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The number of q-ary words with restrictions on the length of the maximal run*

A. V. KOSTOCHKA, V. D. MAZUROV, and L. YA. SAVELYEV

Abstract — It is proved that the number $g(q,s,n)$ of words of length n over a q-letter alphabet such that the length of any subword consisting of one and the same letter is no greater than s is very close to λ^n, where λ is the greatest real root of the polynomial $x^{s+1} - qx^s + q - 1$. A representation of λ in the form of a series is found. The results obtained let us calculate asymptotical values of $g(q,s,n)$ and the function $h(q,s,n) = g(q,s,n) - g(q,s-1,n)$ as $n \to \infty$ for $s > c \log n$, where c is an arbitrary positive constant.

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1. INTRODUCTION

Let $G(q,s,n)$ denote the set of words of length n over the alphabet $\{0,1,\dots,q-1\}$ such that the length of any run (that is, a subword consisting of one and the same letter) is no greater than s, and let $H(q,s,n)$ denote the subset of words from $G(q,s,n)$ containing at least one run of length s. Let

$$g(q,s,n) = |G(q,s,n)|, \qquad h(q,s,n) = |H(q,s,n)|.$$

The problem of finding $g(q,s,n)$ and $h(q,s,n)$ is natural and is connected with a series of other problems. In particular, $g(2,s,n)$ (respectively, $h(2,s,n)$) is equal to the doubled number of representations of n in the form of a sum of ordered summands each of which does not exceed s (respectively, such that the maximal of them is equal to s). The problem of finding $g(q,s,n)$ is a particular case of the problem of finding the probability that a q-ary word of length n contains the maximal run of length no greater than s. V. L. Goncharov [4] derived the general generating function for several probabilistic characteristics connected with maximal runs for the case $q = 2$, where each letter of the word takes one of the values with probability p and the other value with probability $1-p$ independently of the other letters of the word. Using this generating function, he found the limit distribution of the length of the maximal run in typical cases. Similar problems for sequences of dependent symbols were considered by Savelyev [9, 14] and others, some references can be found in [6, 9]. Korshunov [6] found the asymptotics of

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$h(2, s, n)$ as $n \to \infty$ and $s \geq (1/2)\log_2 n + 2\log_2\log_2 n$. In the present paper we use an approach different from the approach in [6] and obtain more general results.

It is clear that

$$h(q, s, n) = g(q, s, n) - g(q, s-1, n) \tag{1}$$

and

$$g(q, s, n) = q^n, \qquad n = 1, 2, \ldots, s. \tag{2}$$

The problem of finding $g(q, s, n)$ is reduced to the investigation of the roots of certain polynomials. Let

$$\begin{aligned} A_{q,s}(x) &= x^s - (q-1)(1 + x + x^2 + \ldots + x^{s-1}), \\ B_{q,s}(x) &= (x-1)A_{q,s}(x) = x^{s+1} - qx^s + q - 1. \end{aligned} \tag{3}$$

The following theorem is the main result of this paper.

Theorem 1. *Let $q \geq 2$, $s \geq 2$ be integers, $\lambda_1 = \lambda_1(q, s)$ be the greatest positive root of the polynomial $A_{q,s}(x)$, and*

$$b(q, s) = \frac{q(\lambda_1 - 1)}{(q-1)((s+1)\lambda_1 - sq)}.$$

Then, for $n \geq s$, the number $g(2, s, n)$ equals the even integer nearest to $b(2, s)\lambda_1^n$, and for $q \geq 3$, the number $g(q, s, n)$ equals the integer nearest to $b(q, s)\lambda_1^n$.

It will be seen later that $b(q, s)$ is close to 1.

