degenerate* if each of its subgraphs has a vertex of degree $\leqslant k$. Thus our term 'strictly k-degenerate' is equivalent to '$(k-1)$-degenerate' in the Lick-White definition. We have made this deviation from their terminology to express some of our results in a more natural way. The smallest k for which G is strictly k-degenerate is sometimes called *the colouring number* $\operatorname{col}(G)$ of G; it can be determined in polynomial time. It is easy to verify by induction that every strictly k-degenerate graph is k-colourable, that is $\chi(G) \leqslant \operatorname{col}(G)$, where $\chi(G)$ is the chromatic number of G; for an extension, see Claim 1.

Let $\boldsymbol{f}$ be a function from $V(G)$ to the set of positive integers. We say that G is *strictly $\boldsymbol{f}$-degenerate* if in every subgraph G' of G there is a vertex v such that $\boldsymbol{d}_{G'}(v) < \boldsymbol{f}(v)$. In other words, G can be completely destroyed by removing the vertices, one at a time, so that each vertex v has at the moment of removal a degree less than $\boldsymbol{f}(v)$. We refer to this process as *eroding* G. Expressed in yet another way, the vertices of a strictly $\boldsymbol{f}$-degenerate graph can be numbered so that each vertex v is adjacent to lesser than $\boldsymbol{f}(v)$ vertices with greater numbers.

Observe that no graph G is strictly $\boldsymbol{d}_G$-degenerate, simply because there is no vertex v such that $\boldsymbol{d}_G(v) < \boldsymbol{d}_G(v)$ to begin with eroding G. However, if $\boldsymbol{f}(v) \geqslant \boldsymbol{d}_G(v)$ for each $v \in V(G)$, $\boldsymbol{f}(w) > \boldsymbol{d}_G(w)$ for some $w \in V(G)$, and G is connected, then G is strictly $\boldsymbol{f}$-degenerate. Indeed, we can first remove w and then use induction because each vertex w' which is adjacent to w in G has $\boldsymbol{f}(w') > \boldsymbol{d}_{G-w}(w')$.

Let $\boldsymbol{F} = (\boldsymbol{f}_1,\dots,\boldsymbol{f}_s)$, where $\boldsymbol{f}_i$ $(1 \leqslant i \leqslant s)$ is a function from $V(G)$ to the non-negative integers. We say that G is *$\boldsymbol{F}$-partitionable* or *$(\boldsymbol{f}_1,\dots,\boldsymbol{f}_s)$-partitionable* if $V(G)$ can be covered by disjoint induced subgraphs $G_1,\dots,G_s$ such that every G_i is strictly $\boldsymbol{f}_i$-degenerate. Such a covering is called *an $\boldsymbol{F}$-partition*.

Claim 1. *If G is strictly $\boldsymbol{f}$-degenerate and $\boldsymbol{f}_1(v) + \cdots + \boldsymbol{f}_s(v) \geqslant \boldsymbol{f}(v)$ for all $v \in V(G)$, then G is $(\boldsymbol{f}_1,\dots,\boldsymbol{f}_s)$-partitionable.*

Proof. Take a vertex v with $\boldsymbol{d}_G(v) < \boldsymbol{f}(v)$ and, by induction, suppose that $G - v$ is partitioned into strictly $\boldsymbol{f}_i$-degenerate subgraphs G_i. There is an i_0 such that v is adjacent to lesser than $f_{i_0}(v)$ vertices from G_{i_0}, because $d_G(v) < f_1(v)+\cdots+f_s(v)$. Then $G_{i_0}+v$ is strictly f_{i_0}-degenerate, because its eroding may be begun with v. □

The subgraphs G_i above may be treated as colour classes. Note that if $f_i(v_j)=0$ then v_j cannot be coloured with i. Indeed, the restriction of f_i to $V(G_i)$ must be positive-valued by the definition of the strict f_i-degeneracy of G_i. Less formally, since G_i is strictly f_i-degenerate, v_j cannot belong to G_i simply because it never has a negative degree and hence can never be removed in the process of eroding G_i.

Thus, the special case of covering $V(G)$ by subgraphs of variable degeneracy in which $f_i(v) \in \{0,1\}$ for all i and v, corresponds to the list colouring of G with the list $L(v)=\{i \mid f_i(v)=1\}$.

In our main Theorem 8 we solve, with polynomial complexity, the problem of determining whether or not G is F-partitionable if $f_1(v)+\cdots+f_s(v) \geqslant d(v)$ for all $v \in V(G)$.

