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Note

Colouring triangle-free intersection graphs of boxes on the plane

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Abstract

We prove that intersection graphs of boxes on the plane with girth 6 and 8 are 3- and 2-degenerate, respectively. This implies that these graphs are 4- and 3-list-colourable, respectively. © 2000 Published by Elsevier Science B.V. All rights reserved.

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1. Introduction

Among other interesting classes of intersection graphs, Asplund and Grünbaum [1] considered class $\mathscr{B}$ of intersection graphs of *boxes*, i.e. rectangles with sides parallel to the axes on the plane. They proved that every graph in $\mathscr{B}$ with clique number at most r is $4r^2-3r$-colourable. For $r=2$, they obtained the exact result:

Theorem 1 (Asplund and Grünbaum [1]). *The maximum chromatic number of a triangle-free graph in $\mathscr{B}$ is six.*

Note that for three-dimensional boxes, the situation is different. Burling [2] constructed a series of triangle-free intersection graphs of three-dimensional boxes with arbitrarily high chromatic number.

The aim of the present paper is to show that graphs in $\mathscr{B}$ with girth greater than 5 can be coloured with fewer than six colours.

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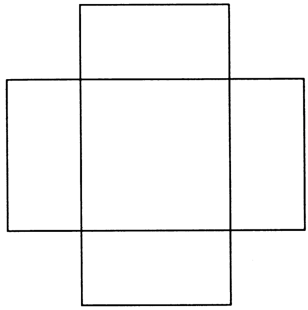

Fig. 1. An example of an x-intersection.

Theorem 2. *If a graph* $G \in \mathscr{B}$ *has girth at least* 6 *or* 8, *then* G *is* 4- *or* 3-*list-colourable, respectively.*

In fact, a stronger statement will be proved. Recall that the *maximum average degree*, $\text{mad}(G)$, of a graph G is $\max_{\emptyset \neq H \subseteq G} 2|E(H)|/|V(H)|$, and a graph is said to be *k-degenerate* if every subgraph has a vertex of degree at most k. Since every graph G with $\text{mad}(G) < k+1$ is k-degenerate and every k-degenerate graph is $(k+1)$-list-colourable, Theorem 2 is an immediate corollary of the following statement.

Theorem 2′. *If a graph* $G \in \mathscr{B}$ *has girth at least* 6 *or* 8, *then* $\text{mad}(G)$ *is strictly less than* 4 *or* 3, *respectively.*

Recall that 'girth at least 4' means 'triangle-free', and Theorem 1 applies. We do not know the situation with girth 5.

In the next section, we introduce the auxiliary notion of a *graphical graph* of a family of boxes and discuss some its properties. Theorem 2′ is proved in the last section.

2. Graphical graphs

Let $\mathscr{A}$ be a family of boxes on the plane whose intersection graph is triangle-free. We shall say that two boxes B_1 and B_2 *x-intersect* if their intersection is nonempty but no corner of B_i is covered by B_{3-i}, $i = 1, 2$ (see Fig. 1).

We also say that B_1 and B_2 *z-intersect* (respectively, *y-intersect*) if exactly one corner of B_i is covered by B_{3-i}, $i = 1, 2$ (respectively, one of them contains exactly two corners of the other) (see Fig. 2).

For every $B \in \mathscr{A}$, let $d_{zy}(B)$ denote the number of boxes $B' \in \mathscr{A}$ such that B and B' z- or y-intersect, and let $d_x(B)$ denote the number of boxes $B'' \in \mathscr{A}$ such that B and B'' x-intersect.

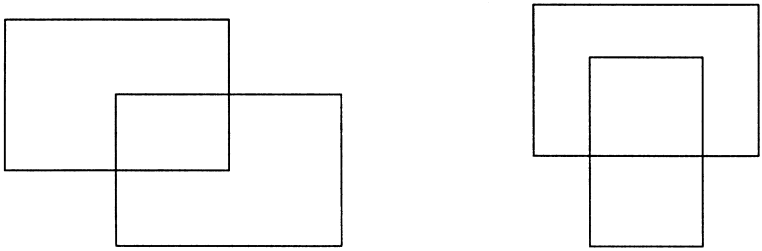

Fig. 2. Examples of a z-intersection (left) and a y-intersection (right).

Let $G \in \mathscr{B}$. There are infinitely many families $\mathscr{A}$ of boxes on the plane such that G is the intersection graph of $\mathscr{A}$. We say that some such $\mathscr{A}$ is a *standard representation* of G if all the sides of all boxes lay on different lines and the sum $\sum_{B \in \mathscr{A}} d_x(B)$ is minimum possible.

