ELSEVIER

Discrete Mathematics 213 (2000) 153–161

DISCRETE MATHEMATICS

www.elsevier.com/locate/disc

On the Hajós number of graphs

G. Gutin[a,b,1], A.V. Kostochka[c,2], B. Toft[a,*,3]

[a] Institute for Mathematik og Datalogi, Odense Universitet, Compusvej 55, 5230 Odense M, Denmark

[b] Brunel University of West London, Uxbridge, Middlesex, UK

[c] Novosibirsk State University, Novosibirsk, Russia

Received 10 February 1997; revised 14 November 1997; accepted 14 February 1998

Abstract

A graph G is said to have property P_m if it contains no subdivision of K_{m+1} and no subdivision of $K_{\lceil m/2\rceil+1,\lfloor m/2\rfloor+1}$. Chartrand et al. (J. Combin Theory 10 (1971) 12–41) (see also Problem 6.3 in Jensen and Toft (Graph Coloring Problems, Wiley, New York, 1995) conjectured that the set of vertices (respectively, edges) of any graph with property P_m can be partitioned into $m-n+1$ subsets such that each of these subsets induces a graph with property P_n, provided $m \geqslant n \geqslant 1$ (respectively, $m \geqslant n \geqslant 2$). We prove that both conjectures fail when $m > cn^2$ for some positive constant c. In fact, we prove that under the condition $m > cn^2$, there exists a graph G with property P_m such that in every colouring of its vertices or edges with m colours there is a monochromatic subgraph H with Hajós number $h(H) > n$, that is, with a subdivision of K_{n+1}. In addition, we prove bounds of Nordhaus–Gaddum type for the Hajós number.

1. Introduction

For a graph G, let *the Hajós number* $h(G)$ denote the maximum k such that G contains a subdivision of K_k.

For a positive integer m, Chartrand et al. [3] denote by P_m the property of a graph G to have $h(G) \leqslant m$ and not to contain a subdivision of the complete bipartite graph $K_{\lceil m/2\rceil+1,\lfloor m/2\rfloor+1}$ either. Let $f_1(m,n)$ (respectively, $f_2(m,n)$) denote the minimum k

* Corresponding author.

E-mail addresses: z.g.gutin@brunel.ac.uk (G. Gutin), sasha@math.nsc.ru (A.V. Kostochka), btoft@imada.sdu.dk (B. Toft)

[1] This work was partially supported by the grant for discrete mathematics of the Danish Natural Science Research Council.

[2] This author finished his part of work during his stay at SFB343 'Diskrete Strukturen in der Mathematik' of Bielefeld University. His work was also partially supported by the grant 96-01-01614 of the Russian Foundation for Fundamental Research.

[3] This work was partially supported by the grant for discrete mathematics of the Danish Natural Science Research Council.

PII: S0012-365X(99)00175-2

such that for any graph G with property P_m, there exists a partition of its vertices (respectively, edges) into k sets such that each set induces a subgraph with property P_n. The conjecture in [3] (see also Problem 6.3 in [11]) that $f_1(m,n) \leqslant m-n+1$ for each $m \geqslant n \geqslant 1$ is sometimes referred to as the (m,n)-conjecture, and the conjecture that $f_2(m,n) \leqslant m-n+1$ for each $m \geqslant n \geqslant 2$ as the $[m,n]$-conjecture. Chartrand et al. proved the (m,n)-conjecture for all n and m satisfying $1 \leqslant n \leqslant m \leqslant 4$ except for the case $(4,1)$ which is the 4CT (the Four Colour Theorem). They also proved the $[m,n]$-conjecture for all n and m satisfying $2 \leqslant n \leqslant m \leqslant 4$, except the case $[4,3]$. The $[4,3]$-conjecture was proved by Heath [10]. The (m,n)- and $[m,n]$-conjectures were mentioned in the excellent survey [20]. Woodall [20] remarked that perhaps the corresponding conjectures in terms of contractions, rather than subdivisions, are more likely to be true.

Indeed, Jørgensen [12], and Hanson and Toft [9] proved that for each n the (m,n)-conjecture fails for almost all graphs. They used the ideas of Erdős and Fajtlowicz [5] who showed that for almost all graphs on v vertices, the Hajós number is at most $(2+\varepsilon)\sqrt{v}$. The first aim of this paper is to show that both the (m,n)-conjecture and the $[m,n]$-conjecture are not true when $m > cn^2$, and that also even the contraction version of the $[m,n]$-conjecture is false for m large enough with respect to n.