2. Extensions of Brooks' theorem

The following result was obtained by Borodin [2] and, independently, by Bollobás and Manvel [1]:

Theorem 2. *Let G be a connected graph with maximum degree $\Delta(G)=\Delta \geqslant 3$ and not the complete graph $K_{\Delta+1}$. Let also $k_1,\dots,k_s$ be positive integers, $s \geqslant 2$, such that $k_1+\cdots+k_s \geqslant \Delta$. Then $V(G)$ can be covered by induced subgraphs $G_1,\dots,G_s$ such that $\mathrm{col}(G_i) \leqslant k_i$ whenever $1 \leqslant i \leqslant s$.*

Brooks' theorem (that $\chi(G) \leqslant \Delta(G)$ if G is as in Theorem 2) follows from Theorem 2 by taking $k_1=\cdots=k_s=1$. The cases of point arboricity (which corresponds to $k_1=\cdots=k_s=2$), and of point partition numbers in general ($k_1=\cdots=k_s$) were solved by Kronk and Mitchem [10] and Mitchem [14].

An extension of Brooks' theorem of a different type, in terms of list colouring (choosability), was obtained by Vizing [17] and, independently, by Erdős et al. [6].

Theorem 3. *Let G be a connected graph, $\Delta(G)=\Delta \geqslant 3$, and $G \neq K_{\Delta+1}$. Let also to each vertex v a list $L(v)$ of admissible colours be assigned such that $|L(v)| \geqslant \Delta$. Then there is a proper colouring such that the colour of each vertex is chosen from its list.*

The purpose of this section is to give a common generalization of Theorems 2 and 3. But this cannot be done simply by replacing the constants k_i by functions f_i in Theorem 2. For we can construct a connected $(s+t)$-regular graph G other than K_{s+t+1} and functions f_1 and f_2 with $f_1(v)+f_2(v)=s+t$ for each $v \in V(G)$ such that G is not (f_1,f_2)-partitionable, as follows:

Construction. *Take any s-regular block B and define* f_1 *and* f_2 *on* $V(B)$ *to be s and t, respectively. Take a block H in which one vertex,* $v(H)$, *has degree t while the others have degree* $s+t$. *Define* f_1 *to be 0 and* f_2 *to be* $s+t$ *on* $H-v(H)$. *Take a copy* H_w *of H for every vertex w of B and identify* $v(H_w)$ *with w.*

We have obtained an $(s+t)$-regular graph G in which $f_1(u)+f_2(u)=s+t$ for each vertex u. Now suppose that G is (f_1,f_2)-partitionable into $G(V_1)$ and $G(V_2)$. If at least one vertex from B is in V_2, then the whole corresponding copy of H is in V_2 and is not strictly f_2-degenerate. Otherwise, all vertices of B are in V_1, but B is not strictly f_1-degenerate.

This obstacle to $\boldsymbol{F}$-partitionability leads to the following definition. Given a graph G and functions f_i where $1\leqslant i\leqslant s$, a *monoblock* H of G is either an end-block of G or G itself if G is 2-connected such that there is an index j (depending on H) with the property that

$$f_i(v)=\begin{cases}0 & \text{for all } i\neq j,\\ d_G(v) & \text{for } i=j\end{cases}$$

for all v in H, except possibly for the cut-vertex if H is an end-block. The 2-connected monoblocks will be called *self-monoblocks*. Clearly, no self-monoblock is $\boldsymbol{F}$-partitionable, because all its vertices must be coloured the same and its eroding cannot start.

Now we are ready to formulate our main result in this section:

Theorem 4. *Let G be a connected graph,* $\Delta(G)=\Delta\geqslant 3$, *and* $G\neq K_{\Delta+1}$. *Let* $f_1,\dots,f_s$ *be nonnegative-integer-valued functions on* $V(G)$ *where* $s\geqslant 2$, $f_1(v)+\cdots+f_s(v)\geqslant\Delta$ *for each* $v\in V(G)$, *and G does not contain a monoblock. Then G is* $(f_1,\dots,f_s)$-*partitionable.*

In turn, this theorem is a special case of Theorem 8 which will be proved in Section 3. Theorem 4 immediately implies the following extension of Brooks' theorem in terms of the list point-partition numbers, in which case the f_i's take only the values 0 and t.