Corollary 1. *For $n \geq s$,*

$$h(2, s, n) = 2(\lfloor 0.5b(2, s)\lambda_1(2, s)^n \rceil - \lfloor 0.5b(2, s-1)\lambda_1(2, s-1)^n \rceil), \tag{4}$$

and for $q \geq 3$,

$$h(q, s, n) = \lfloor b(q, s)\lambda_1(q, s)^n \rceil - \lfloor b(q, s-1)\lambda_1(q, s-1)^n \rceil,$$

where $\lfloor z \rceil$ denotes the integer nearest to z.

Thus, the problem of finding or estimating $g(q, s, n)$ and $h(q, s, n)$ is equivalent to the problem of finding or estimating $\lambda_1(q, s)$. It is clear that

$$\lambda_1(q, 2) = (q-1)(1 + \sqrt{5})/2 \tag{5}$$

and $g(2, 2, n)$ equals the doubled nth Fibonacci number.

The formulae for calculating $\lambda_1(q, s)$ are given in the following theorem.

Theorem 2. *For* $s \geq 2$,

$$\lambda_1(q,s) = q\left(1 - \sum_{k=1}^{\infty}\frac{1}{k}\binom{k(s+1)-2}{k-1}\left(\frac{q-1}{q^{s+1}}\right)^k\right), \tag{6}$$

$$\lambda_1(q,s) = q\exp\left\{-\sum_{k=1}^{\infty}\frac{1}{k}\binom{k(s+1)-1}{k-1}\left(\frac{q-1}{q^{s+1}}\right)^k\right\}. \tag{7}$$

The paper consists of six sections. In Section 2, we investigate properties of roots of some auxiliary polynomials. On the basis of these properties in Section 3, we prove Theorem 1. In Section 4, Theorem 2 is proved. In Section 5 we investigate the asymptotic properties of $g(q,s,n)$ and $h(q,s,n)$ as $n \to \infty$. In Section 6 we discuss the results obtained in the paper.

2. ON THE ROOTS OF THE POLYNOMIAL $A_{q,s}(x)$

Proposition 1. *Let $q \geq 2$, $s \geq 2$ be integers. Then the polynomial $A_{q,s}(x)$ has no multiple roots. It has a real root λ_1 in the interval $(q - q^{1-s}, q)$. The absolute values of all the remaining roots are less than one.*

Proof. The derivative of the polynomial $B_{q,s}(x)$ has exactly two roots, 0 of multiplicity $s-1$ and $\alpha = qs/(s+1)$. Obviously, 0 is not a root of $B_{q,s}(x)$. Suppose that α is a root of $B_{q,s}(x)$. All coefficients of $B_{q,s}(x)$ are integer and the leading coefficient equals 1, therefore any rational root of $B_{q,s}(x)$ has to be integer (see, for example, Section 57 in [7]) and to divide $q-1$. Since s and $s+1$ are mutually prime, $s+1$ divides q, and $\alpha = q_0 s$, where $q_0 = \alpha/s$ is an integer. It follows from the fact that q_0 divides q and $q-1$ that $q_0 = 1$ and $\alpha = s = q-1$. Consequently,

$$0 = B_{q,s}(\alpha) = s^{s+1} - (s+1)s^s + s = -s^s + s,$$

which is impossible.

Thus, $B_{q,s}(x)$ and its derivative have no common roots, that is, the polynomial $B_{q,s}(x)$ has no multiple roots. Consequently, $A_{q,s}(x) = B_{q,s}(x)/(x-1)$ has no multiple roots as well.

Note that

$$\begin{aligned} B_{q,s}(q - q^{1-s}) &= (q - q^{1-s})^s(q - q^{1-s} - q) + q - 1 \\ &= -q(1 - q^{-s})^s + q - 1 < -q(1 - sq^{-s}) + q - 1 = sq^{-s} - 1 < 0, \\ B_{q,s}(q) &= q^{s+1} - q^{s+1} + q - 1 = q - 1 > 0. \end{aligned}$$

Consequently, $A_{q,s}(x)$ has a root λ_1 in the interval $(q-1, q)$.

