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Coloring Relatives of Intervals on the Plane, I: Chromatic Number Versus Girth

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For the intersection graphs of intervals, rays and strings on the plane, we estimate maximum chromatic number in terms of girth.

1. INTRODUCTION

Asplund and Grünbaum [1] and Gyárfás and Lehel [2] started studying many interesting problems on the chromatic number of intersection graphs of intervals and their relatives on the plane. A number of these problems can be obtained in the following framework. For a class $\mathcal{G}$ of intersection graphs and for positive integer k, $k \geq 2$ find or bound

(1) $f(\mathcal{G}, k)$, the maximum chromatic number of a graph in $\mathcal{G}$ with a clique number of at most k; and
(2) $g(\mathcal{G}, k)$, and the maximum chromatic number of a graph in $\mathcal{G}$ with a girth of at least k (here we assume $k \geq 4$).

Note that $f(\mathcal{G}, 2) = g(\mathcal{G}, 4)$. In [3] we studied $f(\mathcal{G}, k)$ for several families $\mathcal{G}$, and the current paper concerns $g(\mathcal{G}, k)$. Our objects are the families:

$\mathcal{I}$ —the intersection graphs of intervals on the plane;
$\mathcal{R}$ —the intersection graphs of rays on the plane;
$\mathcal{S}$ —the intersection graphs of such families of strings (i.e. arcs) on the plane that the intersection of any two strings is a connected subset of the plane.

Obviously, $\mathcal{R} \subseteq \mathcal{I} \subseteq \mathcal{S}$ and thus $f(\mathcal{R}, k) \leq f(\mathcal{I}, k) \leq f(\mathcal{S}, k)$ and $g(\mathcal{R}, k) \leq g(\mathcal{I}, k) \leq g(\mathcal{S}, k)$ for every k. Note also that $f(\mathcal{G}, 2) = g(\mathcal{G}, 4)$ for every family $\mathcal{G}$. The results in McGuiness [5] on coloring intersection graphs of arcwise connected sets imply that $f(\mathcal{R}, k) < \infty$ for each integer k. No such facts are known for $f(\mathcal{I}, k)$ and $f(\mathcal{S}, k)$ and, in fact, the following problems motivated our research:

PROBLEM 1. (Erdös, see [2]). Is $g(\mathcal{I}, 4) < \infty$?

PROBLEM 2. (Kratochvil and Nešetřil; see e.g. [4]). Is $g(\mathcal{S}, 4) < \infty$?

These problems are presently open.

Perhaps somewhat surprisingly we show here that already for $g(\mathcal{S}, 5)$ the situation is more optimistic:

THEOREM 1.

$$g(\mathcal{S}, k) \leq \begin{cases} 6, & k \geq 5; \\ 4, & k \geq 6; \\ 3, & k \geq 8. \end{cases} \tag{1}$$

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Obviously the bound for girth ≥ 8 is the best possible but for the two remaining bounds this is not clear. We state this as

PROBLEM 3. Does there exist a graph of girth ≥ 5 (i.e. triangle and rectangle free) which is an intersection graph of strings (or intervals) and which satisfies $\chi(G) > 3$? (This is equivalent to saying that $g(\mathcal{S}, 5) > 3$.

For $\mathcal{R}$, we improve the bounds of Theorem 1 as follows.

THEOREM 2. *For any integer* $k \geq 6$*,*

$$g(\mathcal{R}, k) = 3;$$

and $g(\mathcal{R}, 5) \leq 4$*.*

The following subfamilies of $\mathcal{I}$ and $\mathcal{R}$ will also be considered:

$\mathcal{I}_m$ —the intersection graphs of intervals on the plane parallel to some m lines;
$\mathcal{R}_m$ —the intersection graphs of rays on the plane parallel to some m lines.

In particular, we are able to prove stronger bounds for $g(\mathcal{I}_2, 5)$ and $g(\mathcal{R}_2, 5)$ than those for $g(\mathcal{I}, 5)$ and $g(\mathcal{R}, 5)$, respectively.

THEOREM 3. $g(\mathcal{I}_2, 5) \leq 5$.

THEOREM 4. $g(\mathcal{R}_2, 5) = 3$.

It is slightly surprising that long cycles do not belong to $\mathcal{R}_m$.