Corollary 5. *Let G be connected, not a complete graph and* $\Delta(G)\leqslant st\geqslant 3$, *where* $s\geqslant 2$. *Let to each vertex v a list* $L(v)$ *of admissible colours be assigned such that* $|L(v)|\geqslant s$. *Then a colour can be chosen from* $L(v)$ *for each v so that each colour class induces a strictly t-degenerate subgraph.*

Proof. Put

$$f_i(v)=\begin{cases}t & \text{if } i\in L(v),\\ 0 & \text{otherwise.}\end{cases}$$

Then

$$\sum_{1 \leqslant i \leqslant s} f_i(v) \geqslant |L(v)| t \geqslant st \geqslant \Delta(G)$$

for each $v \in V(G)$. If G is not st-regular then it is strictly st-degenerate and so the result follows from Claim 1. But if G is st-regular then there are no monoblocks, since $s \geqslant 2$, and the result follows from Theorem 4. □

The case $t=2$ of Corollary 5 is the list point arboricity analogue of Brooks' theorem.

3. Extensions of Gallai's theorem

For a graph G, a *G-list* is a function L that assigns to each vertex v a list $L(v)$ of admissible colours. Then G is *L-choosable* if there exists a proper colouring of the vertices of G with each vertex v getting a colour from $L(v)$.

In connection with Theorem 3, the natural question arises of describing those G-lists such that $|L(v)| \geqslant \boldsymbol{d}_G(v)$ for each $v \in V(G)$ but G is not L-choosable. The answer was given by Borodin [3,4] as follows:

A graph G together with a G-list L is called an *L-graph*. An L-graph G is *constant* if G is either a complete graph or an odd cycle and $L(v)$ is the same set of cardinality $\Delta(G)$ for each $v \in V(G)$. It is easy to see that no constant L-graph is L-choosable.

By an *R-operation* we mean the following:

(a) take disjoint L-graphs G^1 and G^2 and $v_1 \in V(G^1)$, $v_2 \in V(G^2)$ such that $L(v_1) \cap L(v_2) = \emptyset$,

(b) identify v_1 and v_2 to a vertex $v_1 \times v_2$, and

(c) set $L(v_1 \times v_2) = L(v_1) \cup L(v_2)$.

An L-graph G is *R-constructible* if it can be obtained from constant blocks by means of R-operations. It is easy to verify by induction on the number of end-blocks that if an L-graph G is R-constructible, then it is not L-choosable.

Theorem 6 (Borodin [3,4]). *A connected L-graph G satisfying the condition* $|L(v)| \geqslant \boldsymbol{d}_G(v)$ *for each* $v \in V(G)$ *is not L-choosable if and only if G is R-constructible.*

Note that the R-constructibility of an L-graph G is decidable in polynomial time by consecutively deleting end-blocks and reducing L accordingly. Theorem 6 readily implies, in particular, the following result proved independently by Erdős et al. [6]:

Corollary 7 (Erdős et al. [6]). *A connected graph G is L-choosable for each G-list L satisfying the condition* $|L(v)| \geqslant \boldsymbol{d}_G(v)$ *for each* $v \in V(G)$ *if and only if at least one block of G is neither an odd cycle nor a complete graph.*

Proof. If an L-graph G has a block B which is neither an odd cycle nor a complete graph, then G is clearly not R-constructible; hence, by Theorem 6, G is L-choosable.

Conversely, if each block of G is either an odd cycle or a complete graph, then G is R-constructible with an appropriate G-list L; hence, by the easy part of Theorem 6, G is not L-choosable. □

Observe that Theorem 6 and Corollary 7 slightly extend Gallai's theorem [8] which states that in each k-colour-critical graph G every block of the subgraph H induced by the vertices of degree $k-1$ is either a complete graph or an odd cycle. Indeed, suppose H has a block which is neither an odd cycle nor a complete graph. Then $G-H$ can be coloured with colours $1,\dots,k-1$. For each $v \in V(H)$ the list $L(v)$ of remaining admissible colours has cardinality at least $\boldsymbol{d}_H(v)$, and H is L-choosable by Corollary 7. Thus G is $(k-1)$-colourable, and so the result of Gallai's theorem follows.

Note that Brooks' theorem corresponds to the special case of Gallai's theorem when $\Delta(G)=k-1$. Accordingly, our purpose in this section is to extend Theorem 4 in the spirit of Theorem 6, i.e., to give an efficient criterion for the $(\boldsymbol{f}_1,\dots,\boldsymbol{f}_s)$-partitionability of a graph G, assuming

$$\boldsymbol{f}_1(v)+\cdots+\boldsymbol{f}_s(v)\geqslant \boldsymbol{d}_G(v) \quad \text{for every } v \in V(G).$$