Theorem 1. *There exists a constant c such that if $m > cn^2$, then there exists a graph G with property P_m such that for all partitions of its vertices or of its edges into m colour classes, there is a monochromatic subgraph H with Hajós number $h(H) > n$.*

Theorem 2. *For each $n > 1$, there exists an m_0 such that for every $m > m_0$, there exists a graph G with property P_m such that for all partitions of its edges into m colour classes at least one monochromatic subgraph H has K_{n+1} as a minor.* (*It follows that the $[m,n]$-conjecture is false, even in terms of contractions.*)

The second aim of the paper is to find bounds of Nordhaus–Gaddum type for the Hajós number. Zelinka [21] conjectured that $\eta(G)+\eta(\bar{G}) \leqslant n+1$ for each graph G on n vertices, where $\eta(G)$ is the Hadwiger number of G. This is not true (see [15]), but for the Hajós number instead of the Hadwiger number the bound is true.

Proposition 3. *For each positive integer n,*

$$\max\{h(G)+h(\bar{G}) \mid |V(G)|=n\} = n+1, \tag{1}$$

$$\max\{h(G)\times h(\bar{G}) \mid |V(G)|=n\} = \left\lfloor \left(\frac{n+1}{2}\right)^2 \right\rfloor. \tag{2}$$

Then we discuss $\min\{h(G)+h(\bar{G}) \mid |V(G)|=n\}$ whose order of magnitude is $\sqrt{n}$, as it follows from the results of Bollobás and Thomason [2] and Komlós and Szemerédi [13,14]. The exact value is likely to have a complicated formula. We shall give a simple proof that

Theorem 4. *For all n*

$$\min\{h(G)+h(\bar{G}) \mid |V(G)|=n\} > 0.4\sqrt{n}. \tag{3}$$

We suspect that $h(G)\times h(\bar{G})\geqslant n$ for all graphs G on n vertices, but have no proof of this in general. Let T be the set of positive integers t such that any graph G with $h(G)=t$ and n vertices has at most $t^2n/2$ edges. Thomassen's result in [19] implies that $4\in T$. Komlós and Szemerédi [14] proved that for some t_0 every $t>t_0$ is in T.

Theorem 5. *Let G be any graph with $h(G)=t$ and n vertices. If $t\in T$ and n is large in comparison with t or if $t\leqslant 3$, then $h(G)\times h(\bar{G})\geqslant n$.*

Moreover, we can give a better bound when $h(G)\leqslant 2$ (every such graph G is a forest):

Theorem 6. *Let G be a graph on n vertices with $h(G)\leqslant 2$. Then $h(\bar{G})\geqslant\lfloor(2n+1)/3\rfloor$ with two exceptions*:

(a) *G is the path P_4 on four vertices*;

(b) *G is the tree T_7 on 7 vertices, obtained from $K_{1,3}$ by subdividing each of its edges by a vertex.*

The bound in Theorem 6 is best possible.

2. Bounds of Nordhaus–Gaddum type

Proof of Proposition 3. The only part of Proposition 3 needed to be proved is the statement that for each graph G on n vertices,

$$h(G)+h(\bar{G})\leqslant n+1. \tag{4}$$

Indeed, the inequality $h(G)\times h(\bar{G})\leqslant\lfloor((n+1)/2)^2\rfloor$ follows from (4) and the well known inequality on arithmetic and geometric means. Both inequalities are equalities for G being the disjoint union of $K_{\lceil(n+1)/2\rceil}$ and $\bar{K}_{\lfloor(n-1)/2\rfloor}$.