Suppose that $A_{q,s}(x)$ has a root $\lambda \neq \lambda_1$ such that $|\lambda| > 1$. Since λ and λ_1 are roots of $B_{q,s}(x)$, we have

$$q - \lambda = \frac{q-1}{\lambda^s}, \qquad q - \lambda_1 = \frac{q-1}{\lambda_1^s}.$$

Hence,

$$\lambda - \lambda_1 = \frac{q-1}{\lambda_1^s} - \frac{q-1}{\lambda^s} = (q-1)\frac{\lambda^s - \lambda_1^s}{\lambda^s \lambda_1^s},$$

and dividing by $\lambda - \lambda_1$, we obtain

$$1 = (q-1)\frac{\lambda_1^{s-1} + \lambda_1^{s-2}\lambda + \ldots + \lambda^{s-1}}{\lambda^s \lambda_1^s}.$$

Turning to the absolute values, we see that

$$1 \le (q-1)\frac{\lambda_1^{s-1} + \lambda_1^{s-2}|\lambda| + \ldots + |\lambda^{s-1}|}{|\lambda|^s \lambda_1^s} < (q-1)\frac{\lambda_1^{s-1} + \lambda_1^{s-2} + \ldots + \lambda_1 + 1}{\lambda_1^s},$$

because $|\lambda| > 1$. Since λ_1 is a root of $A_{q,s}(x)$, the left-hand side of the inequality is equal to one. This contradiction completes the proof of the proposition.

Lemma 1. *For each root λ of the polynomial $B_{q,s}(x)$, different from λ_1,*

$$\frac{|1-\lambda|}{|q-\lambda|^2} \le \frac{1}{2\sqrt{q}(q-1)}. \tag{8}$$

Proof. By virtue of Proposition 1 it is sufficient to prove that inequality (8) is true for any complex λ, $|\lambda| \le 1$.

Therefore, let $|\lambda| \le 1$, $\lambda = a + bi$, where a and b are real numbers. Let $d = |1 - \lambda|$, $D = |q - \lambda|$. Then

$$d^2 = 1 - 2a + a^2 + b^2,$$
$$D^2 = q^2 - 2qa + a^2 + b^2 = q^2 - 2(q-1)a + d^2 - 1. \tag{9}$$

Therefore, on the part of the circle $|1 - \lambda| = d$ that is contained in the disk $|\lambda| \le 1$, the minimum of D is attained for the greatest value of a, that is, at the intersection with the circle $|\lambda| = 1$. Therefore we assume that $a^2 + b^2 = 1$. Then, according to (9), $2a = 2 - d^2$ and

$$\frac{d}{D^2} = \frac{d}{q^2 + d^2 - 1 - (q-1)(2-d^2)} = \frac{d}{(q-1)^2 + qd^2}.$$

It is clear that the maximum of the last function over positive d is attained at $d = (q-1)/\sqrt{q}$ and equals $1/(2\sqrt{q}(q-1))$. The lemma is proved.

3. THE PROOF OF THEOREM 1

Let $G(i, q, s, n)$ be the subset of the words from $G(q, s, n)$ such that their first letter is i, $i = 0, 1, \ldots, q-1$, and $G(j, i, q, s, n)$ be the subset of the words from $G(i, q, s, n)$ such that their first j letters are i, $i = 0, 1, \ldots, q-1, j = 1, \ldots, s$. The equations that are neede in what follows are derived from the following simple identities:

$$|G(i,q,s,n)| = |G(k,q,s,n)| \qquad \forall i,k = 0,1,\ldots,q-1; \quad (10)$$

$$|G(q,s,n)| = q|G(i,q,s,n)| = q\sum_{j=1}^{s}|G(j,i,q,s,n)| \qquad \forall i = 0,1,\ldots,q-1; \quad (11)$$

and

$$|G(j,i,q,s,n)| = \left|\bigcup_{k\neq i} G(k,q,s,n-j)\right| = (q-1)|G(i,q,s,n-j)| = \frac{q-1}{q}|G(q,s,n-j)|. \quad (12)$$