A graph G together with a set $\boldsymbol{F}=(\boldsymbol{f}_1,\dots,\boldsymbol{f}_s)$ of nonnegative-integer-valued functions on $V(G)$ will be called an $\boldsymbol{F}$-*graph*. The self-monoblocks form one obvious class of obstacles to the $\boldsymbol{F}$-partitionability of an $\boldsymbol{F}$-graph. (In particular, each $\boldsymbol{F}$ with $s=1$ defined on a 2-connected graph G such that $\boldsymbol{f}_1(v)=\boldsymbol{d}(v)$ for each $v \in V(G)$ makes G a self-monoblock.) Another obvious class of obstacles are *constant* blocks, i.e., complete graphs and odd cycles with $\boldsymbol{F}$ being the same on all vertices and $\boldsymbol{f}_1(v)+\cdots+\boldsymbol{f}_s(v)=\boldsymbol{d}_G(v)$ throughout. It turns out that every obstacle to the $\boldsymbol{F}$-partitionability is a superposition of these elementary ones.

Self-monoblocks and constant blocks are collectively called *hard blocks*. All hard blocks are declared to be *hard-constructible*. Further, if there are disjoint hard-constructible $\boldsymbol{F}^j$-graphs G^j where $j=1,2$ with vectors $\boldsymbol{F}^j$ of the same dimension and vertices $v^j \in V(G^j)$, then the $\boldsymbol{F}$-graph G obtained by identifying v^1 with v^2 into a new vertex v^* and taking $\boldsymbol{F}(v^*)=\boldsymbol{F}^1(v^1)+\boldsymbol{F}^2(v^2)$ and $\boldsymbol{F}=\boldsymbol{F}^j$ on G^j-v^* for $j=1,2$, is also defined to be *hard-constructible*.

We shall see that if an $\boldsymbol{F}$-graph G is hard-constructible, then G is not $\boldsymbol{F}$-partitionable. Also, by induction on the number of end-blocks, it is readily decidable in polynomial time whether or not an $\boldsymbol{F}$-graph G is hard-constructible. Our main result in this paper, which is a common extension of Theorems 4 and 6, is

Theorem 8. *Let G be a connected $\boldsymbol{F}$-graph and let $\boldsymbol{f}_1(v)+\cdots+\boldsymbol{f}_s(v)\geqslant \boldsymbol{d}_G(v)$ for every $v \in V(G)$. Then G is $\boldsymbol{F}$-partitionable if and only if G is not hard-constructible.*

Proof. (‘*Only if*’): Suppose an $\boldsymbol{F}$-graph G has the least vertices among all hard-constructible $\boldsymbol{F}$-partitionable graphs. If G is 2-connected, then G is a self-monoblock or a constant block. In a self-monoblock each vertex must be coloured with the same colour j and $\boldsymbol{f}_j(v)=\boldsymbol{d}_G(v)$, so that there is no vertex to begin with eroding G_j. A constant odd cycle is either a self-monoblock with $f_j(v) = 2$ for each $v \in V(G)$ or otherwise has $f_1(v) = f_2(v) = 1$ for each v, and cannot be 2-coloured. Similarly, a constant complete block $G = K_t$ has $f_1(v) + \cdots + f_s(v) = t - 1$ for each $v \in V(G)$, so that in any vertex-s-colouring of G there is a j such that at least $f_j(v) + 1$ vertices of G are coloured with j, and G_j is not strictly $f_j(v)$-degenerate.

Now suppose G is obtained by applying the hard construction to F^j-graphs G^j ($j = 1,2$). Suppose G is F- partitionable and the cut vertex v^* is coloured 1. Then each colour class G_i is strictly f_i-degenerate whenever $i \neq 1$, and so are $G^1 \cap G_i$ and $G^2 \cap G_i$. On the other hand, it follows from the minimality of G that neither of $G^1 \cap G_1$ and $G^2 \cap G_1$ is strictly f_1-degenerate. This implies that there are induced subgraphs H^j of $G^j \cap G_1$ where $j = 1,2$ such that $d_{H^j}(v) \geqslant f_1^j(v)$ in $G^j \cap G_1$ for each $v \in V(H^j)$. If $H = H^1 \cup H^2$, then $d_H(v) \geqslant f_1(v)$ for each $v \in V(H)$, so that G_1 is not strictly f_1-degenerate; this contradiction proves the easy part of Theorem 8.

