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On degrees of vertices in paradoxical trees

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Abstract

We present some necessary and sufficient conditions (in terms of degrees of vertices) for locally finite tree $T=(V,E)$ to be *paradoxical*, i.e. to have a partition $V=V_1\cup V_2$ and one-to-one mappings $f_i: V\to V_i$, $i=1,2$ such that the supremum of the distances between v and $f_i(v)$ over $v\in V$ and $i=1,2$ is finite.

Keywords: Locally finite trees; Vertex degrees

1. Introduction

A classical result of Banach and Tarski [1] states that the unit ball B in $\mathscr{R}^3$ admits a paradoxical decomposition $B=B_1\cup B_2$ such that B, B_1 and B_2 are pairwise congruent. The following notion was introduced in [2]: A metric space (M,d) is *paradoxical*, if there exists a decomposition $M=M_1\cup M_2$ such that M, M_1 and M_2 are equivalent with respect to 'wobbling bijections':

$f: X\to Y$, where $X, Y\subseteq M$, is a *wobbling bijection* if

$$\sup_{x\in X} d(x,f(x))<\infty.$$

It was shown in [2] that a discrete countable metric space is paradoxical if and only if it has at least exponential growth rate. M has *at least exponential growth rate* if there exists r (*the doubling radius*) such that for every finite subset X of M the cardinality of the r-neighbourhood of X is at least twice the cardinality of X.

Particularly interesting cases are countable graphs G with the usual metric given by shortest paths.

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For a subset M of vertices of a graph G, let $N_G^k(M)$ denote the set of vertices in G at a distance at most k from M, and $N_G(M) = N_G^1(M)$. In clear cases we shall omit the subscript.

Deuber et al. [2] proved that a locally finite graph $G=(V,E)$ is paradoxical if and only if there exists a positive integer d_1 such that for each finite $M \subset V$,

$$|N^{d_1}(M)| \geqslant 2|M|. \tag{1}$$

They also observed that a locally finite tree $G=(V,E)$ without pendant vertices is paradoxical if and only if there exists a positive integer d_2 such that

any path induced by the vertices of degree 2 in G has the length at most d_2. (2)

Then Fon-Der-Flaass [3] has found a characterization of paradoxical trees in a wider class. For a vertex v of a locally finite tree $G=(V,E)$, let *finitary valence* (respectively, *infinitary valence*) of v be the number of finite (respectively, infinite) components of $G-v$. Fon-Der-Flaass proved that a locally finite tree $G=(V,E)$ with bounded finitary valence is paradoxical if and only if there exists a positive integer d_3 such that

the size of any connected subgraph induced by the vertices

of infinitary valence at most 2 in G is at most d_3. (3)

He also showed that any locally finite graph can be made paradoxical by adding some pendant vertices and can also be made non-paradoxical by adding some pendant vertices.

In this note we give some necessary and sufficient conditions for locally finite trees to be paradoxical. These conditions involve degrees of vertices and reflect the fact that 'bad' for paradoxicality sets of vertices contain subsets whose sums of degrees are big and those sums in their neighbourhoods (in broad sense) are not so big.

First, we introduce some notation. For a finite subset of vertices M of a locally finite graph G, let $G(M)$ denote the subgraph of G induced by M, $S(M)=\sum_{v\in M}\deg_G(v)$ and $\tilde{S}(M)=\sum_{v\in M}\deg_{G(M)}(v)$. Let also $M'=\{v\in M \mid \deg_G(v)\geqslant 2\}$ and $M''=\{v\in M \mid \deg_{G(M)}(v)\geqslant 2\}$. By $E_H(X)$ we denote the set of edges in the subgraph of the graph H induced by the vertex-set X.