Suppose that for some r and b, K_n contains edge disjoint subdivisions R and B of K_r and K_b, respectively. Let VR and VB denote the set of branching vertices in R and B, respectively, $VRB=VR\cap VB$. If $VRB=\emptyset$, then $r+b\leqslant n$. Otherwise, let $x\in VRB$. Observe that $r-1+b-1=d_R(x)+d_B(x)\leqslant n-1$. Therefore, $r+b\leqslant n+1$. □

The result of Erdős and Fajtlowicz [5] mentioned in the introduction implies that $\min\{h(G)+h(\bar{G}) \mid |V(G)|=n\}\leqslant 5\sqrt{n}$ for large n. On the other hand, Komlós and Szemerédi [13,14] proved that $h(G)\geqslant\sqrt{2e/n}$ when $2e/n$ is large enough ($e=|E(G)|$). Either G or $\bar{G}$ has $e\geqslant n(n-1)/4$, hence for n large enough either $h(G)$ or $h(\bar{G})$ is at least $\sqrt{(n-1)/2}$. Thus the order of magnitude of $\min\{h(G)+h(\bar{G}) \mid |V(G)|=n\}$ is $\sqrt{n}$. We shall not use the results of Komlós and Szemerédi in this connection. Instead, with the help of the following simple lemma, which will also be used in the next section, we shall see that in fact for *all* n, $\min\{h(G)+h(\bar{G}) \mid |V(G)|=n\}$ is not much less than $\sqrt{n}$.

Lemma 1. *Let* $|V(G)|=n$ *and* $|E(G)|\geqslant 0.24(n-1)n$. *Then* $h(G)\geqslant 0.4\sqrt{n}-1.5$.

Proof. Let $A=\{v\in V(G) \mid \deg_G(v)\geqslant 0.43(n-1)\}$ and $a=|A|$. Then $0.43(n-1)(n-a)+(n-1)a>0.48(n-1)n$, and hence $a>n/12$. Let F be the graph with the vertex-set A such that a pair uv is in $E(F)$ iff either uv is in $E(G)$ or the number of paths of length two in G connecting u and v is at least $n/11$. Assume that $A'=\{w_1,w_2,w_3\}$ is an independent set in F. Then A' is an independent set in G, and $|N_G(w_i)\cap N_G(w_j)|\leqslant (n-1)/11$ for $j\neq i$. Since $A'\subset A$, $|N_G(w_1)\cup N_G(w_2)\cup N_G(w_3)|\geqslant 3\cdot 0.43(n-1)-3\cdot(n-1)/11>n-1.5>|V(G)|-|A'|$, a contradiction. Thus, the independence number of F is at most two.

According to an old bound of Erdős and Szekeres [6], F contains some clique Q of size q, where q is the largest integer such that

$$\binom{q+1}{2}\leqslant n/12. \tag{5}$$

We find in G a subdivision of K_q with $V(Q)$ as the set of branching vertices in the following simple way. First we use all the edges of $G(Q)$. Then for each x and y in $V(Q)$ with no edge xy, we assign to the pair xy a path of length two. Assume that we cannot do it for some x and y in $V(Q)$. Since $xy\in E(F)$, there are $m=\lceil n/11\rceil$ paths of length 2 connecting x and y. If, say, r of these paths have vertices in Q as intermediate vertices, then $G(Q)$ has at least $2r$ edges and, since each of the remaining $m-r$ vertices also are already used in some path of length 2, we have $2r+(m-r)\leqslant \binom{q}{2}-1$. Taking (5) into account, we get $n/11<n/12$, a contradiction. Thus, $h(G)\geqslant q$.

Since q is the largest integer satisfying (5), we have $\binom{q+2}{2}>n/12$, and so, $(h(G)+1)(h(G)+2)>n/6$, which implies the statement of the lemma. $\square$

Lemma 1 immediately implies Theorem 4. Indeed, w.l.o.g., we may assume that $|E(G)\geqslant|E(\bar{G})|$. Theorem 4 holds for $G=K_n$, thus we may assume that $\bar{G}$ has an edge. Therefore, $h(G)\geqslant 0.4\sqrt{n}-1.5$ and $h(\bar{G})\geqslant 2$.

Proof of Theorem 5. If $h(G)\leqslant 3$, then G is $h(G)$-colourable [8,4]. In such a colouring one of the colour classes has size at least $|V(G)|/h(G)$. Then $\bar{G}$ contains a complete graph of this size, and hence $h(G)\times h(\bar{G})\geqslant|V(G)|$.