(‘*If*’) Now suppose that G is a graph with the least vertices such that there is an F for which G is a non-F-partitionable and nonhard-constructible F-graph. We first prove the following properties (a)–(d) of G:

(a) *G is 2-connected.* Suppose v^* is a cut-vertex in G and G is the result of identifying subgraphs $G^j \subset G$ on $v^j \in G^j$ to a new vertex v^* ($j = 1,2$). Consider an F-partition of the strictly F-degenerate graph $G - v^*$. Define F^j to coincide with F on $G^j - v^j$ and $f_i^j(v^j)$ to be the number of vertices of $G^j \cap G_i$ adjacent to v^j, where $j = 1,2$. Since G is not hard-constructible, at least one of the F^j-graphs G^j, say G^1, is also not hard-constructible. We combine an F^1-partition of G^1, which exists by the minimality of G, with an F^2-partition of the strictly F^2-degenerate graph $G^2 - v^2$, which exists by Claim 1, to get an F-partition of G. It is enough to observe that if v^* belongs to G_i, then G_i is strictly f_i-degenerate. Indeed, in eroding G_i, we first remove the vertices of $G_i \cap G^1$ in their order in $G_i \cap G^1$; note that when v^* is removed it is adjacent to lesser than $f_i^1(v^*)$ remaining vertices of $G_i \cap G^1$ and hence to lesser than $f_i^1(v^*) + f_i^2(v^*) = f_i(v^*)$ remaining vertices of G_i. Then we remove in G_i the vertices of $G_i \cap G^2$ in their order in $G_i \cap G^2$.

(b) *Each f_i is either constant zero or nowhere-zero.* If, say, $f_1(v) = 0$ and $f_1(w) > 0$ for some $v, w \in V(G)$ then, since G is connected, there are adjacent vertices u, z such that $f_1(u) = 0$, $f_1(z) > 0$. We define F' on $G - z$ by setting $f_i'(w) = f_i(w)$ for all i and w except that, for every w adjacent to z, we put $f_1'(w) = \max\{0, f_1(w) - 1\}$. Then

$$f_1'(w) + \cdots + f_s'(w) \geqslant d_{G-z}(w)$$

for each w and

$$f_1'(u) + \cdots + f_s'(u) = f_1(u) + \cdots + f_s(u) > d_{G-z}(u).$$

Since $G - z$ is connected, it follows that $G - z$ is strictly F'-degenerate and hence F'-partitionable. We put z in G_1 and use any F'-partition of $G - z$ to obtain an F-partition of G (with z removed last, when eroding G_1).

As G is not a monoblock, (b) implies:

(b′) *There are at least two nowhere-zero f_i's, say f_1 and f_2.*

The same argument as in proving (b) implies (c) and (d):

(c) *For any* $z \in V(G)$, *if* $\boldsymbol{F}'$ *is obtained by decreasing* f_1 *or* f_2 *by* 1 *for all vertices adjacent to* z, *then the* $\boldsymbol{F}'$*-graph* $G - z$ *is hard-constructible.*

(d) *For any* $z \in V(G)$, *let* w_1 *and* w_2 *be non-separating vertices in a block* B *of* $G - z$. *Then* z *is adjacent in* G *either to both or to neither of* w_1 *and* w_2.

Indeed, if z is adjacent to w_1 but not to w_2, then at least one of the two possible $\boldsymbol{F}'(w_1)$ obtained by decreasing f_1 or f_2 by 1 for all vertices adjacent to z, differs from $\boldsymbol{F}(w_2)$. But then B obviously could not arise from a hard block in the course of constructing the $\boldsymbol{F}'$-graph $G - z$.

To complete the proof of Theorem 8, take as z a vertex of the minimum degree $\delta = \delta(G)$; furthermore, if G is not δ-regular, then we choose z adjacent to a vertex v of degree greater than δ.

If $G - z$ is 2-connected, then by (d), z is adjacent to all other vertices of G and $G = K_{\delta+1}$. But for any non-constant $\boldsymbol{F}$ such that $f_1(v) + \cdots + f_s(v) \geqslant \delta$ on $G = K_{\delta+1}$, one can construct an $\boldsymbol{F}$-partition of G as follows. Take an i such that f_i is nonconstant; let m be the minimum value of $f_i(v)$ over all $v \in V(G)$. There are vertices x and y such that $f_i(x) = m$ and $f_i(y) > m$. We may suppose $m + 1 \leqslant \delta$, since otherwise G is $\boldsymbol{F}$-partitionable by Claim 1. We give colour i to those $m + 1$ vertices which have the largest f_i, delete these vertices, and decrease f_i on each other vertex w by $\min\{m + 1, f_i(w)\}$ to obtain f_i'. Since the degree of each vertex has gone down by $m + 1$ and, for at least one vertex v, $f_i(v)$ has gone down by only m, the resulting graph is strictly $\boldsymbol{f}'$-degenerate, where $\boldsymbol{f}' = \sum_{1 \leqslant i \leqslant s} f_i'$ and $f_j' = f_j$ if $j \neq i$. Thus G is $\boldsymbol{F}$-partitionable by Claim 1, and this is the required contradiction.

