Europ. J. Combinatorics (1999) **20**, 469–481
Article No. eujc.1998.0295
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Regular Honest Graphs, Isoperimetric Numbers, and Bisection of Weighted Graphs

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The *edge-integrity* of a graph G is $I'(G) := \min\{|S| + m(G - S) : S \subset E\}$, where $m(H)$ denotes the maximum order of a component of H. A graph G is called *honest* if its edge-integrity is the maximum possible; that is, equals the order of the graph. The only honest 2-regular graphs are the 3-, 4-, and 5-cycles. Lipman [13] proved that there are exactly twenty honest cubic graphs. In this paper we exploit a technique of Bollobás [8, 9] to prove that for every $k \geq 6$, almost all k-regular graphs are honest. On the other hand, we show that there are only finitely many 4-regular honest graphs. To prove this, we use a weighted version of the upper bound on the isoperimetric number due to Alon [1]. We believe that this version is of interest by itself.

1. INTRODUCTION

There are several parameters that measure connectivity and vulnerability of graphs. One of them is the edge-integrity introduced by Barefoot *et al.* [6, 7].

DEFINITION 1. The *edge-integrity* of a graph G is

$$I'(G) := \min\{|S| + m(G - S) : S \subset E\},$$

where $m(H)$ denotes the maximum order of a component of H.

DEFINITION 2. A graph G is called *honest* if its edge-integrity is the maximum possible; that is, equal to the order of the graph.

This definition was introduced by Bagga *et al.* [4]. They proved the following [4, 5]:

THEOREM A. *Every graph of diameter 2 is honest.*

THEOREM B. *With the exception of the path of length 3, either G or the complement graph $\bar{G}$ is honest.*

It is easy to see that only 3-, 4-, and 5-cycles are honest 2-regular graphs. Lipman [13] proved:

THEOREM C. *There are exactly twenty honest cubic graphs.*

In [14], Lipman studied the existence of *sparse* honest graphs, i.e., graphs having an average degree less than $\log_2 n$, where n is the number of vertices. He introduced a sufficient condition for honesty (see Theorem 8 of Section 2.2). With the help of this theorem he proved that the Kneser graph $K(7, 3)$ is honest. This is the largest sparse honest graph constructed in [14].

In this paper we continue studying honest sparse graphs. It appears that there are many honest graphs with a constant average degree and an arbitrarily large number of vertices.

†Research supported in part by a USA Israeli BSF grant and by the Fund for Basic Research administered by the Israel Academy of Sciences.
‡This work was partially supported by an Indiana University-Purdue University Fort Wayne Scholar-in-Residence Grant, and by the grant 97-01-01075 of the Russian Foundation for Fundamental Research.

0195–6698/99/060369 + 13 $30.00/0

It is not difficult to construct explicitly large, bounded-degree honest graphs, using the known constructions of expanders. Indeed, by the relation between the spectral properties of a graph and its expansion properties (see, e.g., [2]), if G is a d-regular graph on n vertices and λ is the second largest eigenvalue of its adjacency matrix, then for any set U of m vertices of G, the number of edges between U and its complement is at least $\frac{(d-\lambda)m(n-m)}{n}$. It follows from Theorem 8 of Section 2.2 that if $d - \lambda \geq 2$, then G is honest. In [15, 16], for each prime $p \equiv 1 \pmod 4$, an infinite explicit family of d-regular graphs whose second largest eigenvalue is at most $2\sqrt{d-1}$ is constructed. Thus, for example, by packing two such 5-regular graphs together we obtain explicitly infinitely many 10-regular honest graphs. Our first result here shows that degree 10 is not the best possible.

THEOREM 3. *For every* $k \geq 6$, *almost all* k*-regular graphs are honest.*

On the other hand, we prove:

THEOREM 4. *Any honest graph with maximum degree* 4 *has at most* 10^{60} *vertices.*

The case of 5-regular graphs remains unsettled.

Our main tool in the proof of Theorem 4 is the following extension of the main result of [1].

THEOREM 5. *Let* $G = (V, E)$ *be a multigraph with maximum degree d on n vertices, where n is even and* $n > 40d^9$. *Then there is a partition* $V = V_- \cup V_+$, *where* $|V_-| = |V_+| = n/2$ *such that*

$$e(V_-, V_+) \leq \frac{|E|}{2}\left(1 - \frac{3}{8\sqrt{2d}}\right), \tag{1}$$

where $e(V_-, V_+)$ *is the total number of edges between* V_- *and* V_+.

This inequality is a particular case of the following weighted version of the main result in [1].