By virtue of (11) and (12), for each $n \geq s+1$,

$$g(q,s,n) = (q-1)(g(q,s,n-1) + g(q,s,n-2) + \ldots + g(q,s,n-s)). \quad (13)$$

Thus, for fixed q and s, the sequence $g(q,s,n)$ is the recurring sequence defined by equations (13) and initial values (2). The characteristic polynomial (see [3, 8, 13]) of the sequence $g(q,s,n)$ is $A_{q,s}(x)$. By Proposition 1, all roots of the polynomial $A_{q,s}(x)$ are simple. Consequently (see, for example, [8, 3, 13]), the solution of equations (13) is of the form

$$g(q,s,n) = b_1\lambda_1^n + \ldots + b_s\lambda_s^n, \quad (14)$$

where $\lambda_1, \ldots, \lambda_s$ are the roots of $A_{q,s}(x)$. To find the coefficients $b_1, \ldots, b_s$, we use (2). We obtain the system of equations

$$\begin{array}{rcl} q & = & \lambda_1 b_1 + \lambda_2 b_2 + \ldots + \lambda_s b_s, \\ q^2 & = & \lambda_1^2 b_1 + \lambda_2^2 b_2 + \ldots + \lambda_s^2 b_s, \\ \ldots & & \\ q^s & = & \lambda_1^s b_1 + \lambda_2^s b_2 + \ldots + \lambda_s^s b_s. \end{array} \quad (15)$$

By the Cramer rule, $b_i = D_i/D$, $i = 1, \ldots, n$, where

$$D = \begin{vmatrix} \lambda_1 & \lambda_2 & \ldots & \lambda_s \\ \lambda_1^2 & \lambda_2^2 & \ldots & \lambda_s^2 \\ \ldots & \ldots & \ldots & \ldots \\ \lambda_1^s & \lambda_2^s & \ldots & \lambda_s^s \end{vmatrix},$$

and D_i is obtained from D by substituting the column of right-hand sides of system (15) for the ith column. It is clear that

$$D = \lambda_1\lambda_2\ldots\lambda_s W(\lambda_1, \lambda_2, \ldots, \lambda_s),$$
$$D_i = \lambda_1\ldots\lambda_{i-1}q\lambda_{i+1}\ldots\lambda_s W(\lambda_1, \ldots, \lambda_{i-1}, q, \lambda_{i+1}, \ldots, \lambda_s),$$

where $W(z_1, \dots, z_s)$ denotes the Vandermonde determinant in the variables $z_1, \dots, z_s$. Since

$$W(z_1, \dots, z_s) = \prod_{i<j}(z_j - z_i),$$

we obtain

$$\frac{D_i}{D} = \frac{q}{\lambda_i}\frac{\prod_{j\neq i}(q - \lambda_j)}{\prod_{j\neq i}(\lambda_i - \lambda_j)}.$$

Because $\lambda_1, \lambda_2, \dots, \lambda_s$ are the roots of $A_{q,s}(x)$, we have

$$\prod_{j\neq i}(q - \lambda_j) = \frac{A_{q,s}(q)}{q - \lambda_i} = \frac{1}{q - \lambda_i}.$$

Since $A_{q,s}(\lambda_i) = B_{q,s}(\lambda_i) = 0$, we find that

$$\prod_{j\neq i}(\lambda_i - \lambda_j) = \lim_{x\to\lambda_i}\frac{A_{q,s}(x)}{x - \lambda_i} = \lim_{x\to\lambda_i}\frac{B_{q,s}(x)}{(x-1)(x - \lambda_i)} = \lim_{x\to\lambda_i}\frac{1}{x-1}\lim_{x\to\lambda_i}\frac{B_{q,s}(x) - B_{q,s}(\lambda_i)}{x - \lambda_i}$$
$$= \frac{1}{\lambda_i - 1}\left[B_{q,s}(x)\right]'_{x=\lambda_i} = \frac{(s+1)\lambda_i^s - sq\lambda_i^{s-1}}{\lambda_i - 1}.$$