Let $t\in T$, $t\geqslant 4$ and

$$n>4t^3. \tag{6}$$

For a graph $H=(V,E)$ the *edge-density* $\mathrm{ed}(H)$ is defined as $2|E|/|V|\cdot(|V|-1)$. Note that if $x\in V$ and $d(x)\geqslant 2|E|/|V|$ then $\mathrm{ed}(H-x)\leqslant\mathrm{ed}(H)$. By the definition of T,

$$\mathrm{ed}(G)\leqslant t^2/(n-1).$$

Observe also that, for each $k \leqslant |V(H)|$, the average edge-density over all induced subgraphs of H with k vertices equals $\mathrm{ed}(H)$. Thus there is a k-subset W such that $\mathrm{ed}(H[W]) \leqslant \mathrm{ed}(H)$.

Denote $f = \left\lceil \sqrt{3nt/10} \right\rceil$. By (6), $f > t^2$, which by the definition of T, is at least the average degree of G. Thus, successively removing from G vertices of degree at least f, we come up with a graph $G - X$ of maximum degree at most $f - 1$ and such that $\mathrm{ed}(G - X) \leqslant \mathrm{ed}(G)$. Since $f \cdot |X| \leqslant |E(G)|$, the number $|X|$ of deleted vertices is at most $t^2 n/(2f) \leqslant \sqrt{5t^3 n/6}$.

In $G - X$, let W be a vertex-subset of size $\lceil n/t \rceil$ with minimum edge-density. By the above remarks,

$$\mathrm{ed}(G[W]) \leqslant \mathrm{ed}(G - X) \leqslant \mathrm{ed}(G) \leqslant t^2/(n-1).$$

Since $\lceil n/t \rceil \leqslant (n-1)/t + 1$, it follows that

$$\begin{aligned} e(G[W]) &\leqslant \frac{t^2}{n-1} 0.5 \lceil n/t \rceil (\lceil n/t \rceil - 1) \\ &\leqslant 0.5t \lceil n/t \rceil \leqslant 0.5(n + t - 1). \end{aligned}$$

For each edge xy of $G[W]$ we want to find a vertex z of $G - X - W$ such that xz and yz are not edges in G. Furthermore, the vertices z corresponding to different xy should be different. If this can be accomplished, then there is a subdivision of the complete graph on $\lceil n/t \rceil$ vertices in $\bar{G}$ with the vertices of W as branch vertices. Hence $h(G) \times h(\bar{G}) \geqslant n$.

If the number of possible z for each xy is at least $0.5(n + t - 1)$, then this can indeed be accomplished. Remembering that each vertex in $G - X$ has degree at most $f - 1 < \sqrt{3nt/10}$ we need only to have the following inequality fulfilled:

$$n - |X| - |W| - 2(f - 1) \geqslant 0.5(n + t - 1).$$

But

$$n - |X| - |W| - 2(f - 1) \geqslant n - \sqrt{5t^3 n/6} - \lceil n/t \rceil - 2\sqrt{3nt/10},$$

which exceeds $0.5(n + t - 1)$ provided $t \geqslant 4$.

Indeed,

$$\begin{aligned} n - |X| - |W| - 2(f - 1) &\geqslant n - \sqrt{5t^3 n/6} - n/t - 1 - 2\sqrt{3nt/10} \\ &> 0.5(n + t - 1) \end{aligned}$$

as it is equivalent to

$$\left(\frac{1}{2} - \frac{1}{t}\right) n + \left(t\sqrt{\frac{5t}{6}} - \sqrt{\frac{6t}{5}}\right)\sqrt{n} - \frac{1+t}{2} > 0,$$

whose left-hand side is a quadratic polynomial of $\sqrt{n}$ which is always positive since its first coefficient and discriminant are positive ($t \geqslant 4$). □

To support further our conjecture on the minimum possible product of the Hajós number of a graph and its complement, we find the minimum possible Hajós number of the complement of a forest T (i.e., graph T with $h(T)\leqslant 2$) on n vertices. It is straightforward to check that for the exceptional graphs in the statement of Theorem 6, we have $h(\bar{P}_4)=h(P_4)=2$, $h(\bar{T}_7)=4$. Before proving the theorem, we show that in all other cases its bound is attained on paths.