Let $G = (V, E)$ be a simple weighted graph; that is, a graph with no loops and no multiple edges, with a non-negative weight $w(e)$ assigned to each edge. Assume $V = \{1, 2, \ldots, n\}$ and let d_i denote the degree of i. For two disjoint subsets U, U' of V, let $w(U, U')$ denote the total weight of the edges between U and U'.

For any positive integer k, define

$$\epsilon_{2k} = \epsilon_{2k+1} = \frac{\binom{2k}{k}}{2^{2k+1}}.$$

It is not difficult to check, as is done in [17], that for every positive integer d, $\epsilon_d \geq \frac{1}{2\sqrt{2}\sqrt{d}}$.

THEOREM 6. *Let* $G = (V, E)$ *be a weighted graph as above, where* $V = \{1, 2, \ldots, n\}$, n *is even and* d_i *is the degree of* i. *If* $n > 40d_i^9$, *then there is a partition* $V = V_- \cup V_+$ *where* $|V_-| = |V_+| = n/2$ *such that*

$$w(V_-, V_+) \leq \sum_{ij \in E} \frac{w(ij)}{2}\left(1 - \frac{3}{8}\epsilon_{d_i} - \frac{3}{8}\epsilon_{d_j}\right) \leq \sum_{ij \in E} \frac{w(ij)}{2}\left(1 - \frac{3}{16\sqrt{2d_i}} - \frac{3}{16\sqrt{2d_j}}\right). \tag{2}$$

The idea of the proof is that of [1] with two twists. We believe that Theorem 6 is of independent interest.

The structure of the paper is as follows: in the next section we introduce notation and discuss related results. In Section 3 we prove Theorem 3. In Section 4 we prove Theorem 4 using Theorem 5. The last section is devoted to the proof of Theorem 6 which immediately implies Theorem 5.

2. Background

2.1. (n, r)-configurations. For $r \geq 3$ and $n > r$, let $\mathbf{G}(n, r - reg)$ denote the set of all r-regular graphs with vertex set $V = \{1, 2, \ldots n\}$. We always assume that $rn = 2k$ is an even number, and so k is the number of edges in a graph. We say that *almost all r-regular graphs have a certain property Q* if the portion of graphs in $\mathbf{G}(n, r - reg)$ not possessing Q is $o(|\mathbf{G}(n, r - reg)|)$.

It is not too easy to calculate $|\mathbf{G}(n, r - reg)|$ (see e.g., [9]). In order to facilitate studying $\mathbf{G}(n, r - reg)$, Bollobás [8] (for a more detailed description see [9]) introduced a very convenient model of (n, r)-configurations.

Let $W = \bigcup_{j=1}^{n} W_j$ be a fixed set of $2k = rn$ labeled vertices, where $|W_j| = r$ for each j. An (n, r)-*configuration* F is a partition of W into k pairs of vertices, called *edges* of F. Let Φ be the *set of* (n, r)-configurations. Clearly

$$|\Phi| = N(k) = (2k - 1)!!.$$

(Recall that for any positive odd integer m, $m!! = m \cdot (m - 2) \cdot \ldots \cdot 3 \cdot 1$.) For $F \in \Phi$, let $\phi(F)$ be the multigraph with vertex set $V = \{1, 2, \ldots, n\}$, in which each i and j are joined by the same number of edges as W_i and W_j are joined in F. In other words, $\phi(F)$ is obtained from F by merging each W_i into a vertex i. Clearly, $\phi(F)$ is an r-regular multigraph on V (sometimes with loops). Most important is the fact that the portion of $F \in \Phi$ such that $\phi(F)$ is a simple graph is at least c_r, where $c_r > 0$, for every sufficiently large n, and each simple graph on V corresponds to the same number of (n, r)-configurations (namely, to $(r!)^n$). Thus if we prove that almost all (n, r)-configurations have a certain property Q, then almost all r-regular graphs have Q as well.

2.2. Edge-integrity vs. isoperimetric number. Another parameter that measures connectivity and vulnerability of graphs is the isoperimetric number of a graph introduced by Buser [11] and studied by several authors, including Bollobás [10]. For a graph G and $U \subset V(G)$, let $f(U)$ denote the number of edges between U and $V(G) \setminus U$.

Definition 7. The *isoperimetric number of G* is

$$i(G) = \min\left\{ \frac{f(U)}{|U|} \ : \ U \subset V \right\},$$

where the minimum is taken over all subsets U of V with $|U| \leq |V|/2$.

The isoperimetric number of G turns out to be related to its edge-integrity, and, thus, to its honesty. It is easy to see that if the isoperimetric number of a graph G is less than 1, then the graph is not honest. Thus, to prove Theorem 4, we shall derive from Theorem 6 that only finitely many 4-regular graphs have isoperimetric number 1 or larger.