Recalling that $\lambda_i^s(q - \lambda_i) = q - 1$, we obtain

$$\frac{D_1}{D} = \frac{q}{\lambda_1}\frac{1}{q - \lambda_1}\frac{\lambda_1 - 1}{\lambda_1^{s-1}((s+1)\lambda_1 - sq)} = \frac{q(\lambda_1 - 1)}{(q-1)((s+1)\lambda_1 - sq)}.$$

Since $|\lambda_i| < 1$ for $i \geq 2$, we see that for such i and $n \geq s$

$$|b_i\lambda_i^n| = |\lambda_i^n D_i/D| \leq \frac{|\lambda_i|^{n-s}q|1 - \lambda_i|}{|q - \lambda_i||-(s+1)\lambda_i + sq|}.$$

Note that

$$|-(s+1)\lambda_i + sq| = |s(q - \lambda_i) - \lambda_i| \geq s|q - \lambda_i| - 1 > (s-1)|q - \lambda_i|.$$

Taking Lemma 1 into account, we find that for $i \geq 2$ and $n \geq s$,

$$|b_i\lambda_i^n| \leq \frac{|\lambda_i|^{n-s}q|1 - \lambda_i|}{(s-1)|q - \lambda_i|^2} \leq \frac{q}{2\sqrt{q}(q-1)(s-1)}.$$

Hence,

$$|g(q,s,n) - b_1\lambda_1^n| \leq (s-1)\frac{q}{2\sqrt{q}(q-1)(s-1)} = \frac{\sqrt{q}}{2(q-1)}. \tag{16}$$

Taking into account that $q - 1 > \sqrt{q}$ for $q \geq 3$, we see that the assertion of the theorem is proved for $q \geq 3$. The numbers of words in $G(2,s,n)$ that begin with 0 and 1 are the same, therefore $g(2,s,n)$ is even. Thus, the assertion of the theorem follows from (16).

4. FORMULAE FOR $\lambda_1(q, s)$

To calculate $\lambda_1(q, s)$, we use the Lagrange theorem on inverting series (see, for example, Chapter XII, Section 4 in [12]).

Theorem 3. *Let $f(y)$ and $\phi(y)$ be functions of y that are analytic in some neighbourhood of a point a, and let y be the solution of the equation*

$$y = a + x\phi(y). \tag{17}$$

Then f can be represented as the series

$$f(y) = f(a) + \sum_{k=1}^{\infty} \frac{x^k}{k!} \frac{d^{k-1}}{dy^{k-1}} \left[f'(y)\phi(y)^k\right]_{y=a}. \tag{18}$$

It is clear that λ is a root of $B_{q,s}(x)$ if and only if $\mu = q/\lambda$ is a root of the equation

$$y = 1 + \frac{q-1}{q^{s+1}} y^{s+1}. \tag{19}$$

Equation (19) is a particular form of equation (17) with $a = 1$, $x = (q-1)/q^{s+1}$, $\phi(y) = y^{s+1}$. Applying Theorem 3 to (19) with $f(y) = q/y$ and noting that

$$f'(y)\phi(y)^k = -\frac{q}{y^2} y^{k(s+1)} = -q\, y^{k(s+1)-2},$$

$$\frac{1}{k!} \frac{d^{k-1}}{dy^{k-1}} \left[f'(y)\phi(y)^k\right]_{y=a} = -\frac{q}{k} \binom{k(s+1)-2}{k-1} y^{ks-1},$$

for the root μ of equation (19) close to one, we obtain the formula

$$\frac{q}{\mu} = q - q \sum_{k=1}^{\infty} \frac{1}{k} \binom{k(s+1)-2}{k-1} \left(\frac{q-1}{q^{s+1}}\right)^k.$$

By the Cauchy criterion, this series converges for any $s \geq 2$, $q \geq 2$. It is clear from this formula that $q - 1 < q/\mu < q$. As we have mentioned above, q/μ is a root of $B_{q,s}(x)$. Consequently, $q/\mu = \lambda_1(q, s)$, and formula (6) is proved.