Lemma 2. *For a path P_n on n vertices, we have*

$$h(\bar{P}_n)\leqslant \left\lfloor \frac{2n+1}{3} \right\rfloor. \tag{7}$$

Proof. Let $P=v_1,\dots,v_n$ be a path, and M be the set of branching vertices in a subdivision of K_m in $\bar{P}$. For each edge xy in $P[M]$, we need a path in $\bar{P}$ connecting x and y, and containing a vertex in $V(P)\backslash M$. Thus,

$$|E(P[M])|\leqslant n-m. \tag{8}$$

Among sets M of cardinality m with the minimum possible $|E(P[M])|$, consider an M' with the minimum sum of numbers of its elements. Observe that M' possesses the following properties:

(a) if $v_{i+1}\notin M'$, $v_{i+2}\in M'$, then $v_i\in M'$ (otherwise we move v_{i+2} to v_{i+1});
(b) if $v_{i+1}\in M'$, $v_{i+2}\in M'$, then $v_i\in M'$ (otherwise we move v_{i+1} to v_i):
(c) $v_1\in M'$.

It follows that, for some k, $M'=\{v_1,\dots,v_k,v_{k+2},v_{k+4},\dots\}$. Thus, $k=|E(P[M'])|+1$, and

$$m\leqslant (n+k)/2. \tag{9}$$

Substituting (8) into (9), we get $m\leqslant(2n+1)/3$. □

Proof of Theorem 6. We use induction on n. For $n\leqslant 4$, the statement is easy (including exception (a)).

Let G be a counterexample with the smallest number n ($n\geqslant 5$) of vertices. Then G possesses the following properties.

(i) *G is connected*: Assume first that G has an isolated vertex v. By the minimality of G, $h(\bar{G}-v)\geqslant\lfloor 2(n-1)/3\rfloor$ (even if $G-v$ is P_4 or T_7). Hence $h(\bar{G})\geqslant 1+\lfloor 2(n-1)/3\rfloor=\lfloor(2n+1)/3\rfloor$, a contradiction. If G is not connected and has no isolated vertices, choose pendant vertices, say x and y, in distinct components of G. We may assume that the neighbours of x and y are a and b, respectively, and that the component containing x is of maximum order in G. Let $H=G-\{x,a,y\}$. By the minimality of G, $h(\bar{H})\geqslant\lfloor(2(n-3)+1)/3\rfloor$. Now we prove that $h(\bar{G})\geqslant 2+h(\bar{H})$, which easily implies $h(\bar{G})\geqslant\lfloor(2n+1)/3\rfloor$. Consider a subdivision F of $K_{h(\bar{H})}$ in $\bar{H}$. Now, adding branch vertices x and y, and the path yab if b is a branch vertex in F, we obtain a subdivision of $K_{2+h(\bar{H})}$ in $\bar{G}$.

(ii) *If $n\neq 7,10$, then no two pendant vertices in G share a neighbour*: Indeed, if pendant vertices x and y both are adjacent to a vertex a, the graph $H=G-\{x,a,y\}$ is neither P_4 nor T_7. So, $h(\bar{H})\geqslant\lfloor(2(n-3)+1)/3\rfloor$, and, since x and y are adjacent to all vertices in $\bar{G}-a$, we have $h(\bar{G})\geqslant 2+h(\bar{H})$.

(iii) *The diameter of G is at least four*: Otherwise, G has at least $n-2$ pendant vertices and so $h(\bar{G}) \geqslant n-2$.

Choose two pendant vertices x and y on distance at least four. Let their neighbours be a and b, respectively. We construct H as we did in the proof of (i). If $h(\bar{H}) \geqslant \lfloor (2(n-3)+1)/3 \rfloor$, then we are done as in (i). Thus the only situations we have to deal with are that $H = P_4$ or $H = T_7$. Let $H' = G - \{x, b, y\}$. Because of the symmetry between H and H', we assume further that H' also is either P_4 or T_7. Since G has no cycles, b and a are pendant vertices in H and H', respectively.

Case 1: $H = H' = P_4$. This is possible only if G is either the path P_7 (and we know that $h(\bar{P}_7) = 5$) or T_7.

Case 2: $H = H' = T_7$. Then G is the tree obtained from two disjoint copies of the path P_5 by joining their central vertices with an edge, and b and a are vertices of degree two on distance two in this tree. Taking instead of y a pendant vertex z of distance five from x, we obtain that the graph $G - \{x, a, z\}$ is not isomorphic to T_7. □

3. On subdivisions of graphs

Jørgensen [12], and Hanson and Toft [9] observed the following fact:

Lemma 3. *Almost all graphs on v vertices have property $P_{3\lceil\sqrt{v}\rceil}$.*

The next fact is well known and follows from standard calculations (cf. e.g. [1]).