On the other hand, the fact that the isoperimetric number of a graph G is greater than 1 does not imply that the graph is honest, as can be seen by the following example. Let the graphs G_1, G_2, and G_3, be three disjoint copies of $K_8 - \{e\}$, the complete graph on eight vertices with a missing edge $e = (a_1, a_2)$, (b_1, b_2), (c_1, c_2), respectively. The degrees of the graphs at these vertices a_i, b_i, and c_i $(i = 1, 2)$ are 6, while all the other vertices have degree 7. We add 15 edges such that between any two disjoint parts G_i, G_j, $(i \neq j, i, j = 1, 2, 3)$ there are five edges connecting them, and the obtained graph G is 8-regular. Now, it is easy to check that G is not honest, and $i(G) = 10/8$.

Still, the following theorem of Lipman (which we exploit in the proof of Theorem 3) implies that each graph G with $i(G) \geq 2$ is honest.

THEOREM 8 (LIPMAN [14]). *Let G be a graph with n vertices. Suppose that for every integer $m \le n/2$ and every set of vertices C with $|C| = m$,*

$$f(C) \ge \left\lceil \frac{2m(n-m)}{n} \right\rceil.$$

Then G is honest.

Bollobás [10] proved the following:

THEOREM 9 ([10]). *Let r and $0 < \eta < 1$ be such that*

$$2^{4/r} < (1-\eta)^{1-\eta}(1+\eta)^{1+\eta}.$$

Then almost all r-regular graphs have isoperimetric number at least $(1-\eta)r/2$.

Theorem 9 implies that for each $k \ge 9$, the isoperimetric number of almost all k-regular graphs is at least 2.06. Since all graphs with isoperimetric number greater than or equal to 2 are honest, it follows that for every $k \ge 9$, almost all k-regular graphs are honest.

3. PROOF OF THEOREM 3

PROOF. Let $r \ge 6$ be a fixed integer and let $mr - f$ be an even integer. Let $t(n, r, m, f)$ be the number of (n, r)-configurations such that a given subset of vertices of size m is connected with the rest by exactly f edges. Then

$$t(n, r, m, f) = \binom{mr}{f}\binom{(n-m)r}{f} f!(mr - f - 1)!!((n-m)r - f - 1)!!.$$

Hence the portion of (n, r)-configurations in which at least one subset of vertices of size m is connected with the rest by at most $2m(n-m)/n$ edges is estimated from above by

$$T(n, r, m) = \binom{n}{m} \sum_{\{f \le 2m(n-m)/n \mid rm - f \text{ is even}\}} \binom{mr}{f}\binom{(n-m)r}{f} \times f!(mr - f - 1)!!((n-m)r - f - 1)!!/(nr-1)!!. \qquad (3)$$

First, let $m \le 100$ and $f < 2m$. Then there exists a number $C = C(m, r)$ such that

$$\binom{n}{m} t(n, r, m, f)/(nr-1)!! \le Cn^m (n-m)^f (nr)^{-(mr+f)/2} < Cn^{(f-(r-2)m)/2}.$$

It follows that for $r \ge 6$ and $m \le 100$, $T(n, r, m) \le 2C\, n^{-m}$.

Now, we consider $100 < m \le n/2$. We show that $t(n, r, m, f)$ is an increasing function in f. For $2m \le f < 2 + 2m(n-m)/n$, consider the ratio

$$\begin{aligned}
\frac{t(n, r, m, f-2)}{t(n, r, m, f)} &= \frac{\binom{mr}{f-2}\binom{(n-m)r}{f-2}(f-2)!(mr - f + 1)!!((n-m)r - f + 1)!!}{\binom{mr}{f}\binom{(n-m)r}{f} f!(mr - f - 1)!!((n-m)r - f - 1)!!} \\
&= \frac{(f-1)f \cdot (f-1)f \cdot (mr - f + 1) \cdot ((n-m)r - f + 1)}{(mr - f + 1)(mr - f + 2)((n-m)r - f + 1)((n-m)r - f + 2)(f-1)f} \\
&= \frac{(f-1)f}{(mr - f + 2)((n-m)r - f + 2)} < 1/4.
\end{aligned}$$

This means that

$$T(n, r, m) < 2\binom{n}{m} t(n, r, m, f_0)/(nr-1)!!, \qquad (4)$$

where f_0 is the maximum integer less than $2m(n-m)/m$ such that $mr - f_0$ is even. Our general aim is to show that $\sum_{m=1}^{\lceil n/2\rceil} T(n,r,m) = o(1)$. We already saw that $\sum_{m=1}^{100} T(n,r,m) = O(1/n)$.