Lemma 1. *Let a triangle-free $G \in \mathscr{B}$ have minimum degree at least 2. Let $\mathscr{A}$ be a standard representation of G. Then $d_{zy}(B) \geqslant 2$ for every $B \in \mathscr{A}$.*

Proof. Let $B \in \mathscr{A}$. If $d_x(B) = 0$, then we are done, since $d_x(B) + d_{zy}(B) \geqslant 2$. Suppose that B and B' x-intersect. Then $B \setminus B'$ is the disjoint union of two boxes, say, B_1 and B_2. Since G is triangle-free, no box in $\mathscr{A} - B$ meets both B_1 and B_2. If B_1 does not meet other boxes, then instead of B we can use a shorter box $\hat{B}$ which y-intersects B'. This contradicts the condition that $\mathscr{A}$ is a standard representation of G. It follows that each of B_1 and B_2 meets another box, say B_1' and B_2', in $\mathscr{A}$. Repeating (if needed) the argument, we obtain finally that each of B_1 and B_2 z- or y-intersects some box in $\mathscr{A}$. □

The following notion (similar to one used in [3]) will be helpful. The vertices of the *graphical graph*, $\tilde{G}(\mathscr{A})$, of a family $\mathscr{A}$ of boxes on the plane are the corners of all boxes and the crossing points of the sides of the boxes; the edges of $\tilde{G}(\mathscr{A})$ are the pieces of sides of boxes connecting these points (see Fig. 3).

By the construction, $\tilde{G}(\mathscr{A})$ is always a plane graph. We say that a face of $\tilde{G}(\mathscr{A})$ *is of type i*, if its interior is covered by exactly i boxes of $\mathscr{A}$. The set of faces of type i will be denoted by $T_i = T_i(\mathscr{A})$. The faces of type 0 will be called *outer* and the other faces will be called *interior*. For every face F of $\tilde{G}$, let $r(F)$ denote its *rank*, i.e. the number of edges on the boundary of F. Observe that (since the intersection graph of $\mathscr{A}$ is triangle-free) there are no faces of type i for $i \geqslant 3$ and that the faces of type 2 are always of rank four.

Let $G \in \mathscr{B}$ be triangle-free, $\mathscr{A}$ be a standard representation of G and $\tilde{G} = \tilde{G}(\mathscr{A})$ be the graphical graph of $\mathscr{A}$. Let $n = n(G)$ and $m = m(G)$ denote the number of vertices and edges of G, respectively. Furthermore, let $v = v(\tilde{G})$, $e = e(\tilde{G})$, $f = f(\tilde{G})$ and $f_i = f_i(\tilde{G})$ denote the number of vertices, edges, faces and faces of rank i in $\tilde{G}$,

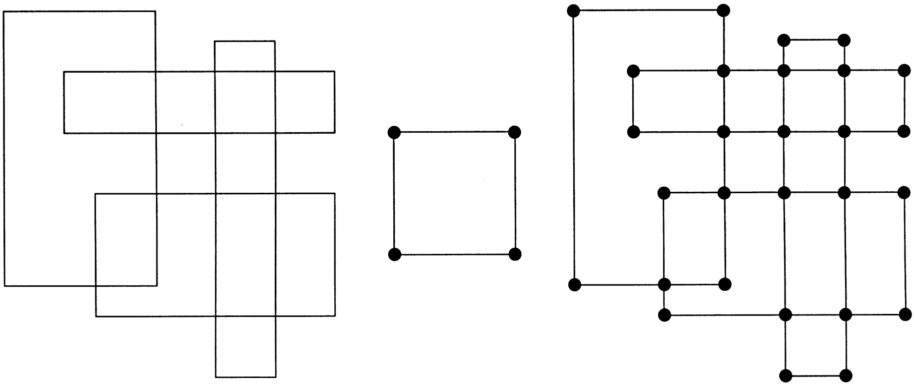

Fig. 3. An example: a family of boxes (left), its intersection graph (in the center) and its graphical graph (right).