Lemma 4. *Almost all graphs on v vertices have the property that for every $k \geqslant (\log v)^2$ each subgraph on k vertices has at least $0.24k(k-1)$ edges.*

Proof of Theorem 1. We may assume that $n > 12$ since for smaller n the theorem would follow from the case $n{=}13$. We choose $c > 100$ so that for each $v > c$, according to Lemmas 3 and 4, there exists a graph G_v on v vertices satisfying $P_{3\lceil\sqrt{v}\rceil}$ and such that every subgraph of G_v with $k \geqslant (\log v)^2$ vertices has at least $0.24k(k-1)$ edges.

Let $m > cn^2$ and $v = \lfloor (m/3)^2 \rfloor$. Define $k = \max\{\lceil (\log v)^2 \rceil, 9n^2\}$. Consider the graph G_v. By the definition, it has the property P_m and each of its subgraphs on k vertices has at least $0.24k(k-1)$ edges. We show that for each partition $\{V_1, \ldots, V_s\}$ of $V(G_v)$ into $s = \lfloor v/k \rfloor$ parts, at least one of V_i-s induces a subgraph with the Hajós number at least $n+1$. In a partition $\{V_1, \ldots, V_s\}$ of $V(G_v)$, at least one set, say V_1, has cardinality at least k. Let G' be the subgraph of G_v induced by some subset of V_1 of cardinality k. By the properties of G_v, G' has at least $0.24k(k-1)$ edges and thus, by Lemma 1, $h(G') \geqslant 0.4\sqrt{k} - 1.5 \geqslant 0.4 \cdot 3n - 1.5$. For $n \geqslant 13$, the last expression is at least $n+1$. Thus, the number of parts needed to partition $V(G_v)$ into subgraphs with Hajós number at most n is at least $1+\lfloor v/k \rfloor$. If $k=\lceil (\log v)^2 \rceil$, this is at least

$$0.5\left(\frac{m}{6\log m}\right)^2 > m.$$

If $k=9n^2$, then $1+\lfloor v/k \rfloor > m^2/81n^2 > m$, as $c>100$. The theorem for vertex partitions is proved.

For edge partitions, let G_v be as above for a large c. Since all subgraphs of size k have edge density at least 0.48, G_v itself has edge density at least 0.48 (see the remark in the proof of Theorem 5 b)). Therefore, $|E(G_v)| \geqslant 0.24 \cdot v(v-1)$. In a partition of the edge-set of G_v into m parts at least one part F has $|E(F)| \geqslant 0.24\, v(v-1)/m$. For this F the average degree is at least $0.48(v-1)/m \geqslant 2v/5n \geqslant 2m^2/45m = 2m/45 > 2c\,n^2/45$, which is sufficiently large for c sufficiently large. By the theorem of Komlós and Szemerédi [14] $h(F_1) \geqslant \sqrt{2cn^2/45} > n$ for c sufficiently large.

Thus G_v cannot be partitioned into m spanning subgraphs each with Hajós number at most n. □

Note that for any fixed n and m large enough, we have $k=\lceil (\log v)^2 \rceil$ and hence need at least $0.5(m/6\log m)^2$ parts to partition $V(G_v)$ into subgraphs of Hajós number at most n.

Proof of Theorem 2. It was observed by several authors (e.g., in [16,7]) that almost every graph on v vertices has Hadwiger number at most $v/\sqrt{\log v}$. Almost repeating any of these arguments, one easily sees that for $k=\lceil v/\sqrt{\log v} \rceil$, almost every graph on v vertices does not contain $K_{k,k}$ as a minor. Thus, for some w and each integer $v>w$, there exists a graph H_v such that

(a) $|V(H_v)|=v$, $|E(H_v)| > v^2/4$;

(b) for $k=\lceil v/\sqrt{\log v} \rceil$, H_v contains neither K_{k+1} nor $K_{k,k}$ as a minor.

By a result of Mader [17,18], for each positive integer n, there exists a constant c_n such that each graph G with $|E(G)| \geqslant c_n |V(G)|$ has Hajós number at least $n+1$ (the best known upper bound for c_n is in [14]).