By (4) and Stirling's formula, we have

$$T(n,r,m) < 2\frac{\binom{n}{m}\binom{mr}{f_0}\binom{(n-m)r}{f_0} f_0!(mr-f_0-1)!!((n-m)r-f_0-1)!!}{(nr-1)!!}$$
$$< \frac{2n\, n^n (mr)^{mr}((n-m)r)^{r(n-m)}}{m^m(n-m)^{n-m} f_0^{f_0}(mr-f_0)^{0.5(mr-f_0)}((n-m)r-f_0)^{0.5((n-m)r-f_0)}(rn)^{0.5rn}}.$$

Since the derivative of $f^f(mr-f)^{0.5(mr-f)}((n-m)r-f)^{0.5((n-m)r-f)}$ with respect to f when f is around $2m(n-m)/2$ is negative, substituting $2m(n-m)/2$ instead of f_0 gives an upper bound for $T(n,r,m)$. Dividing both the numerator and the denominator by $n^{(r+1)n}$, we obtain

$$T(n,r,m) < \frac{2n}{(m/n)^m(1-m/n)^{n-m}(2(1-m/n)m/n)^{2m(n-m)/n}}$$
$$\times \frac{(rm/n)^{rm}}{(rm/n-2(1-m/n)m/n)^{0.5rm-(n-m)m/n}}$$
$$\times \frac{(r(1-m/n))^{r(n-m)}}{r^{rn/2}(r(1-m/n)-2(1-m/n)m/n)^{0.5r(n-m)-(n-m)m/n}}.$$

Let $\alpha = m/n$. Then

$$T(n,r,m) < T_1(n,r,\alpha) = \frac{2n\,(r\alpha)^{r\alpha n}}{\alpha^{\alpha n}(1-\alpha)^{(1-\alpha)n}(2\alpha(1-\alpha))^{2\alpha(1-\alpha)n}r^{rn/2}}$$
$$\times\frac{(r(1-\alpha))^{r(1-\alpha)n}}{(\alpha r-2\alpha(1-\alpha))^{(0.5r-(1-\alpha))\alpha n}((1-\alpha)r-2\alpha(1-\alpha))^{(0.5r-\alpha)(1-\alpha)n}}.$$

Let $T_2(n,r,\alpha) = \frac{1}{n}\log(T_1(n,r,\alpha)/(2n))$. We have

$$T_2(n,r,\alpha) = 0.5r\log r + \alpha(0.5r-2+\alpha)\log\alpha + (1-\alpha)(0.5r-1-\alpha)\log(1-\alpha)$$
$$-2\alpha(1-\alpha)\log 2 - \alpha(0.5r-1+\alpha)\log(r-2+2\alpha) - (1-\alpha)(0.5r-\alpha)\log(r-2\alpha).$$

Now we take three derivatives of $T_2(n,r,\alpha)$ with respect to α;

$$\frac{\partial T_2(n,r,\alpha)}{\partial\alpha} = (0.5r-2+2\alpha)\log\alpha - (0.5r-2\alpha)\log(1-\alpha) + (4\alpha-2)\log 2$$
$$-(0.5r-1+2\alpha)\log(r-2+2\alpha) + (0.5r+1-2\alpha)\log(r-2\alpha);$$
$$\frac{\partial^2 T_2(n,r,\alpha)}{\partial\alpha^2} = 2\log\alpha + \frac{0.5r-2+2\alpha}{\alpha} + 2\log(1-\alpha) + \frac{0.5r-2\alpha}{1-\alpha} + 4\log 2$$
$$-2\log(r-2+2\alpha) - \frac{r-2+4\alpha}{r-2+2\alpha} - 2\log(r-2\alpha) - \frac{r+2-4\alpha}{r-2\alpha};$$
$$\frac{\partial^3 T_2(n,r,\alpha)}{\partial\alpha^3} = \frac{2}{\alpha} - \frac{0.5r-2}{\alpha^2} - \frac{2}{1-\alpha} + \frac{0.5r-2}{(1-\alpha)^2} - \frac{4}{r-2+2\alpha} - \frac{2r-4}{(r-2+2\alpha)^2}$$
$$+\frac{4}{r-2\alpha} + \frac{2r-4}{(r-2\alpha)^2} = 2(1-2\alpha)\Bigg(\frac{2\alpha(1-\alpha)-0.5r+2}{2\alpha^2(1-\alpha)^2}$$
$$-\frac{4}{(r-2+2\alpha)(r-2\alpha)} - \frac{4(r-2)(r-1)}{(r-2+2\alpha)^2(r-2\alpha)^2}\Bigg).$$

