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Total Colourings of Planar Graphs with Large Girth

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It is proved that if G is a planar graph with total (vertex–edge) chromatic number χ'', maximum degree Δ and girth g, then $\chi'' = \Delta + 1$ if $\Delta \geq 5$ and $g \geq 5$, or $\Delta \geq 4$ and $g \geq 6$, or $\Delta \geq 3$ and $g \geq 10$. These results hold also for graphs in the projective plane, torus and Klein bottle.

1. INTRODUCTION

The *total chromatic number* $\chi'' = \chi''(G)$ of a graph G is the smallest number of colours that suffice to colour the vertices and edges of G so that no two adjacent or incident elements have the same colour. It is clear that $\chi'' \geq \Delta + 1$, and Behzad [1] and Vizing [14] conjectured that $\chi'' \leq \Delta + 2$, for every graph G with maximum degree Δ. This conjecture was verified by Rosenfeld [12] and Vijayaditya [13] for $\Delta = 3$ and by Kostochka [9–11] for $\Delta \leq 5$. For planar graphs the conjecture was verified by Borodin [2] for $\Delta \geq 9$, and now remains open only for $6 \leq \Delta \leq 7$ [8]. But for planar graphs it is often possible to determine χ'' precisely, as shown by the following theorem. Recall that the *girth* of a graph is the length of its shortest cycle.

THEOREM. *Let G be a planar graph with maximum degree Δ and girth g. Then $\chi''(G) = \Delta + 1$ in each of the following cases:*

(a) $\Delta \geq 11$*;*
(b) $\Delta \geq 7$ *and* $g \geq 4$*;*
(c) $\Delta \geq 5$ *and* $g \geq 5$*;*
(d) $\Delta \geq 4$ *and* $g \geq 6$*;*
(e) $\Delta \geq 3$ *and* $g \geq 10$*.*

The first result in this direction was obtained by Borodin [2], who proved a weaker version of (a) with $\Delta \geq 14$. Chen and Wu [5] proved weaker versions of (b) with $\Delta \geq 8$ and $g \geq 4$, of (c) with $\Delta \geq 6$ and $g \geq 5$, and of (d) with $\Delta \geq 4$ and $g \geq 8$. The purpose of the present paper is to prove (c)–(e). We proved (a) and (b) in [3] and [4] respectively, using different and involved methods. We believe that further improvements to these results will need new ideas.

On the other hand, few planar graphs are known for which $\chi'' > \Delta + 1$. Clearly if $\Delta = 2$ then $\chi'' = 3$ if and only if every cycle in G has length divisible by 3. In [6] there are examples with $\Delta = 3$, $g = 4$ and $\chi'' = 5$. The problem of closing the gap between these examples and the bounds in the Theorem would seem to be hard.

We prove the Theorem in Section 2. In Section 3, we prove that (c), (d) and (e) hold also for graphs in the projective plane, torus and Klein bottle. In [4] we proved that (b) also holds in these surfaces, as does the weaker version of (a) with $\Delta \geq 12$ (and moreover, in each case, with the list total chromatic number χ''_{list} in place of χ''). But our proof of (a) in [3] does not seem to extend naturally to these surfaces. Further extensions involving the maximum average degree are discussed briefly in Section 4.

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2. Proof of the Theorem

We have seen that (a) and (b) are already known. So let G be a minimal counterexample to any of (c)–(e); clearly G is 2-connected. We may suppose that if $H \subsetneq G$ then $\chi''(H) \leq \Delta + 1$, since this follows from the minimality of G if H satisfies the hypotheses of the theorem, and otherwise it follows from the result [9, 12, 13] that $\chi''(H) \leq \Delta(H) + 2$ if $\Delta(H) < 5$. Our proof uses an application of Euler's formula (Lemma 1) and some structural information derived from the minimality of G (Lemmas 2–4) to obtain a contradiction, in one case using the discharging method.

Throughout, G has n vertices, m edges and r faces, the sets of which are denoted by V, E and F respectively. The degree of vertex v is denoted by $d(v)$, a *d-vertex* is a vertex v with $d(v) = d$, and n_d denotes the number of d-vertices in G. The number of edges incident to face f is denoted by $r(f)$.