THEOREM 5. *Cycle* C_n *of length* n *belongs to* $\mathcal{R}_m$ *iff* $3 \leq n \leq 6m$*.*

This fact together with Theorems 4 and 2 and the fact that $f_1(\mathcal{R}_2, 2) = 4$, proved in [3], give the exact values of $g(\mathcal{R}_2, k)$ and exact values for $g(\mathcal{R}_m, k)$, $k \geq 6$:

COROLLARY.

$$(1)\ g(\mathcal{R}_2, k) = \begin{cases} 4, & k = 4; \\ 3, & 5 \leq k \leq 11; \\ 2, & k \geq 12, \end{cases}$$

$$(2)\ g(\mathcal{R}_m, k) = \begin{cases} 3, & 6 \leq k \leq 6m - 1; \\ 2, & k \geq 6m. \end{cases}$$

2. NOTATIONS AND PRELIMINARIES

Let P be a Cartesian plane. All our strings, intervals, rays and lines are supposed to be subsets of P. By a *string* we mean a non-closed continuous curve on the plane without self-crossings, i.e. the image $s = s(f)$ of a one-to-one continuous mapping $f : [0, 1] \to P$. It will be convenient to consider the point $f(0)$ as the *origin* $o(s)$ of s, and the point $f(1)$ as its *terminal* $t(s)$. Intervals are also considered to have origins and terminals.

Each ray r can be represented by the quadruple (x, y, u, w), where the point (x, y) is its *origin* $o(r)$ and (u, w) is its *vector* $v(r)$. In other words, $r = \{(x, y) + \lambda \cdot (u, w) \mid \lambda \geq 0\}$.

Say that a family F of intervals, rays or lines is an *m-direction family* if there are m (straight) lines $l_1, \ldots, l_m$ such that any member of F is parallel to some l_i, $1 \leq i \leq m$.

For a family F of subsets of P, its *intersection graph* $G = G_F$ is the undirected graph with the vertex set F such that for $r, p \in V$,

$$(r, p) \in E(G) \leftrightarrow r \cap p \neq \emptyset.$$

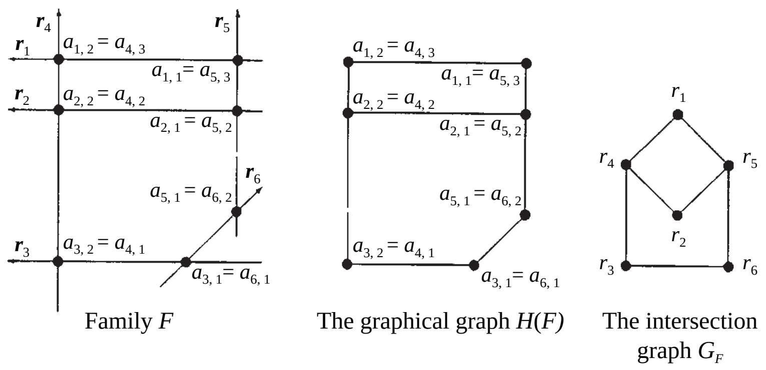

FIGURE 1. A ray family F and corresponding graphs.

Certainly, for a graph G, there could be very different families F' and F'' such that $G = G_{F'} = G_{F''}$. Any such family is called a *representation* of G.

We denote by $\mathcal{S}$ the collection of intersection graphs of string families on P such that the intersection of any two strings is a connected subset of P. Evidently, any intersection graph of intervals and rays belongs to $\mathcal{S}$. The intersection graphs of L*-shapes* described by Gyàrfàs and Lehel [2] also belong to $\mathcal{S}$.

We are interested in triangle-free graphs. Clearly for any string representation F of a triangle-free graph $G \in \mathcal{S}$, each point of P belongs to at most two strings. Moreover, if s_1 and s_2 are intersecting strings and $q = s_1 \cap s_2$ (q is a connected curve by the definition of S) then no point of q belongs to any other member of F. Thus, contracting q into a point we obtain a family with the same intersection graph. It follows that for each triangle-free graph $G \in \mathcal{S}$, there exists a string representation $U(G)$ such that *any two strings have at most one point in common*. This also refers to ray representations. We call such representations U*-representations* (by strings, intervals or rays).

Let $F = \{s_1, \ldots, s_n\}$ be a U-representation of a triangle-free graph $G \in \mathcal{S}$. Let for $i = 1, \ldots, n$, the points $a_{i,1}, \ldots, a_{i,q_i}$ be the common points of s_i with other members of F numbered starting from its origin $o(s_i)$. Since G has no triangles, any $a_{i,j}$ lies on exactly two strings. Now we can define *the graphical graph* $H[F]$ as follows. The vertex set of H is the union $\bigcup_{i=1}^{n}\{a_{i,1}, \ldots, a_{i,q_i}\}$ and the edge set of H is $\{(a_{i,j}, a_{i,j+1}) \mid 1 \le i \le n, 1 \le j \le q_i\}$. An example (for a ray-intersection graph) is given in Figure 1.