It is not hard to check that for $0 < \alpha < 0.5$ and $r \geq 6$,

$$\frac{\partial^3 T_2(n, r, \alpha)}{\partial \alpha^3} < 0.$$

It follows that $\frac{\partial T_2(n,r,\alpha)}{\partial \alpha}$ is concave up. Note also that $\frac{\partial T_2(n,r,0.5)}{\partial \alpha} = 0$ for every r. Since a concave up function has at most two zeros, we conclude that either $\frac{\partial T_2(n,r,0.5)}{\partial \alpha}$ is negative on $(0, 0.5)$, or it is first negative and then positive. In other words, either $T_2(n, r, \alpha)$ is monotonically decreasing at $(0, 0.5)$, or it first decreases and then monotonically increases. In both cases, in order to find $\max\{T_2(n, r, \alpha) \mid 100/n \leq \alpha \leq 0.5\}$, it is enough to check the values for $\alpha = 100/n$ and $\alpha = 0.5$. We have

$$T_2(n, r, 0.5) = 0.5(r \log r - (r - 2) \log 2 - (r - 1) \log(r - 1) < -0.02$$

for each $r \geq 6$. It is a routine computation to check that

$$T_2(n, r, 100/n) = -100 \log n\,(0.5r - 2)/n + O(1/n).$$

Thus, for a fixed $r \geq 6$ and large n, $T_2(n, r, \alpha) < -50 \log n$ for any $0 < \alpha < 0.5$. It follows that $T(n, r, m) < 2n^{-49}$ for each fixed $r \geq 6$ and any $100 < m \leq n/2$. This, together with Theorem 8, proves the theorem. □

4. PROOF OF THEOREM 4

We shall use the following fact.

LEMMA 10 ([12]). *Let T be a tree with maximum degree q. Then for any $k \leq |V(T)|$, the vertex set $V(T)$ can be divided into two parts V_1 and V_2 such that:*

(a) $|V_1| = k$;
(b) *the subgraph $T\langle V_2\rangle$ induced by V_2 is a tree;*
(c) *the number of components of $T\langle V_1\rangle$ is at most $1 + \log_{\frac{q-1}{q-2}} k$.*

PROOF. Let $n > 10^{60}$ and $G = (V, E)$ be a multigraph on n vertices with maximum degree 4. Let $k = 10^5$ and $m = \lfloor n/2k \rfloor$. If G is not connected, then it is not honest. Otherwise G has a spanning tree T. Applying Lemma 10 to T, and to the subsequent trees guaranteed by the lemma, $2m$ times for each $j = 1, \dots, 2m$, we find a disjoint subset W_j of V such that

$$|W_j| = k \text{ and } |E_G(W_j)| \geq k - 1 - \lfloor \log_{3/2} k \rfloor > k - 30. \tag{5}$$

Denote $W_0 = V \setminus \bigcup_{j=1}^{2m} W_j$. Let H be obtained from $G - W_0$ by merging each W_j into a vertex, say w_j and deleting loops. Then by (5),

$$\deg_H(w_j) \leq 4|W_j| - 2|E_G(W_j)| \leq 2(k + 30) = 200\,060 \qquad \text{for every } w_j \in V(H).$$

Applying Theorem 5 to H, we conclude that there is a partition (U_+, U_-) of $V(H)$ such that $|U_+| = |U_-| = m$ and

$$\begin{aligned} |E_H(U_+, U_-)| &\leq \frac{|E(H)|}{2}\left(1 - \frac{3}{16\sqrt{100\,030}}\right) \leq 200\,060\frac{m}{2}\left(1 - \frac{1}{2000}\right) \\ &\leq 1.0003 \cdot (n/2) \cdot (1 - 0.0005) < (1 - 0.0002) \cdot (n/2). \end{aligned}$$

Let $Z = \bigcup_{w_j \in U_+} W_j$. Then $|Z| = km > 0.5n - 100\,000$ and

$$\begin{aligned} f(Z) &< |E_H(U_+, U_-)| + f(W_0) < (1-0.0002)(n/2) + 4 \cdot 200\,000 \\ &\le (1-0.0002)(|Z| + 100\,000) + 8 \cdot 10^5 < 0.9999|Z| - (0.0001|Z| - 10^6) \\ &< 0.9999|Z|. \end{aligned}$$

The set Z witnesses that $i(G) < 0.9999$, and hence G is not honest. □

5. Proof of Theorem 6

5.1. Lemmas. We now prove several lemmas that will enable us to modify the proof of [1] and adapt it to our purpose. Let $w_1, w_2, \ldots, w_d \ge 0$ be d real numbers whose sum is 1, and let $\delta_1, \ldots, \delta_d$ be d independent, identically distributed random variables , each taking the values -1 and $+1$ with equal probability. Let $X = X(w_1, w_2, \ldots, w_d)$ be the random variable $X = \left|\sum_{i=1}^d \delta_i w_i\right|$.