Thus, $G - z$ is not 2-connected. By (a) and (d), z is adjacent to all nonseparating vertices of all end-blocks in $G - z$. Because $\delta(G - z) \geqslant \delta - 1$, in each end-block there are at least δ vertices, i.e., at least $\delta - 1$ nonseparating ones. Since there are at least two end-blocks, we have $\boldsymbol{d}(z) = \delta \geqslant 2(\delta - 1)$, which yields $\delta \leqslant 2$. But $\delta \geqslant 2$ by (a). Thus $\delta = 2$ and $G - z$ has exactly two end-blocks, each isomorphic to K_2. By the choice of z, G is 2-regular, hence a cycle, and by (b'), $\boldsymbol{F} = (1, 1, 0, \ldots, 0)$ throughout G. Since G is not hard-constructible, G is a cycle of even length and is easily $\boldsymbol{F}$-partitionable. □

As a natural special case of Theorem 8, we have a Gallai-type result for the list point-partition numbers which extends Corollaries 5 and 7. We define an L-graph G to be $L \times t$*-choosable* if for each $v \in V(G)$ a colour can be chosen so that each colour class induces a strictly t-degenerate subgraph. Clearly, the cases $t = 1$ and 2 of this definition correspond to the list colouring and list point arboricity, respectively.

Corollary 9. *A connected graph* G *is* $L \times t$*-choosable for each* G*-list* L *satisfying the condition* $|L(v)| \geqslant \boldsymbol{d}_G(v)/t$ *for each* $v \in V(G)$ *if and only if at least one block of* G *differs from* K_{st+1} *for all* $s \geqslant 1$, *from a* t*-regular graph, and from an odd cycle if* $t = 1$.

Proof. Applying Theorem 8 for $t \geqslant 2$, just observe that an L-graph G is $L \times t$-choosable if and only if G is $\boldsymbol{F}$-partitionable where

$$\boldsymbol{f}_i(v) = \begin{cases} t & \text{if } i \in L(v), \\ 0 & \text{otherwise.} \end{cases}$$

In particular, for such an $\boldsymbol{F}$, each monoblock is t-regular, and each constant complete block has degree st. □

4. Additional degree constraints

Borodin [4] used a simple argument to deduce from Theorem 2 the following stronger result, which was proved independently by Bollobás and Manvel [1]:

Theorem 2′. *Let G be a connected graph with maximum degree $\Delta(G) = \Delta \geqslant 3$ and not the complete graph $K_{\Delta+1}$. Let also $k_1, \ldots, k_s$ be positive integers, $s \geqslant 2$, such that $k_1 + \cdots + k_s \geqslant \Delta$. Then $V(G)$ can be covered by induced subgraphs $G_1, \ldots, G_s$ such that* $\mathrm{col}(G_i) \leqslant k_i$ *and $\Delta_{G_i} \leqslant k_i$ whenever $1 \leqslant i \leqslant s$.*

We will show that our main Theorem 8 and its special cases are also ‘self-strengthening’. For a covering c of $V(G)$ by disjoint induced subgraphs of G and for $v \in V(G)$, let $c(v)$ denote the number of the subgraph (the colour class) in c containing v. Given a vector $\boldsymbol{F} = (\boldsymbol{f}_1, \ldots, \boldsymbol{f}_s)$, let

$$R_c(\boldsymbol{F}) = \sum_{v \in V(G)} (\boldsymbol{d}_{G_{c(v)}}(v) - 2\boldsymbol{f}_{c(v)}(v)).$$

In what follows, we write $\boldsymbol{d}_{G_i}(w)$ to denote the number of vertices of G_j that are adjacent to w, even when $w \notin G_j$. The following simple fact will help us.