Theorem 1. *Let $G=(V,E)$ be a locally finite tree and α be a real number*, $0\leqslant\alpha<2$. *Then the following conditions are equivalent*:

(i) *G is paradoxical, i.e., there exists a positive integer d_1 such that for each finite $M\subset V$,* (1) *holds*;

(ii) *there exists a positive integer d_4 such that for each finite $M\subset V$,*

$$\tilde{S}(N^{d_4}(M)) - \alpha|N^{d_4}(M)| \geqslant 2(S(M)-\alpha|M|); \tag{4}$$

(iii) *there exists a positive integer d_5 such that for each finite $M\subset V$,*

$$S(N^{d_5}(M)) - \alpha|N^{d_5}(M)| \geqslant 2(S(M)-\alpha|M|). \tag{5}$$

Note that for $\alpha=2$ the statement is already false. In particular, the expression $\tilde{S}(N^{d_4}(M)) - 2|N^{d_4}(M)|$ is negative for any finite subset M of any locally finite tree and any positive integer d_4, while the expression $S(M) - 2|M|$ might be positive for some M. We shall see in Section 2 that for $\alpha = 2$, condition (iii) also can be violated in paradoxical trees. To express the importance of vertices of degree 2 we give yet another characterization of paradoxical trees.

Theorem 2. *Let $G=(V,E)$ be a locally finite tree. Then the following conditions are equivalent*:

(i) *G is paradoxical, i.e., there exists a positive integer d_1 such that for each finite $M \subset V$,* (1) *holds*;

(ii) *there exists a positive integer d_6 such that for each finite $M \subset V$,*

$$S((N^{d_6}(M))') - 2|(N^{d_6}(M))'| > 2(S(M') - 2|M'|); \tag{6}$$

(iii) *there exists a positive integer d_7 such that for each finite $M \subset V$,*

$$\tilde{S}((N^{d_7}(M))'') - 2|(N^{d_7}(M))''| > 2(S(M') - 2|M'|). \tag{7}$$

Note that strict inequality in (6) and (7) cannot be replaced by the non-strict one, because the latter is fulfilled for infinite paths. Note also that inequality (6) cannot be replaced by the inequality

$$S(N^{d_6}(M)) - 3|N^{d_6}(M)| > 2(S(M) - 3|M|) \tag{8}$$

even for trees without vertices of degree two. Namely, there are many paradoxical trees such that for each d_6 they contain sets M violating (8).

2. Proof of Theorem 1

From now on, $G=(V,E)$ is a locally finite tree.

Lemma 1. *For each finite $M \subset V$,*

(a) $|N(M)| > 0.5S(M)$,

(b) $|N(M)| > S(M) - |M|$.

Proof. As G has no cycles, $S(M) \leqslant \tilde{S}(N(M)) = 2|E_G(N(M))| < 2|N(M)|$. This proves (a).

Now, the number of edges incident to M is $S(M) - |E_G(M)| > S(M) - |M|$. But this number is less than $|N(M)|$. This proves (b). □

(i) ⇒ (ii). Denote $x = \lceil \log_2 6/(2-\alpha) \rceil$. Then

$$|N^{xd_1+1}(M)| = |N^{xd_1}(N(M))| \geqslant 2^x |N(M)| \geqslant \frac{6}{2-\alpha}|N(M)|.$$

Since the number of components of $G(N^{xd_1+1}(M))$ is at most $|M|$,

$$\tilde{S}(N^{xd_1+1}(M)) = 2|E_G(N^{xd_1+1}(M))| \geqslant 2(|N^{xd_1+1}(M)| - |M|).$$

Thus, taking into account Lemma 1, we obtain

$$\begin{aligned}\tilde{S}(N^{xd_1+1}(M)) - \alpha|N^{xd_1+1}(M)| &\geqslant (2-\alpha)|N^{xd_1+1}(M)| - 2|M| \\ &\geqslant 6|N(M)| - 2|M| > 4|N(M)| > 2S(M).\end{aligned}$$

(ii) $\Rightarrow$ (iii). Obvious.

(iii) $\Rightarrow$ (i). Denote $y = \lceil \log_2 14/(2-\alpha) \rceil$. We shall show that

$$|N^{2yd_5+2}(M)| \geqslant \tfrac{7}{6}|M| \tag{9}$$

which implies $|N^{10yd_5+10}(M)| \geqslant (\frac{7}{6})^5|M| > 2|M|$.