Lemma 11. *For each $w_1, w_2, \ldots, w_d$ as above, the expectation of $X(w_1, \ldots, w_d)$ satisfies*

$$E(X(w_1, \ldots, w_d)) \ge E(X(1/d, 1/d, \ldots, 1/d)).$$

Proof. Given a sequence $w_1, \ldots, w_d$ of d non-negative reals whose sum is 1, and assuming two elements of the sequence, say w_1 and w_2 differ, let $u_1, \ldots, u_d$ be the sequence defined by $u_1 = u_2 = (w_1 + w_2)/2$, and $u_i = w_i$ for all $i > 2$. By the triangle inequality, for every real x,

$$\begin{aligned} |x + w_1 + w_2| &+ |x - w_1 - w_2| + |x + w_1 - w_2| + |x - w_1 + w_2| \\ &\ge |x + w_1 + w_2| + |x - w_1 - w_2| + 2|x| \\ &= |x + u_1 + u_2| + |x - u_1 - u_2| + |x + u_1 - u_2| + |x - u_1 + u_2|. \end{aligned}$$

This implies, by breaking the expectation of $E(X(w_1, \ldots, w_d))$ into the sum of 2^{d-2} terms each being a sum of four terms as above, that $E(X(w_1, \ldots, w_d)) \ge E(X(u_1, \ldots, u_d))$. Repeating this argument we obtain the desired result at the limit. □

Lemma 12. *With the numbers ϵ_d defined in the introduction*

$$E(X(1/d, 1/d, \ldots, 1/d)) = 2\epsilon_d.$$

Proof. We describe the proof for odd d, the computation for even d is similar. For an odd d, observe that

$$\sum_{i=0}^{(d-1)/2} i\binom{d}{i} = d \sum_{i=1}^{(d-1)/2} \binom{d-1}{i-1} = \frac{d\left(2^{d-1} - \binom{d-1}{(d-1)/2}\right)}{2}.$$

Therefore

$$\sum_{i=0}^{(d-1)/2} \binom{d}{i}(d-2i) = d2^{d-1} - d\left(2^{d-1} - \binom{d-1}{(d-1)/2}\right) = d\binom{d-1}{(d-1)/2}.$$

It follows that

$$E(X(1/d, 1/d, \ldots, 1/d)) = \frac{1}{2^d}\sum_{i=0}^{d}\binom{d}{i}|d-2i|\frac{1}{d} = \frac{2d\binom{d-1}{(d-1)/2}}{d2^d} = 2\epsilon_d.$$

□

LEMMA 13. *Let $w_1, \ldots, w_d$ be non-negative reals, and suppose that the sum $\sum_{i=1}^{d} \delta_i w_i$ is never zero for each of the 2^d choices of $\delta_i \in \{-1, 1\}$. Let δ_i be independent, identically distributed random variables each taking the values -1 and $+1$ with equal probability, and define $\frac{1}{2} + \epsilon(i)$ to be the following probability:*

$$\frac{1}{2} + \epsilon(i) = \mathrm{Prob}\left(\mathrm{sign}(\delta_i) = \mathrm{sign}\left(\sum_{i=1}^{d} \delta_i w_i\right)\right).$$

Then

$$\sum_{i=1}^{d} \epsilon(i) w_i \geq \left(\sum_{i=1}^{d} w_i\right) \epsilon_d.$$

PROOF. Clearly it suffices to prove the assertion of the lemma for the case $\sum_{i=1}^{d} w_i = 1$, as both sides are linear with respect to this sum. In this case,

$$\begin{aligned} E(X(w_1, w_2, \ldots, w_d)) &= \frac{1}{2^d} \sum_{\delta_i \in \{-1,1\}} \left(\sum_{i=1}^{d} \delta_i w_i\right) \mathrm{sign}\left(\sum_{i=1}^{d} \delta_i w_i\right) \\ &= \sum_{i=1}^{d} w_i \left(\left(\frac{1}{2} + \epsilon(i)\right) - \left(\frac{1}{2} - \epsilon(i)\right)\right) = 2 \sum_{i=1}^{d} w_i \epsilon(i). \end{aligned}$$

The result now follows from the previous two lemmas. □

5.2. The proof. We now prove Theorem 6. Given a weighted graph $G = (V, E)$ on n vertices as in the theorem, we must show that there is a partition $V = V_- \cup V_+$, where $|V_-| = |V_+| = n/2$ and $w(V_-, V_+)$ satisfies (2).