Observation 10. *Let $w \in V(G)$ and $i \neq j$ be such that $c(w) = i$, $\boldsymbol{d}_{G_i}(w) \geqslant \boldsymbol{f}_i(w)$ and $\boldsymbol{d}_{G_j}(w) < \boldsymbol{f}_j(w)$. Then moving w from G_i to G_j decreases $R_c(\boldsymbol{F})$.*

Proof. When we erase the colour from w, the function $R_c(\boldsymbol{F})$ decreases by $\boldsymbol{d}_{G_i}(w) - 2\boldsymbol{f}_i(w)$ due to the contribution of w and it decreases by 1 $\boldsymbol{d}_{G_i}(w)$ times due to the contributions of the neighbours of w in G_i. The total loss of $R_c(\boldsymbol{F})$ is $2\boldsymbol{d}_{G_i}(w) - 2\boldsymbol{f}_i(w) \geqslant 0$. When we then colour w with j, this results in an increment of $2(\boldsymbol{d}_{G_j}(w) - \boldsymbol{f}_j(w)) < 0$. □

The idea of ‘self-strengthening’ is expressed by the following

Claim 11. *If a graph G is $\boldsymbol{F}$-partitionable where $\boldsymbol{f}_1(v) + \cdots + \boldsymbol{f}_s(v) \geqslant \boldsymbol{d}_G(v)$ for all $v \in V(G)$, then there is an $\boldsymbol{F}$-partition such that $\boldsymbol{d}_{G_i}(v) \leqslant \boldsymbol{f}_i(v)$ for every $v \in V(G_i)$ and $1 \leqslant i \leqslant s$.*

Proof. Among all $\boldsymbol{F}$-partitions c of G, we take a partition c^* minimizing $R_c(\boldsymbol{F})$ and assert that c^* is what we need.

Assume that a vertex w has $c^*(w)=i$ and $\boldsymbol{d}_{G_i}(w) > \boldsymbol{f}_i(w)$. Then for some j, due to the degree constraints, $\boldsymbol{d}_{G_j}(w) < \boldsymbol{f}_j(w)$, and therefore $G_j + w$ is strictly $\boldsymbol{f}_j$-degenerate. Hence the partition c' obtained from c^* by moving w from G_i to G_j is also an $\boldsymbol{F}$-partition of G. But by Observation 10, $R_{c'}(\boldsymbol{F}) < R_{c^*}(\boldsymbol{F})$, a contradiction to the choice of c^*. □

Note that a graph G having $\boldsymbol{d}_G(v) \leqslant \boldsymbol{f}(v)$ for each $v \in V(G)$ is strictly $\boldsymbol{f}$-degenerate if and only if in each of its connected components there is a vertex v of degree less than $\boldsymbol{f}(v)$. Hence, with the help of Claim 11 we can obtain Theorem 2′ along with the following statements.

Theorem 4′. *Let G be a connected graph, $\Delta(G) = \Delta \geqslant 3$, and $G \neq K_{\Delta+1}$.* Let $\boldsymbol{f}_1, \ldots, \boldsymbol{f}_s$ *be nonnegative-integer-valued functions on $V(G)$, $s \geqslant 2$, $\boldsymbol{f}_1(v) + \cdots + \boldsymbol{f}_s(v) \geqslant \Delta$ for each $v \in V(G)$, and suppose G does not contain a monoblock. Then G is $(\boldsymbol{f}_1, \ldots, \boldsymbol{f}_s)$-partitionable in such a way that $\boldsymbol{d}_{G_i}(v) \leqslant \boldsymbol{f}_i(v)$ whenever $1 \leqslant i \leqslant s$* and $v \in V(G_i)$.

Corollary 5′. *Let G be connected, not a complete graph and $\Delta(G) \leqslant st \geqslant 3$, where $s \geqslant 2$. Let to each vertex v a list $L(v)$ of admissible colours be assigned such that $|L(v)| \geqslant s$. Then a colour can be chosen from $L(v)$ for each v so that each colour class induces a strictly t-degenerate subgraph of maximum degree not greater than t.*

Theorem 8′. *Let G be a connected $\boldsymbol{F}$-graph and let $\boldsymbol{f}_1(v) + \cdots + \boldsymbol{f}_s(v) \geqslant \boldsymbol{d}_G(v)$ for every $v \in V(G)$. Then for every $v \in V(G)$, the graph G can be covered by disjoint induced subgraphs G_i such that $\boldsymbol{d}_{G_i}(v) \leqslant \boldsymbol{f}_i(v)$ whenever $1 \leqslant i \leqslant s$ and $v \in V(G_i)$ and each connected component of G_i contains a vertex z with $\boldsymbol{d}_{G_i}(z) < \boldsymbol{f}_i(z)$, if and only if G is not hard-constructible.*

Note that Observation 10 immediately implies the following result by Borodin and Kostochka [5] extending Gerencsér's [9] and Lovász' [13] results on covering graphs by subgraphs of bounded maximum degree.