Consider an arbitrary integer $n>1$. Let $m_0=m_0(n)$ be the smallest integer satisfying the properties that

(i) $w \leqslant m_0\sqrt{\log m_0}/3$;

(ii) $\sqrt{\log m_0} > 12c_n$.

Now, let m be an arbitrary integer greater than $m_0(n)$ and $v=\lceil m\sqrt{\log m}/3 \rceil$. By (i), there is a graph H_v satisfying (a) and (b). Since $m > 2v/\sqrt{\log v}$, H_v has neither K_{m+1} nor $K_{\lceil m/2 \rceil+1, \lfloor m/2 \rfloor+1}$ as a minor. On the other hand, for each partition $\{E_1,\dots,E_m\}$ of the edge-set of H_v, at least one E_i contains more than $v^2/4m \geqslant v \cdot m\sqrt{\log m}/3/4m \geqslant v \cdot c_n$ edges. Thus, we cannot partition the edge-set of H_v into m spanning subgraphs possessing P_n. □

Acknowledgements

We would like to thank the referees for their helpful remarks and suggestions.

References

[1] N. Alon, J. Spencer, The Probabilistic Method, Wiley, New York, 1992.
[2] B. Bollobás, A. Thomason, Proof of a conjecture of Mader, Erdős and Hajnal on topological complete subgraphs, European J. Combin. 19 (1998) 883–887.
[3] G. Chartrand, D.P. Geller, S.T. Hedetniemi, Graphs with forbidden subgraphs, J. Combin. Theory 10 (1971) 12–41.
[4] G.A. Dirac, A property of 4-chromatic graphs and some remarks on critical graphs, J. London Math. Soc. 27 (1952) 85–92.
[5] P. Erdős, S. Fajtlowicz, On the conjecture of Hajós, Combinatorica 1 (1981) 141–143.
[6] P. Erdős, G. Szekeres, A combinatorial problem in geometry, Compositio Math. 2 (1935) 463–470.
[7] Fernandez de la Véga, On the maximum density of graphs which have no subcontractions to K^S, Discrete Math. 46 (1983) 109–110.
[8] H. Hadwiger, Über eine Klassifikation der Streckenkomplexe, Vierteljahrssch. Naturf. Grs. Zürich 88 (1943) 133–142.
[9] D. Hanson, B. Toft, The (m,n)-conjecture is false, Bull. Inst. Combin. Appl. 11 (1994) 59–66.
[10] L.S. Heath, Edge coloring planar graphs with two outerplanar subgraphs, Proceedings of the Second Annual ACM-SIAM Symposium on Discrete Algorithms, 1991, pp. 195–202.
[11] T.R. Jensen, B. Toft, Graph Coloring Problems, Wiley, New York, 1995.
[12] L.K. Jørgensen, Some probabilistic and extremal results on subdivisions, J. Graph Theory 13 (1989) 75–85.
[13] J. Komlós, E. Szemerédi, Topological cliques in graphs, Combin. Probab. Comput. 3 (1994) 247–256.
[14] J. Komlós, E. Szemerédi, Topological cliques in graphs II, Combin. Probab. Comput. 5 (1996) 79–90.
[15] A.V. Kostochka, On Hadwiger numbers of a graph and its complement, in: A. Hajnal, L. Lovász, V.T. Sós (Eds.), Finite and Infinite Sets, Colloq. Math. Soc. János Bolyai 37 (1981) 537–545.
[16] A.V. Kostochka, The minimum Hadwiger number for graphs with a given mean degree of vertices, Metody Diskret. Analiz. 38 (1982) 37–58 (in Russian).
[17] W. Mader, Homomorphieeigenschaften und mittlere Kantendichte von Graphen, Math. Ann. 174 (1967) 265–268.
[18] W. Mader, Hinreichende Bedingungen für die Existens von Teilgraphen, die zu einem vollständigen Graphen homöomorph sind, Math. Nachr. 53 (1972) 145–150.
[19] C. Thomassen, Some homeomorphism properties of graphs, Math. Nachr. 64 (1974) 119–133.
[20] D.R. Woodall, Improper colouring of graphs, in: R. Nelson, R.J. Wilson (Eds.), Graph Colourings, Pitman Research Notes, vol. 218, Longman, London, 1990, pp. 45–63.
[21] B. Zelinka, Hadwiger number of finite graphs, Math. Slov. 26 (1976) 23–30.