Note that $H(F)$ is considered to be a *plane graph*, i.e. a graph embedded into P, and its edges are the parts of strings connecting $a_{i,j}$ with $a_{i,j+1}$. Let v, e and f denote the number of vertices, edges and faces in $H(F)$, and m denote the number of edges in $G = G_F$. Each $a_{i,j}$ corresponds to an edge in G_F. Hence

$$v = m. \tag{2}$$

Each string s_i, $i = 1, \ldots, n$ produces $q_i - 1$ edges in $H(F)$, and so

$$e = \sum_{i=1}^{n}(q_i - 1) = 2v - n. \tag{3}$$

3. COLORING STRING GRAPHS

If the class of string graphs $\mathcal{S}$ contains a k-chromatic graph of girth l then it also contains a k-vertex-critical graph G of girth l. Since its minimum degree is at least $k - 1$, we have

$$m \ge n(k - 1)/2. \tag{4}$$

Let $F = \{s_1, \ldots, s_n\}$ be a U-representation of G. Then each cycle in $H(F)$ corresponds to a closed walk in G such that any edge is traversed at most once. Hence the girth of $H(F)$ is at least l, and

$$2e \geq l \cdot f. \tag{5}$$

From Euler's formula, and (5) we obtain

$$v - e + 2e/l > 0.$$

Hence by (2) and (3),

$$m/n < \frac{l-2}{l-4},$$

and comparing this with (4), we have

$$k - 1 < \frac{2(l-2)}{l-4}. \tag{6}$$

For $l = 5$, (6) gives $k \leq 6$, for $l = 6$, it gives $k \leq 4$, and for $l = 8$ it gives $k \leq 3$. Theorem 1 is proved. □

4. Coloring Rays

Analogously to the previous section, let G be a k-vertex-critical ray-intersection graph of girth l, and let $F = \{r_1, \ldots, r_n\}$ be a U-representation by rays. We may assume $k \geq 3$.

Lemma 1. *The size of the outer face of $H(F)$ is at least n.*

Proof. We associate with each ray $r_i \in F$ an edge $d_i \in E(H(F))$ on the boundary of the outer face of $H(F)$ in the following way:

(1) If the degree of $a_{i,q(i)}$ in $H(F)$ is 2 then we put $d_i = (a_{i,q(i)-1}, a_{i,q(i)})$.
(2) If the degree of $a_{i,q(i)}$ in $H(F)$ is 3 then for some $j \neq i$ and $2 \leq l \leq q(j) - 1$, $a_{i,q(i)} = a_{j,l}$. In this case, we put $d_i = (a_{j,l-1}, a_{j,l})$.

It is easy to see that all d_is are distinct edges of the outer face. □

Now, instead of (5) by Lemma 1

$$2e \geq g(f-1) + n,$$

and hence

$$f \leq 1 + (4v - 3n)/g. \tag{7}$$

From Euler's formula and (3) we have

$$v - (2m - n) + f = 2,$$

which together with (7) gives $2 + (m - n) \leq 1 + (4m - 3n)/g$, and $g < (4m - 3n)/(m - n)$. In other words

$$m/n < (g-3)/(g-4). \tag{8}$$

Comparing (8) with (4), we obtain

$$k - 1 < 2(g-3)/(g-4).$$

This gives $k \leq 4$ for $g = 5$ and $k \leq 3$ for $g \geq 6$. Thus, Theorem 2 is proved. □

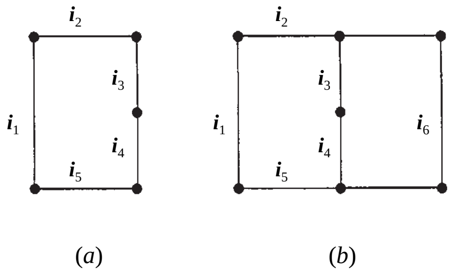

FIGURE 2. Girth 5-faces.

5. Two Direction Families of Girth Five

Let a k-vertex-critical graph $G \in \mathcal{I}_2$ have girth five and F be its U-representation by horizontal and vertical intervals. Since geometrically any face in $H(F)$ is a polygon with an even number of corners, each 5-face in $H(F)$ is a 4-gon which one side contains the common point of some two vertical or two horizontal members of F (see Figure 2(a)).