Similarly, to derive formula (7), it is sufficient to apply Theorem 3 to (19) with $f(y) = \ln(q/y)$ (cf. [11, p. 187]). Then we obtain

$$\ln \lambda_1(q, s) = \ln q - \sum_{k=1}^{\infty} \frac{1}{k} \binom{k(s+1)-1}{k-1} \left(\frac{q-1}{q^{s+1}}\right)^k.$$

5. ON THE ASYMPTOTICS OF $g(q, s, n)$ AND $h(q, s, n)$

As approximations to λ_1, we can take the partial sums of series (6). We introduce the notation

$$\varphi(x) = \frac{q(x-1)}{(q-1)((s+1)x - sq)} x^n,$$
$$a_k = \frac{1}{k}\binom{k(s+1)-2}{k-1},$$
$$z_m = z_m(q, s) = q\left(1 - \sum_{k=1}^{m} a_k \left(\frac{q-1}{q^{s+1}}\right)^k\right).$$

Theorem 1 asserts that $\varphi(\lambda_1(q, s))$ almost coincides with $g(q, s, n)$. We estimate the error of approximation of $\varphi(\lambda_1(q, s))$ by $\varphi(z_m^n)$. Since

$$a_k \le \frac{(ks + k/2)^{k-1}}{k!} \le \frac{(e(s+1/2))^k}{k(s+1/2)},$$

we obtain

$$z_m - \lambda_1 = q \sum_{k=m+1}^{\infty} a_k \left(\frac{q-1}{q^{s+1}}\right)^k \le q \left(\frac{e(s+1/2)(q-1)}{q^{s+1}}\right)^{m+1}.$$

We recall that $\lambda_1 > q - q^{1-s}$ by Proposition 1. It is easy to check that for $6 \le s \le n-1$ and $x > q - q^{1-s}$ the function $\varphi(x)$ increases and the fraction

$$\frac{q(x-1)}{(q-1)((s+1)x - sq)} = \frac{q}{(q-1)(s+1)} \frac{x-1}{x - sq/(s+1)}$$

decreases as x increases. Consequently, since $z_m > \lambda_1$,

$$1 > \frac{\varphi(\lambda_1)}{\varphi(z_m)} \ge \frac{\lambda_1^n}{z_m^n} = \left(1 - \frac{z_m - \lambda_1}{z_m}\right)^n \ge 1 - n\frac{z_m - \lambda_1}{z_m} \ge 1 - 2n\left(\frac{e(s+1/2)(q-1)}{q^{s+1}}\right)^{m+1}.$$

For any $q \ge 2$, $s \ge 4$,

$$e(s+1/2)(q-1)q^{-s-1} < 1/2,$$

therefore for such q, s and $m > 2\log_2 n$,

$$1 > \frac{\varphi(\lambda_1)}{\varphi(z_m)} > 1 - 2n\, 2^{-1-2\log_2 n} > 1 - 1/n.$$

If $n \to \infty$ and $s \ge 0.5\log_q n + 2\log_q \log_q n$, then $\varphi(z_1)$ is asymptotically equal to $g(q, s, n)$. Indeed, in this case,

$$1 > \frac{\varphi(\lambda_1)}{\varphi(z_1)} \ge 1 - 2n\left(\frac{e(s+1/2)(q-1)}{q^{s+1}}\right)^2$$
$$\ge 1 - 2n\left(\frac{e(0.5\log_q n + \log_q \log_q n + 1/2)(q-1)}{q\sqrt{n}\log_q^2 n}\right)^2 \ge 1 - 2\left(\frac{e}{\log_q n}\right)^2.$$

Hence the results of [6] follow. If in this case instead of $\varphi(z_1)$ we use $\varphi(z_m)$ for a fixed $m > 1$, then we obtain a better approximation. The behaviour of $g(q, s, n)$ as $n \to \infty$ and $s > c \log n$ for any fixed $c > 0$ can be investigated similarly.