Lemma 1.

(i) $\sum_{v \in V}(d(v) - 4) + \sum_{f \in F}(r(f) - 4) < 0.$
(ii) $\sum_{v \in V}((t - 2)d(v) - 2t) < 0$ *if* $0 < t \leq g$.

Proof. Euler's formula $n - m + r = 2$ can be rewritten in the form $(2m - 4n) + (2m - 4r) = -8$, which implies (i), and in the form $((2t - 4)m - 2tn) + (4m - 2tr) = -4t$, which implies (ii) since $tr \leq gr \leq 2m$ for a graph with girth g. □

The next lemma is used in the proofs of (c) and (d).

Lemma 2. *Suppose* $\Delta \geq 4$. *Then*

(i) every 2*-vertex is adjacent to two* Δ*-vertices;*
(ii) $n_2 < n_\Delta$.

Proof. Suppose v is a 2-vertex adjacent to u and w, where $d(w) < \Delta$. We can totally colour $G - v$ with $\Delta + 1$ colours (by the minimality of G) and then colour uv, vw and v in that order, since the number of colours that we may not use is at most $(\Delta - 1) + 1 = \Delta$ for uv, $(\Delta - 2) + 1 + 1 = \Delta$ for vw, and $2 + 2 = 4$ for v, while $\Delta + 1 \geq 5$. This contradicts the choice of G as a counterexample and so proves (i).

The proof of (ii) uses the following important idea from [2]. Suppose G contains a *2-alternating cycle C*, that is, a cycle $v_1 v_2 \ldots v_{2k} v_1$ of even length such that $d(v_1) = d(v_3) = \cdots = d(v_{2k-1}) = 2$. Then, by the minimality of G, we can totally colour $G - \{v_1, v_3, \ldots, v_{2k-1}\}$ with $\Delta + 1$ colours. Each edge of C now has at most $(\Delta - 2) + 1 = \Delta - 1$ colours that may not be used on it, hence at least two that may, and so the problem of colouring the edges of C is equivalent to colouring the vertices of an even cycle, given a choice of two colours at each vertex; it is well known [7, 15] that this is possible. The 2-vertices of C are now easily coloured since $\Delta + 1 \geq 5$, and this contradicts the choice of G as a counterexample to the theorem. Thus there are no 2-alternating cycles; and (ii) follows easily from this and (i) if one considers the forest of all edges joining 2-vertices to Δ-vertices. □

The proof of (d) is now complete, since Lemma 1(ii) with $t = 6$ implies $n_2 > n_4 + 2n_5 + 3n_6 + \cdots$, contrary to Lemma 2(ii). The proof of (c) is not so easy.

Lemma 3. *If* $\Delta \geq 5$, *then no* 3*-vertex is adjacent to two* 3*-vertices.*

Proof. Suppose u, v, w are 3-vertices with uv, $vw \in E$. We can totally colour $G - uv$ with $\Delta + 1$ colours by the minimality of G, and then erase the colours on u and v. Now there are two colours available for each of u, uv and v, and the only problem arises if they are the same two colours in each case. But the edge vw has at most four restrictions on its colour, and $\Delta + 1 \geq 6$, so by recolouring vw if necessary we can complete the colouring and obtain the required contradiction. □

We now complete the proof of (c). Assign a 'charge' of $d(v)-4$ units to each vertex v of G and of $r(f)-4$ units to each face f of G. By Lemma 1(i), the sum of the charges assigned is nonpositive (in fact, strictly negative). We now redistribute the charge, without changing its sum, in such a way that the sum is provably positive, and this contradiction will prove (c).

The rules for redistribution are as follows:

R1: Each 2-vertex receives $\frac{1}{2}$ from each adjacent Δ-vertex and each incident face.
R2: Each 3-vertex receives $\frac{1}{3}$ from each incident face.