If $|N(M)| \geqslant \frac{7}{6}|M|$, there is nothing to prove. So, we may assume

$$|N(M)| < \tfrac{7}{6}|M|. \tag{10}$$

Case 1: $|M'| \geqslant |M|/6$. Since $S(M') - \alpha|M'| \geqslant (2-\alpha)|M'|$, we have

$$\begin{aligned}|N^{yd_5+1}(M)| &\geqslant |N^{yd_5+1}(M')| \geqslant 0.5S(N^{yd_5}(M')) \\ &\geqslant 0.5 \cdot 2^y(S(M') - \alpha|M'|) \geqslant \frac{7}{2-\alpha}[(2-\alpha)|M'|] = 7|M'| \geqslant \frac{7}{6}|M|.\end{aligned}$$

Case 2: $|M'| < |M|/6$. Let $M_1 = M \backslash M' = \{v \in M \mid \deg_G(v) = 1\}$ and $M_0 = N(M) \backslash M_1$. Then $|M_1| > 5|M|/6$ and, by (10),

$$|M_0| = |N(M) \backslash M| + |M'| < |M|/6 + |M|/6 = |M|/3.$$

Since $M_1 \subseteq N(M_0)$, we have

$$S(M_0) - \alpha|M_0| > |M_1| - \alpha|M|/3 \geqslant |M|/6.$$

Thus,

$$\begin{aligned}|N^{2yd_5+2}(M)| &\geqslant |N^{2yd_5+1}(M_0)| \\ &\geqslant 0.5S(N^{2yd_5}(M_0)) \\ &\geqslant 0.5 \cdot 2^{2y}(S(M_0) - \alpha|M_0|) \\ &\geqslant 0.5\left(\frac{14}{2-\alpha}\right)^2 \frac{|M|}{6} \\ &\geqslant \frac{14^2|M|}{2 \cdot 2^2} > 2|M|.\end{aligned}$$

The theorem is proved.

An example showing that $\alpha = 2$ does not work in Theorem 1 is as follows. Let the tree T consist of the path $(v_1, v_2, \ldots)$ and for each $k = 1, 2, \ldots$ the vertex v_{5k} be adjacent to 5^k pendant vertices. Then T is paradoxical, since for each finite set M of vertices, on distance at most 5 from v_i with maximum i which is in $N(M)$ there are more pendant vertices then $|M|$. On the other hand, for every $d\leqslant 1$, let $M_d=\{v_1,\dots,v_{4d+2}\}$. Then $S(M_d)-2|M_d|\geqslant 4$ and, since exactly one edge connects $N^d(M_d)$ with the rest of T, $S(N^d(M_d))-2|N^d(M_d)|=-1$.

3. Proof of Theorem 2

Lemma 2. *Let $T=(V,E)$ be a finite tree without vertices of degree 2. Then $S(V')-2|V'|\geqslant 0.5(|E|-1)$.*

Proof. For trees with at most four vertices, the statement is obvious. Let $T=(V,E)$ be a smallest counterexample and v be a pendant vertex in T. Delete v and if its neighbour w becomes a vertex of degree 2, contract one edge incident with w. The resulting tree T_0 has at least $|V|-2$ vertices, none of which has degree 2. Clearly, $S(V'(T_0))-2|V'(T_0)|\leqslant S(V')-2|V'|-1$. By the minimality of T,

$$S(V'(T_0))-2|V'(T_0)|\geqslant 0.5(|E(T_0)|-1)\geqslant 0.5|V|-2.$$

This gives a contradiction. □

Lemma 3. *Let $T=(V,E)$ be a finite forest with at most m components and such that the length of any path induced by the vertices of degree 2 in T is at most d. Then $S(V')-2|V'|\geqslant 0.5(|E|/(d+2)-m)>|V|/(2d+4)-m$.*

Proof. First, let $m=1$. Construct $\tilde{T}$ by replacing each path connecting vertices of degree not equal to 2 whose internal vertices have degree 2 with an edge. Then $\tilde{T}$ has no vertices of degree 2 and $|E(\tilde{T})|\geqslant|E|/(d+2)$. By Lemma 2, $S(V'(\tilde{T}))-2|V'(\tilde{T})|\geqslant 0.5(|E|/(d+2)-1)$. But $S(V'(\tilde{T}))-2|V'(\tilde{T})|=S(V')-2|V'|$.