Theorem 12 (Borodin and Kostochka 5, [Lemma 2']). *Let G be a graph, and suppose that $\boldsymbol{f}_1(v) + \cdots + \boldsymbol{f}_s(v) > \boldsymbol{d}_G(v)$ for every $v \in V(G)$. Then $V(G)$ can be covered by disjoint induced subgraphs G_i such that $\boldsymbol{d}_{G_i}(v) < \boldsymbol{f}_i(v)$ whenever $1 \leqslant i \leqslant s$ and $v \in V(G)$.*

In view of Theorems 8′ and 12, the following question seems interesting to us:

Question 13. *For which pairs $(G, \boldsymbol{F})$ such that $s \geqslant 2$ and $\boldsymbol{f}_1(v) + \cdots + \boldsymbol{f}_s(v) \geqslant \boldsymbol{d}_G(v)$ for every $v \in V(G)$ can the set $V(G)$ be covered by s induced subgraphs $G_1, \ldots, G_s$*

such that

(a) *whenever* $1 \leqslant i \leqslant s-1$, *each connected component of* G_i *contains a vertex* z *with* $\boldsymbol{d}_{G_i}(z) < \boldsymbol{f}_i(z)$ *and has* $\boldsymbol{d}_{G_i}(v) \leqslant \boldsymbol{f}_i(v)$ *for all other vertices*;

(b) $\boldsymbol{d}_{G_s}(v) < \boldsymbol{f}_s(v)$ *for all* $v \in V(G_s)$?

5. Graphs on surfaces

The *point-partition number* $\alpha_k(G)$ of a graph G is defined by Lick and White [12] as the minimum number of induced k-degenerate subgraphs which cover $V(G)$. In particular, $\alpha_0(G)$ and $\alpha_1(G)$ are the chromatic number and the point arboricity of G. For a closed surface $\mathscr{S}^N$ with Euler characteristic N, the point partition number $\alpha_k(S^N)$ is the maximum value of $\alpha_k(G)$ over all graphs G embeddable on $\mathscr{S}^N$. Recall that $H(N) = \lfloor 7 + \sqrt{49 - 24N}/2 \rfloor$ is the Heawood number of $\mathscr{S}^N$.

All $\alpha_k(S^N)$ were found by Lick and White [12], apart from the cases: $k = 0$ for the plane, which was the Four Colour Problem, and $k = 1$ and 2 for the Klein bottle $\mathscr{K}$. Borodin [2] proved $\alpha_1(\mathscr{K}) = 3$ and $\alpha_2(\mathscr{K}) = 2$, and extended Lick and White's result in the spirit of Theorem 2. Now we further extend this as follows:

Theorem 13. *Let* G *be a graph embedded on* $\mathscr{S}^N$ *other than the plane and let* $\boldsymbol{F} = (\boldsymbol{f}_1, \ldots, \boldsymbol{f}_s)$ *be a vector such that* $\boldsymbol{f}_1(v) + \cdots + \boldsymbol{f}_s(v) \geqslant H(N)$ *for all* $v \in V(G)$. *Then* G *is* $\boldsymbol{F}$*-partitionable. Moreover, if* G *is embeddable on the Klein bottle and* $\boldsymbol{f}_1(v) + \cdots + \boldsymbol{f}_s(v) \geqslant 6$ *for all* $v \in V(G)$, *then* G *is* $\boldsymbol{F}$*-partitionable unless* $(G, \boldsymbol{F})$ *has a subgraph which is a* 6*-regular monoblock.*

Proof. We use Claim 1, Theorem 2 and the facts that each graph embeddable on $\mathscr{S}^N$ other than the plane is strictly $H(N)$-degenerate and that, moreover, a graph on the Klein bottle is either 6-regular (and different from K_7 by Franklin's theorem [7]) or has a vertex of degree at most 5. □

Borodin [2] conjectured that every planar graph can be partitioned into two subgraphs that are strictly 3-degenerate and strictly 2-degenerate graphs respectively and also into two subgraphs that are strictly 4-degenerate and strictly 1-degenerate. The first of these conjectures was confirmed by Thomassen [15]. He also proved [16] the 5-choosability of plane graphs. For the general case of variable degeneracy we have the following.

Conjecture 14. Let G be a planar graph and let $\boldsymbol{F} = (\boldsymbol{f}_1, \ldots, \boldsymbol{f}_s)$, $s \geqslant 2$, be a vector with $\boldsymbol{f}_1(v) + \cdots + \boldsymbol{f}_s(v) = 5$ for each $v \in V(G)$ and such that for each i, $1 \leqslant i \leqslant s$, the subgraph induced by the vertices v having $\boldsymbol{f}_i(v) = 5$ is strictly 5-degenerate. Then G is $\boldsymbol{F}$-partitionable.

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