It is easy to see that the sum of the charges on the vertices is now strictly positive: each 2-vertex started with -2 and has gained 2, each 3-vertex started with -1 and has gained 1, each d-vertex ($3 < d < \Delta$) started non-negative and has not changed, and the Δ-vertices collectively started with at least n_Δ and have given up $n_2 < n_\Delta$ (by Lemma 2(ii)). It remains to prove that the sum of the charges on the faces is non-negative.

Let us label three types of face according to their degree sequences $N = 33\Delta2\Delta$, $Z_2 = \Delta\Delta2\Delta2$ and $Z_3 = 33x3x$, where x denotes anything greater than 3. Then N-faces have charge $-\frac{1}{6}$, Z_2-faces and Z_3-faces have charge 0, and all other faces are called P-faces and have positive charge. Indeed, each P-face could give $\frac{1}{2}$ instead of $\frac{1}{3}$ to each incident 3-vertex without going negative: this is because Lemmas 2(i) and 3 ensure that a pentagonal P-face has, at most, two non-x vertices, and if $r(f) \geq 6$ then no three consecutive vertices of f can receive $\frac{1}{2}$, so that f gives up at most $\frac{2}{3}r(f) \times \frac{1}{2} = \frac{1}{3}r(f)$, whereas its initial charge was $r(f) - 4 \geq \frac{1}{3}r(f)$ if $r(f) \geq 6$.

Now suppose that instead of giving $\frac{1}{2}$ to each incident 3-vertex, each P-face gives only $\frac{1}{3}$ as originally proposed, but also gives $\frac{1}{12}$ to each bounding edge joining a 3-vertex with a Δ-vertex. Suppose also that each N-face gives $-\frac{1}{12}$ to each of its two $(\Delta, 3)$-edges. Then every face has non-negative charge, and it remains to show that the total number of $-\frac{1}{12}$s on edges does not exceed the number of $\frac{1}{12}$s.

To this end, let wv_1 be a $(\Delta, 3)$-edge in the boundary of an N-face f_0, where $d(w) = \Delta$. By Lemma 3, the other face f_1 incident with wv_1 is of type Z_3 or P. If it is P then it gives $\frac{1}{12}$ to wv_1 and we are done; so suppose it is Z_3. Then f_1 has an edge $wv_2 (v_2 \neq v_1)$ where $d(v_2) = 3$. Again, the other face f_2 incident with wv_2 is of type Z_3 or P. Continue in this way around w until we reach an edge wv_i between a Z_3-face f_{i-1} and a P-face f_i. Then wv_i receives $\frac{1}{12}$ from P, which matches the $-\frac{1}{12}$ on wv_1. Since every $-\frac{1}{12}$ is matched with a different $\frac{1}{12}$ in this way, the proof of (c) is complete.

Finally we prove (e). The truth of (d) means that in proving (e) we may assume $\Delta = 3$. A $3(k)$-*vertex* is a 3-vertex adjacent to exactly k 2-vertices.

LEMMA 4. *Suppose* $\Delta = 3$. *Then*

(i) no 2*-vertex is adjacent to two* 2*-vertices;*
(ii) no 2*-vertex is adjacent to a* 2*-vertex and a* 3(2)*-vertex;*
(iii) no 3*-vertex is adjacent to three* 2*-vertices.*

PROOF. (i) is an easy exercise for the reader (cf. Lemma 3). The proof of (ii) and (iii) involves the following *recolouring procedure*. Let $tuvwx$ and vyz be paths in G (distinct letters representing distinct vertices) where $d(w) = 2$. Suppose we have a total 4-colouring of G except for a possible clash of colours at u (i.e. u and its neighbouring edges do not necessarily all have different colours). Suppose w.l.o.g. uv, v, vw, vy are coloured $1, 2, 3, 4$ respectively. Then we can change the colours of (some of) uv, v, vw and w as in Table 1, according to the colours of w, wx, x and y, without creating a clash anywhere other than at u. To verify Table 1, it suffices to check that wx and x do not change; that any two occurrences of the same colour in the same line of the table are separated by at least two other colours (columns wx and x being repeated for this purpose); that colour 4 is not used on uv, v or vw;

TABLE 1.