respectively. Then the following equalities hold:

$$2m = \sum_{B\in\mathscr{A}} d_x(B) + \sum_{B\in\mathscr{A}} d_{zy}(B), \tag{1}$$

$$v = 4n + 2\sum_{B\in\mathscr{A}} d_x(B) + \sum_{B\in\mathscr{A}} d_{zy}(B), \tag{2}$$

$$e = 4n + 4\sum_{B\in\mathscr{A}} d_x(B) + 2\sum_{B\in\mathscr{A}} d_{zy}(B), \tag{3}$$

$$2e = \sum_{i=4}^{\infty} i \cdot f_i. \tag{4}$$

If a box participates in j x-intersections, then it contains exactly $j+1$ faces of type 1 in $\tilde{G}$. Hence

$$|T_1| = n + \sum_{B\in\mathscr{A}} d_x(B). \tag{5}$$

Each z-intersection creates a new face of type 2 and increases the sizes of both faces of type 1 adjacent to this new face by two. Each y-intersection creates a new face of type 2 and increases the size of one of the faces of type 1 adjacent to this new face by four. This yields the following equation:

$$\sum_{F\in T_1} r(F) = 4|T_1| + 2\sum_{B\in\mathscr{A}} d_{zy}(B). \tag{6}$$

3. Proof of Theorem 2′

Suppose that the theorem does not hold. Let $g \in \{6, 8\}$ and G be a counter-example to the theorem for this g with the smallest number of vertices. Let $\mathscr{A}$ be a standard representation of G and $\tilde{G} = \tilde{G}(\mathscr{A})$ be the graphical graph of $\mathscr{A}$.

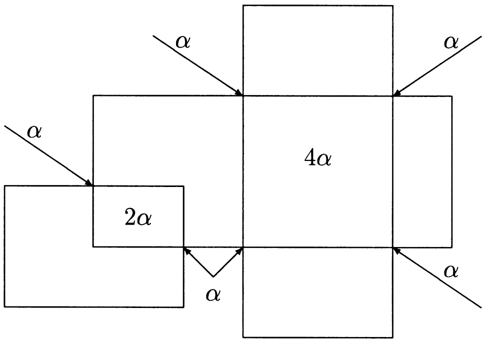

Fig. 4.

Due to the minimality of G, we have $2|E(G)|/|V(G)|\geqslant 4$ if $g=6$ and $2|E(G)|/|V(G)|\geqslant 3$ if $g=8$. If G has a vertex B of degree at most one, then $G-B$ is also a counter-example to the theorem, a contradiction. Thus, the minimum degree of G is at least two. Hence Lemma 1 and Eqs. (1)–(6) hold for G, $\mathscr{A}$ and $\tilde{G}$. Let T_2^x denote the set of faces in T_2 created by x-intersections, and $T_2^{zy}=T_2\setminus T_2^x$. In order to estimate $f=\sum_{F\in T_0\cup T_1\cup T_2} r(F)$ we use discharging, i.e. giving each F a number $r'(F)$ following certain rules so that $\sum_{F\in T_0\cup T_1\cup T_2} r'(F)=\sum_{F\in T_0\cup T_1\cup T_2} r(F)$.

α-Discharging procedure: *For each* $F\in T_1$, *set* $r'(F)=r(F)$. *For each* $F\in T_2^x$, *set* $r'(F)=r(F)+4\alpha$. *For each* $F\in T_2^{zy}$, *set* $r'(F)=r(F)+2\alpha$. *For each* $F\in T_0$, *set* $r'(F)=r(F)-a(F,\pi/2)\alpha$, *where* $a(F,\phi)$ *denotes the number of angles equal to* ϕ *on the boundary of* F.

Since each angle of $\pi/2$ on the boundary of a face $F\in T_0$ is vertical to an angle on the boundary of a face of type 2, we can consider our procedure as if every $F\in T_0$ sends through each angle equal to $\pi/2$ on its boundary the share α to the face of type 2 incident with the vertex of this angle (see Fig. 4).

This proves that $\sum_{F\in T_0\cup T_1\cup T_2} r'(F)=\sum_{F\in T_0\cup T_1\cup T_2} r(F)$.

Lemma 2. *If*

$$\alpha\leqslant\frac{6g-24}{4+3g}, \tag{7}$$

then for every $F\in T_0$, $r'(F)\geqslant 4+4\alpha/3$.