To approximate $\varphi(\lambda_1(q, s))$, we can use the representation of $\lambda_1(q, s)$ of the form (7). The following assertion can be proved in the same way as the inequalities proved above.

Proposition 2. *Let $q \geq 2$, $s \geq 2$, $m \geq 1$, be integers and*

$$u(q, m, s) = \sum_{k=1}^{m} \frac{1}{k} \binom{k(s+1)-1}{k-1} \left(\frac{q-1}{q^{s+1}}\right)^k.$$

Then

$$g(q, s, n) \sim q^n e^{-n\, u(q,m,s)}$$

as $n \to \infty$ and

$$s \geq \frac{1}{m+1} \log_q n + \log_q \log_q n.$$

6. CONCLUDING REMARKS

On representation of $\lambda_1(q, s)$ in radicals. Formula (5) is a simple finite formula for $\lambda_1(q, 2)$. Unfortunately, it is impossible to express $\lambda_1(q, s)$ in radicals for many of s. We give an outline of the proof that it is impossible for $s = 5$, $q = 2$.

It is not difficult to check that $f = x^5 - x^4 - x^3 - x^2 - x - 1$ is indecomposable modulo 5 and

$$f \equiv (x^3 + x^2 - x + 1)(x^2 + x - 1) \pmod 3.$$

It is obvious that the polynomials x^3+x^2-x+1 and x^2+x-1 are indecomposable modulo 3. According to [1, Section 66], the Galois group of the polynomial f, considered as the group of permutations of its roots, contains a cycle of length five and a transposition. Consequently, it coincides with the symmetric group of degree five, and in particular, is unsolvable.

On the difference $g(2, s, n) - b(2, s)\lambda_1(2, s)$. No less than a half of the roots of the polynomial $A_{q,s}$ are placed in the left half-plane (see Sections 2 and 3 of Chapter XV in [2]), therefore it is not difficult to see that $|g(2, s, n) - b(2, s)\lambda_1^n| < 1/2$. Thus, the following refinement of Theorem 1 is true.

Theorem 1′. *Let $q \geq 2$, $s \geq 2$ be integers, $\lambda_1 = \lambda_1(q, s)$ be the greatest positive root of the polynomial $A_{q,s}(x)$, and let*

$$b(q, s) = \frac{q(\lambda_1 - 1)}{(q-1)((s+1)\lambda_1 - sq)}.$$

Then for $n \geq s$, the number $g(q, s, n)$ equals the integer nearest to $b(q, s)\lambda_1^n$.

On the Pizo numbers. A real number $a > 1$ is called a Pizo number if it is a root of a polynomial $f_a(x)$ with integer coefficients and the leading coefficient equal to one, such that the absolute values of all the remaining roots of $f_a(x)$ is less than one. In particular, each integer $a > 1$ (as the root of the binomial $x - a$) is a Pizo number. The Pizo numbers possess a series of remarkable properties (see, for example, [5, Chapter VIII]). In fact, Proposition 1 asserts that $a = \lambda_1(q, s)$ is a Pizo number, where $f_a(x) = A_{q,s}(x)$.

On other statements of problems. The theory of runs in random sequences is closely related to the theory of random allocations. This relation was investigated in detail by Trunov [10].

The problem of finding $g(q, s, n)$ can be formulated as the problem of finding the number of words of length n that contain no subwords of length $s+1$ consisting of one and the same letters. Evdokimov called our attention to the fact that in certain applications estimates of the number of words of length n with different sets of forbidden words are needed.

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