Case	Old colour				New colour (changes marked)					
	w	wx	x	y	uv	v	vw	w	wx	x
1	1	2	3	?	$\underline{3}$	2	$\underline{1}$	$\underline{4}$	2	3
2	1	$\frac{2}{4}$	$\frac{4}{2}$	?	$\underline{3}$	2	$\underline{1}$	$\underline{3}$	$\frac{2}{4}$	$\frac{4}{2}$
3	1	4	3	1	$\underline{2}$	$\underline{3}$	$\underline{1}$	$\underline{2}$	4	3
4	1	4	3	3	$\underline{2}$	$\underline{1}$	3	$\underline{2}$	4	3
5	4	1	?	1	1	$\underline{3}$	$\underline{2}$	4	1	?
6	4	1	?	3	$\underline{3}$	$\underline{1}$	$\underline{2}$	4	1	?
7	4	2	?	?	$\underline{3}$	2	$\underline{1}$	4	2	?

and that if v changes then it is different from y. (If $v(w)$ does not change then the colour of $y(x)$ is irrelevant.)

To prove (ii), suppose that $d(t) = d(u) = 2$. Colour $G - tu$ and erase the colours on t and u. Then there are two colours available for each of t, tu and u, and the only problem arises if they are the same two colours in each case. In this case apply the above recolouring procedure, which changes the ordered pair of colours on uv and v. Now we can colour t, tu and u.

To prove (iii), suppose that $d(u) = d(y) = 2$. Colour $G - tu$ and erase the colour on u. If the only colour that can be assigned to tu is the same as that of uv, apply the recolouring procedure. This will change the colour of uv except in Case 5; in this case interchange the roles of the w and y branches (and hence of colours 3 and 4) to get into Case 1, 2 or 4 before doing the recolouring. Then we can colour tu.

We can now colour u unless it touches four different colours. In this case tu must have the same colour as vw or vy, w.l.o.g. vy. Applying the recolouring procedure will change the unordered pair of colours assigned to uv and v (without making uv the same as tu) except in Case 4; in this case uncolour vy, vw and w and then colour uv with 3, v with 4, and each of vy, vw and w in turn with 1 or 2. At last we can colour u.

To complete the proof of (e), let G_{23} be the bipartite subgraph of G comprising V and all edges of G that join a 2-vertex to a 3-vertex. Then G_{23} has no isolated 2-vertices, by Lemma 4(i), and maximum degree at most 2, by Lemma 4(iii), and any component of it that is a path with more than one edge must end in two 3-vertices by Lemma 4(ii). It follows that $n_3 \geq n_2$, whereas Lemma 1(ii) with $t = 10$ forces $n_2 > n_3$. This contradiction completes the proof of the Theorem. □

3. OTHER SURFACES OF NON-NEGATIVE CHARACTERISTIC

The only argument in the previous section that depends on G being planar is the proof of Lemma 1. However, this works equally well for graphs in the projective plane, and so the results of (c), (d) and (e) all hold for such graphs. For graphs in the torus and Klein bottle, Lemma 1 holds with weak inequality ($\leq$) instead of strict inequality ($<$). This is enough for the proofs of (c) and (d), but it is not enough for (e). Thus we need more work to prove (e).

By the last paragraph of the previous section, any graph $G = (V, E)$ in the torus or Klein bottle that is a minimal counterexample to (e) must have a very restricted form, with $n_2 = n_3$, and with each component of G_{23} being either a copy of K_2 or a 2-alternating cycle. Recall the definition of 2-*alternating cycle* from the proof of Lemma 2, and observe that the proof there of their nonexistence does not work when $\Delta = 3$ since it assumes $\Delta + 1 \geq 5$. However, the result remains true:

LEMMA 5. *G contains no 2-alternating cycles.*

PROOF. Let C be a 2-alternating cycle with vertices $v_1, u_1, v_2, \ldots, v_k, u_k, v_1$ where each vertex v_i is adjacent also to v_i'. By the minimality of G, there is a total 4-colouring c of $G - \{u_1, u_2, \ldots, u_k\}$. Recolour each vertex v_i if necessary so that $c(v_i) \notin \{c(v_i'), c(v_i v_i'), c(v_{i+1}v_{i+1}')\}$ (subscripts modulo k). The edges of C can now be coloured as in Lemma 2. If all vertices v_i have the same colour, then the vertices u_i are easily coloured since each touches at most three other colours. So suppose that not all vertices v_i have the same colour, say $c(v_k) \neq c(v_1)$. In this case we recolour the edges of C more carefully.