Proof. Let $F\in T_0$. Condition (7) is equivalent to the inequality

$$(2-\alpha)g-4\geqslant 4+\tfrac{4}{3}\alpha.$$

Note that the number of boxes adjacent to F is at most $a(F,\pi/2)$. Therefore, $a(F,\pi/2)\geqslant g$ and

$$(2-\alpha)a(F,\pi/2)-4\geqslant 4+\tfrac{4}{3}\alpha.$$

In other words,

$$(2a(F,\pi/2)-4)-\alpha a(F,\pi/2)\geqslant 4+\tfrac{4}{3}\alpha. \tag{8}$$

Observe that for every face F whose angles are only $\pi/2$ and $3\pi/2$, $a(F,\pi/2)-a(F,3\pi/2)=4$, that is,

$$r(F)=2a(F,\pi/2)-4.$$

This together with (8) yields that $r(F)-\alpha a(F,\pi/2)\geqslant 4+\frac{4}{3}\alpha$ which proves the lemma. □

Let (7) hold. Then due to the discharging procedure and Lemma 2,

$$2e=\sum_{F\in T_0\cup T_1\cup T_2} r'(F)\geqslant\left(4+\frac{4\alpha}{3}\right)|T_0|+\sum_{F\in T_1} r(F)+(4+2\alpha)|T_2|+2\alpha|T_2^x|.$$

By (6) and (5),

$$\begin{aligned}\sum_{F\in T_1} r(F)&=4|T_1|+2\sum_{B\in\mathscr{A}} d_{zy}(B)\\ &=\left(4+\frac{4\alpha}{3}\right)|T_1|-\frac{4\alpha}{3}\left(n+\sum_{B\in\mathscr{A}} d_x(B)\right)+2\sum_{B\in\mathscr{A}} d_{zy}(B).\end{aligned}$$

Therefore,

$$2e\geqslant\left(4+\frac{4\alpha}{3}\right)f-\frac{4\alpha}{3}\left(n+\sum_{B\in\mathscr{A}} d_x(B)\right)+2\sum_{B\in\mathscr{A}} d_{zy}(B)+\frac{2\alpha}{3}|T_2^{zy}|+\frac{8\alpha}{3}|T_2^x|.$$

Since $|T_2^x|=\frac{1}{2}\sum_{B\in\mathscr{A}} d_x(B)$ and $|T_2^{zy}|=\frac{1}{2}\sum_{B\in\mathscr{A}} d_{zy}(B)$, this is equivalent to

$$2e\geqslant\left(4+\frac{4\alpha}{3}\right)f-\frac{4\alpha}{3}n+\left(2+\frac{\alpha}{3}\right)\sum_{B\in\mathscr{A}} d_{zy}(B).$$

Recall that $v-e+f=2$ according to Euler's formula. Hence

$$2e>\left(4+\frac{4\alpha}{3}\right)(e-v)-\frac{4\alpha}{3}n+\left(2+\frac{\alpha}{3}\right)\sum_{B\in\mathscr{A}} d_{zy}(B).$$

Substituting the expressions for v and e from (2) and (3), we get

$$\begin{aligned}&8n+8\sum_{B\in\mathscr{A}} d_x(B)+4\sum_{B\in\mathscr{A}} d_{zy}(B)\\ &\quad>\left(4+\frac{4\alpha}{3}\right)\left(2\sum_{B\in\mathscr{A}} d_x(B)+\sum_{B\in\mathscr{A}} d_{zy}(B)\right)-\frac{4\alpha}{3}n+\left(2+\frac{\alpha}{3}\right)\sum_{B\in\mathscr{A}} d_{zy}(B).\end{aligned}$$

In other words,

$$\begin{aligned}\left(8+\frac{4\alpha}{3}\right)n&>\frac{8\alpha}{3}\sum_{B\in\mathscr{A}} d_x(B)+\left(2+\frac{5\alpha}{3}\right)\sum_{B\in\mathscr{A}} d_{zy}(B)\\ &=\frac{16\alpha}{3}m+(2-\alpha)\sum_{B\in\mathscr{A}} d_{zy}(B).\end{aligned}$$

By Lemma 1, $\sum_{B\in\mathscr{A}} d_{zy}(B) \geqslant 2n$ and hence

$$\left(8+\frac{4\alpha}{3}\right)n > \frac{16\alpha}{3}m + (2-\alpha)2n,$$

i.e.

$$\frac{m}{n} < \frac{6+5\alpha}{8\alpha} = \frac{3}{4\alpha} + \frac{5}{8}. \tag{9}$$

Let $g \geqslant 8$. Then $\alpha = \frac{6}{7}$ satisfies (7), and from (9) we derive that

$$\frac{m}{n} < \frac{7}{8} + \frac{5}{8} = \frac{3}{2}.$$

Similarly, for $g \geqslant 6$, $\alpha = \frac{6}{11}$ satisfies (7), and

$$\frac{m}{n} < \frac{11}{8} + \frac{5}{8} = 2.$$

A contradiction to the choice of G proves Theorem $2'$.

Remark. Maybe the chromatic number of intersection graphs of three-dimensional boxes with large girth is bounded.

Acknowledgements

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