The basic idea is very simple: we first assign each vertex v a random sign $h(v) \in \{-1, 1\}$ and if $h(v)$ is not equal to $\mathrm{sign}\left(\sum_{u \in N(v)} w(vu)h(u)\right)$, then we randomly decide whether to reverse its sign or leave it as it is. It is then shown that the expected total weight of edges between the negative vertices and the positive vertices is not too large. One difficulty in the process of obtaining a rigorous proof along these lines is that we have to keep the two classes of equal size. This causes several problems, and we overcome them by combining, as in [1], the FKG Inequality with some combinatorial ideas. The main difference between the proof in [1] and the proof here, is that in the simple case considered in [1], one can obtain a sufficiently good upper bound for the probability that each edge separately is a crossing edge, and the desired result, thus, follows by linearity of expectation. Here one has to average over all edges incident with a vertex, using the lemmas of the previous subsection. An additional convenient trick is to first apply, if needed, a small perturbation to the weights to make sure that no linear combination of the weights of the edges incident with a vertex with $-1, 1$ coefficients vanishes. This will ensure that the sum $\sum_{u \in N(v)} w(vu)h(u)$ will always have a well-defined sign. As the perturbation can be arbitrarily small, it is obvious that it makes no difference and, hence, we may and will assume from now on that the weights satisfy this generic assumption.

We need the following lemma, proved in [1].

LEMMA 14. *Let H be a graph on $n = 2m$ vertices, with maximum degree Δ, and suppose $n > 40\Delta^3$. Then there is a perfect matching $M = \{(u_i, v_i) : 1 \leq i \leq m\}$ of all vertices of H satisfying the following properties.*

(i) *Each edge of M is **not** an edge of H.*

(ii) *There is no alternating cycle of length* 4 *or* 6 *consisting of edges of* H *and* M *alternately.*

Returning to the proof of Theorem 6, consider the following randomized procedure for constructing a partition of the set of vertices of $G = (V, E)$ into two equal parts V_- and V_+. First, let H be the graph on V in which two vertices are adjacent if their distance in G is at most 3. By assumption, the maximum degree Δ in H satisfies $n > 40\Delta^3$, and hence, by Lemma 14 there is a matching $M = \{(u_i, v_i) : 1 \le i \le m\}$ satisfying the assertion of the lemma. Let $h : V \mapsto \{-1, 1\}$ be a random function obtained by choosing, for each i, $1 \le i \le m$, randomly and independently, one of the two possibilities ($h(u_i) = -1$ and $h(v_i) = 1$) or ($h(u_i) = 1$ and $h(v_i) = -1$), both choices being equally probable. Call a vertex $v \in V$ *stable* if $h(v) = \mathrm{sign}(\sum_{u \in N(v)} w(vu)h(u))$, otherwise call it *active*. Call a pair of vertices (u_i, v_i) matched under M an *active pair* if both u_i and v_i are active, otherwise, call it a *stable pair*. Let $h' : V \mapsto \{-1, 1\}$ be the random function obtained from h by randomly modifying the values of the vertices in active pairs as follows. If (u_i, v_i) is an active pair then choose randomly either ($h'(u_i) = -1$ and $h'(v_i) = 1$) or ($h'(u_i) = 1$ and $h'(v_i) = -1$), both choices being equally probable. Otherwise, define $h'(u_i) = h(u_i)$ and $h'(v_i) = h(v_i)$. Finally, define $V_- = h'^{-1}(-1)$ and $V_+ = h'^{-1}(1)$.

It is obvious that $|V_-| = |V_+| = m$ $(= n/2)$. To complete the proof, we prove an upper bound for the expected value of $w(V_-, V_+)$. Fix an edge of G; by renaming the vertices if needed, we may assume, without loss of generality, that its two vertices are u_1 and u_2, which are matched under M to v_1 and v_2, respectively. Our objective is to estimate the probability that $h'(u_1) \ne h'(u_2)$. This is done by estimating the conditional probability of this event assuming that $h(u_1) = h(u_2)$ and the conditional probability assuming that $h(u_1) \ne h(u_2)$. Before starting to estimate these probabilities, note that by the choice of M, the sets $\{v_1\} \cup N(v_1)$ and $\{v_2\} \cup N(v_2)$ of the closed neighborhoods of v_1 and v_2, respectively, are disjoint and both of them do not intersect the set $\{u_1, u_2\} \cup N(u_1) \cup N(u_2)$. Moreover, the only edges of M whose two ends lie in the set

$$\{u_1, u_2, v_1, v_2\} \cup N(u_1) \cup N(u_2) \cup N(v_1) \cup N(v_2)$$

are the two edges $\{u_1, v_1\}$ and $\{u_2, v_2\}$. These facts, illustrated in Figure 1, will be useful as they imply that various events are independent. Thus, for example, the event (v_1 *is active and* $h(v_1) = -1$) is independent of the event (v_2 *is active and* $h(v_2) = 1$), as those are determined by disjoint sets of random choices. (Note that for this to hold it is not enough that the closed neighborhoods of v_1 and v_2 are disjoint; one also needs the fact that there are no edges of M joining these two neighborhoods.)