If $m>1$, we apply the lemma to each component and after summing obtain that $S(V')-2|V'|\geqslant|E|/(2(d+2))-m/2$. □

Lemma 4. *Let $T=(V,E)$ be a finite forest with at most m components and having a subset R of vertices such that any path of length $d+1$ induced by the vertices of degree 2 in T contains a vertex in R. Then $S(V')-2|V'|>|V|/(2d+4)-m-|R|$.*

Proof. If $R=\emptyset$, we are done by Lemma 3. Let the lemma be valid for all forests with $|R|<r$, and T be a forest with $|R|=r$. Let $v\in R$. By definition, $\deg(v)=2$. Split v into two vertices of degree one. We obtain a forest with one more vertex and one more component. On the other hand, the size of R decreases. By induction, we are done. □

Lemma 5. *Let $G=(V,E)$ be a locally finite tree without any path of length $d+1$ induced by the vertices of degree 2 in G. Let M be any finite subset of V and d be arbitrary positive integer. Denote by T the forest induced by $N^d(M)$. Then the sum* *of the number of components of T and the minimum number of vertices covering all paths of length $d+1$ induced by the vertices of degree 2 in T is at most $|M|$.*

Proof. Clearly, the number of components of $T(N^{d-1}(M))$ is at most $|M|$. Any vertex having degree 2 in T but not in G must be in $N^d(M)\backslash N^{d-1}(M)$. And any such vertex connects two distinct components of $T(N^{d-1}(M))$. $\square$

(i)$\Rightarrow$(iii). Assume that G is paradoxical. As it was observed in [2], then there exists d_2 such that (2) holds. We may assume that $d_2 > 8$. Consider an arbitrary finite $M \subset V$. Denote $\tilde{M} = N^{d_1 d_2+1}(M)$. Since G is paradoxical and by Lemma 1,

$$|\tilde{M}| \geqslant 2^{d_2}|N(M)| > 2^{d_2}(S(M) - |M|).$$

By Lemmas 4 and 5, for $d_2 > 8$ we have

$$\begin{aligned}\tilde{S}(\tilde{M}) - 2|\tilde{M}| &> |\tilde{M}|/(2d_2+4) - |M| \\ &> \frac{2^{d_2} - 2d_2 - 4}{2d_2+4}(S(M) - |M|) \\ &> 2(S(M') - |M'|).\end{aligned}$$

(iii) $\Rightarrow$ (ii). To see the validity of this implication, it is enough to observe that for each finite $L \subset V$, $\tilde{S}(L'') - 2|L''| \leqslant S(L') - 2|L'|$.

(ii) $\Rightarrow$ (i). Let a positive integer d_6 be such that for each finite $M \subset V'$, (6) holds. Then for $d_2 = 2d_6 + 1$, condition (2) holds. Indeed, if G contains a path P of length $2d_6 + 2$ with the central vertex, say, v, then (6) does not hold for $M = \{v\}$.

Let $z = 2 + \lceil \log_2 8d_6 + 12 \rceil$. We shall show that

$$|N^{2zd_6+2}(M)| \geqslant \left(1 + \frac{1}{16d_6+24}\right)|M|. \tag{11}$$

This will imply that

$$|N^{(2zd_6+2)(16d_6+24)}(M)| \geqslant \left(1 + \frac{1}{16d_6+24}\right)^{16d_6+24}|M| > 2|M|,$$

i.e., G satisfies (1) with $d_1 = (2zd_6+2)(16d_6+24)$.

If $|N(M)| \geqslant (1 + 1/(16d_6+24))|M|$, there is nothing to prove. So, we may assume

$$|N(M)\backslash M| < \frac{1}{16d_6+24}|M|. \tag{12}$$

Then $G(N(M))$ has at most $|M|/(16d_6+24)$ components and $N(M)\backslash M$ covers all long paths of vertices of degree 2 in $G(N(M))$. By Lemma 4,

$$S((N(M))') - 2|(N(M))'| > |M|/(4d_6+6) - |M|/(8d_6+12) = |M|/(8d_6+12).$$

Finally,

$$
\begin{aligned}
|N^{zd_6+2}(M)| &= |N^{zd_6+1}(N(M))| \\
&\geqslant 0.5S(N^{zd_6}(N(M))) \\
&\geqslant 0.5S((N^{zd_6}(N(M)))') \\
&\geqslant 0.5 \cdot 2^z(S(N(M)') - 2|N(M)'|) \\
&\geqslant 0.5 \cdot 4(8d_6 + 12)|M|/(8d_6 + 12) = 2|M|.
\end{aligned}
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