Since $c(v_k) \notin \{c(v_1), c(v_1 v_1')\}$, we can define $c(u_k v_1) := c(v_k)$, and now $c(v_1 u_1)$ is uniquely determined. For each i in turn $(2 \leq i \leq k)$, define $c(u_{i-1}v_i) := c(v_{i-1})$ if possible; if not, since $c(v_{i-1}) \neq c(v_i v_i')$, necessarily $c(v_{i-1}) = c(v_i)$, and we choose $c(u_{i-1}v_i) \notin \{c(v_{i-1}u_{i-1}), c(v_i), c(v_i v_i')\}$. In each case, $c(v_i u_i)$ is uniquely determined. Note that $c(v_k u_k) \neq c(v_k) = c(u_k v_1)$, so that we have a proper colouring of the edges of C. Now each vertex u_i touches at most three other colours, and so the vertices u_i are easily coloured, and this contradicts the choice of G as a counterexample to the theorem.

It follows from Lemma 5 and the remarks preceding it that the subgraph G_3 of G induced by the 3-vertices is 2-regular and hence is a union of disjoint cycles—call them *D-cycles*—and the vertices of G_3 are connected in pairs by paths of three edges lying outside G_3—call them *D-paths*—whose internal vertices both have degree 2. Contract every D-path to form a 4-regular graph G'. Recall that G has girth $g \geq 10$. If $G' \cong K_5$ then G has a single D-cycle of length 10, which is impossible since one cannot connect two vertices of a 10-cycle by a path of three edges without creating a shorter cycle. Thus $G' \ncong K_5$, and it follows from Brooks's theorem that G' is vertex-4-colourable. Hence the vertices of all the D-cycles in G can be 4-coloured in such a way that the end vertices of each D-path have the same colour. That the edges of the D-cycles can now be coloured is precisely Lemma 1 in [9] (or, alternatively, it can be deduced from the well-known fact [7, 15] that a cycle is (vertex) list-2-colourable unless it has odd length and every vertex has the same choice of two colours). For each D-path $v_0v_1v_2v_3$, the colours of v_0v_1 and v_2v_3 are now uniquely determined. We can set $c(v_1v_2) := c(v_0) = c(v_3)$, and now v_1 and v_2 are easily coloured. Thus G has a total 4-colouring, and this contradiction completes the proof of (e) for graphs in the torus and Klein bottle. □

4. FURTHER EXTENSIONS

The *maximum average degree* $\text{mad}(G)$ of a graph G is the maximum value of $2|E(H)|/|V(H)|$ taken over all subgraphs H of G. We proved in [4] that $\chi'' = \Delta + 1$ for every graph G with $\Delta \geq 7$ and $\text{mad}(G) \leq 4$. This implies both (b) and its extensions to other surfaces, since if G is a graph with girth $g \geq 4$ embedded in a surface of non-negative characteristic then $\text{mad}(G) \leq 4$.

In a similar way, we can extend (d) and (e). Note that if $\text{mad}(G) \leq 3$ then

$$\sum i n_i = 2|E(G)| \leq 3|V(G)| = 3 \sum n_i,$$

so that $n_2 \geq n_4 + 2n_5 + 3n_6 + \cdots$; and if $\text{mad}(G) \leq 5/2$ and $\Delta = 3$ then a similar argument gives $n_2 \geq n_3$. But in proving (d) and (e), the only use we made of the embedding of G was when proving these inequalities. Thus our proofs show also that $\chi'' = \Delta + 1$ for every graph G with $\Delta \geq 4$ and $\text{mad}(G) \leq 3$, or with $\Delta \geq 3$ and $\text{mad}(G) \leq 5/2$.

It is not so easy to obtain analogous extensions of (a) and (c). Problems of this type are discussed in [4].

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