Moreover, a common point of two vertical or two horizontal members of F cannot belong to two 5-faces as in Figure 2(b); for otherwise the intervals $\boldsymbol{i}_1$, $\boldsymbol{i}_2$, $\boldsymbol{i}_6$ and $\boldsymbol{i}_5$ form a 4-cycle in G. Thus the number of 5-faces in $H(F)$ does not exceed the number p of common points of intervals in the same direction. Evidently, $p < n$. Consequently, instead of (5) we have

$$2e \geq 6f - p > 6f - n. \tag{9}$$

Now, proceeding along the lines of the proof of Theorem 1, we obtain $m/n < 5/2$ and then $k - 1 < 5$. This proves Theorem 3. □

Let F be a U-representation of a 4-vertex-critical graph $G \in \mathcal{R}_2$ of girth at least five by horizontal and vertical rays. Any component of the subgraph of G induced by horizontal (resp., vertical) rays has either one or two vertices. In the latter case, the union of the corresponding rays must be a line, and the intersection must be a point. We will call such a line l an *F-line* with the origin $o(l)$ and two rays l^+ and l^-. The signs $+$ and $-$ are taken in accordance with the orientation of the axes of P.

By Lemma 1, in our case, inequality (9) can be refined as follows:

$$2e \geq 6(f - 1) - p + n,$$

and hence

$$3n + p - 6 \geq 2m.$$

The last inequality is equivalent to the following:

$$\sum_{x \in V(G)} (\deg_G x - 3) \leq p - 6. \tag{10}$$

Recall that each summand $\deg_G x - 3$ is non-negative, since G is 4-critical.

LEMMA 2. *If F forms at most two horizontal F-lines or at most two vertical F-lines then the chromatic number of G_F is at most three.*

PROOF. Consider the case when there are exactly two horizontal F-lines l_1 and l_2. The case when there is at most one horizontal F-line is analogous and even easier. Let $o(l_i) = (a_i, b_i)$, $l = 1, 2$. We may assume that $a_1 \leq a_2$. For each vertical F-line l, one can choose a ray $r(l) \in \{l^+, l^-\}$ disjoint from both l_2^+ and l_1^-. We can color by 1 the set $\{l_2^+, l_1^-\} \cup \{r(l) \mid l$ is a vertical$F$-line$\}$. The remaining horizontal rays we color by 2, and vertical rays by 3. □

LEMMA 3. *Let F have p_1 horizontal F-lines and p_2 vertical F-lines. Then* $\max\{p_1, p_2\} \le 4$.

PROOF. Assume that $p_1 \le p_2$, $p_2 \ge 5$. By Lemma 2, $p_1 \ge 3$.

Let l be a horizontal F-line. Then l intersects at least p_2 horizontal members of F, and the vertices l^+ and l^- in G are adjacent. Hence $\deg_G l^+ + \deg_G l^- \ge p_2 + 2$. Consequently, $\sum_{x \in V(G)} (\deg_G x - 3) \ge p_1(p_2 - 4)$, and by (10),

$$p_1(p_2 - 4) \le p_1 + p_2 - 6.$$

This is impossible for $p_1 \ge 3$, $p_2 \ge 5$. □

Denote by V_4 the set of vertices of G of degree at least 4. By Gallai's theorem on critical graphs, each block of $G - V_4$ is either a complete graph or an odd cycle. By Lemma 3 and (10), $|V_4| \le 2$. Thus, for any $x \in V(G) \setminus V_4$, $\deg_{G-V_4} x \ge 3 - 2 = 1$. Moreover, if there are two vertices x and y of degree 1 in $G - V_4$ then $|V_4| = 2$ and both x and y are adjacent to both elements of V_4. But this implies the existence of a 4-cycle in G, a contradiction.

Consequently, any pendant block in $G - V_4$ is an odd cycle. Let x, y and z be three consecutive vertices on such a cycle having degree two in $G - V_4$. Since $|V_4| \le 2$, there exists $u \in V_4$ adjacent to at least two of x, y and z. This again implies the existence of a 3-cycle or a 4-cycle in G, a contradiction. Theorem 4 is proved. □

6. RAY-INTERSECTION FAMILIES OF LARGE GIRTH

As rays (unlike lines) are oriented configurations, we can consider *oriented directions*. Let $\mathcal{R}^m$ denote the set of intersection graphs of translates of some fixed m rays. Obviously, $\mathcal{R}_m \subset \mathcal{R}^{2m}$ for any m. Thus the 'only if' part of Theorem 5 is implied by the 'only if' part of the following fact.

THEOREM 6. *Cycle C_n of length n belongs to $\mathcal{R}^m$, $m \ge 3$, iff* $3 \le n \le 3m$.

The 'if' parts of both theorems follow from the following construction.

CONSTRUCTION. Choose m oriented directions (vectors) $d_i = (\cos\frac{2\pi i}{m}, \sin\frac{2\pi i}{m})$, $i = 0, \dots, m-1$. Note that for even m, the vector d_i is opposite to $d_{i+m/2}$, $i = 0, \dots, m/2 - 1$. The origins of the three rays in the direction d_i, $i = 0, \dots, m-1$ are $(\cos\frac{2\pi i}{m}, \sin\frac{2\pi i}{m})$, $(\cos\frac{2\pi(i+1)}{m}, \sin\frac{2\pi(i+1)}{m})$ and $(2\cos\frac{2\pi(i-1)}{m}, 2\sin\frac{2\pi(i-1)}{m})$. This gives a realization of C_{3m}. It is easy to change it slightly to obtain realizations of C_{3m-1} and C_{3m-2}.

An example (for $m = 4$) is given on Figure 3.

To prove the 'only if' part of Theorem 6, assume that the rays $r_1, \dots, r_n$ in m oriented directions form a representation of C_n, $n > 4$. For each i, $i = 1, \dots, n$, the set $P \setminus (r_i \cup r_{i+1})$ has two connected parts: $P_1(i)$ (containing the origins of r_i and r_{i+1}) and $P_2(i)$. An example is given in Figure 4.

The following fact is obvious but helpful.

OBSERVATION. Either all r_j, $j \in \{1, \dots, n\} \setminus \{i-1, i, i+1, i+2\}$ are contained in $P_1(i)$ or all of them are contained in $P_2(i)$.

Now we prove the final lemma which yields the 'only if' part of Theorem 6.

LEMMA 4. *At most three rays go in each oriented direction.*

PROOF. Assume that there are four rays, r_{i_1}, r_{i_2}, r_{i_3} and r_{i_4} (numbered from left to right), going downward.

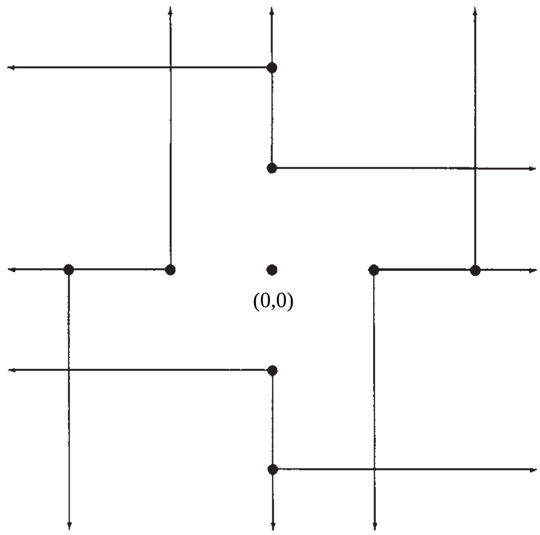

FIGURE 3. 4 oriented directions.

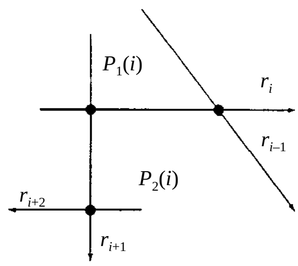

FIGURE 4.

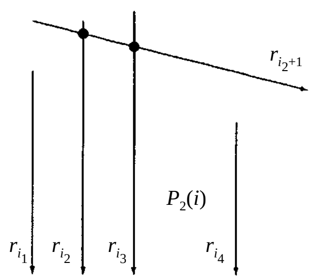

FIGURE 5. Case 1.

Case 1. The first coordinate of the vector of r_{i_2+1} is positive (i.e. it goes to the right-hand side of r_{i_2}) (see Figure 5).

Since G is 2-regular, r_{i_2+1} does not meet either r_{i_3} or r_{i_4}. So one of them (say, r_{i_4}) is contained in $P_2(i_2)$. But then either r_{i_1} is contained in $P_1(i_2)$ of (if $i_1 = i_2 + 2$) r_{i_1+1} is contained in $P_1(i_2)$. This contradicts the Observation.

Because of the symmetry between i_2+1 and i_2-1, it is enough now to consider the following situation.

Case 2. The first coordinates of vectors of both r_{i_2+1} and r_{i_2-1} are negative (i.e. the rays go to the left-hand side of r_{i_2}).

Since $n > 4$, at least one of r_{i_2+1} and r_{i_2-1} (say, r_{i_2+1}) misses r_{i_1}. It must also miss either r_{i_3} or r_{i_4} (say, r_{i_4}). But then r_{i_1} is contained in $P_2(i_2)$ and r_{i_4} is contained in $P_1(i_2)$. This is the final contradiction. □

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