In order to estimate the conditional probability $\mathrm{Prob}[h'(u_1) \ne h'(u_2) | h(u_1) = h(u_2)]$ note, first, that in case $h(u_1) = h(u_2)$ then if at least one of the pairs (u_1, v_1) or (u_2, v_2) is active, then this probability is precisely a half. On the other hand, if they are both stable, it is zero. Therefore,

$$\begin{aligned} &\mathrm{Prob}[h'(u_1) \ne h'(u_2) | h(u_1) = h(u_2)] \\ &\qquad = \frac{1}{2} - \frac{1}{2}\mathrm{Prob}[(u_1, v_1), (u_2, v_2) \text{ stable } | h(u_1) = h(u_2)]. \end{aligned} \tag{6}$$

Clearly

$$\begin{aligned} &\mathrm{Prob}[(u_1, v_1), (u_2, v_2) \text{ stable } | \; h(u_1) = h(u_2)] = \mathrm{Prob}[(u_1, v_1) \text{ stable } | h(u_1) = h(u_2)] \\ &\cdot \mathrm{Prob}[(u_2, v_2) \text{ stable } | h(u_1) = h(u_2), (u_1, v_1) \text{ stable}] \end{aligned} \tag{7}$$

FIGURE 1. A typical edge u_1u_2.

Furthermore,

$$\text{Prob}[(u_1, v_1) \text{ stable} \mid h(u_1) = h(u_2)] = \text{Prob}[v_1 \text{ stable } |h(u_1) = h(u_2)]$$
$$+ \text{Prob}[v_1 \text{ active } |h(u_1) = h(u_2)] \cdot \text{Prob}[u_1 \text{ stable } |h(u_1) = h(u_2), v_1 \text{ active}].$$

Since, by the choice of M, the set $\{u_1, u_2\}$ does not intersect $N(v_1)$ and none of its members is matched under M to a member of $N(v_1)$, it follows that

$$\text{Prob}[v_1 \text{ stable } |h(u_1) = h(u_2)] = \text{Prob}[v_1 \text{ stable}] = \frac{1}{2}.$$

Let e denote the edge u_1u_2 and define $\epsilon(u_1, e)$ by the equation

$$\frac{1}{2} + \epsilon(u_1, e) = \text{Prob}\left(h(u_2) = \text{sign}\left(\sum_{u \in N(u_1)} w(u_1u)h(u)\right)\right).$$

Note that by Lemma 13, if $e_1, \ldots, e_d$ is the set of all edges incident with u_1, the inequality

$$\sum_{i=1}^{d} w(e_i)\epsilon(u_1, e_i) \geq \left(\sum_{i=1}^{d} w(e_i)\right)\epsilon_d, \tag{8}$$

holds. This will be useful in the end of the proof. We claim that

$$\text{Prob}[u_1 \text{ stable } |h(u_1) = h(u_2), v_1 \text{ active}] = \text{Prob}[u_1 \text{ stable } |h(u_1) = h(u_2)] = \frac{1}{2} + \epsilon(u_1, e).$$

To see this, note first, that by the choice of M the event (v_1 *active*) is determined only by the values of $|h(w) - h(v_1)|$ for $w \in N(v_1)$ and, hence, does not influence the conditional probability $\text{Prob}[u_1 \text{ stable } |h(u_1) = h(u_2)]$. The above expression for the last conditional probability, thus, follows from the definition of $\epsilon(u_1, e)$.

Substituting the expressions above we conclude that

$$\text{Prob}[(u_1, v_1) \text{ stable } |h(u_1) = h(u_2)] = \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2} + \epsilon(u_1, e)\right) = \frac{3}{4} + \frac{1}{2}\epsilon(u_1, e). \tag{9}$$

We can now apply a similar reasoning to estimate the conditional probability

$$\text{Prob}[(u_2, v_2) \text{ stable } |h(u_1) = h(u_2), (u_1, v_1) \text{ stable}].$$

The crucial point is that when $h(u_1) = h(u_2)$, the event ((u_2, v_2) *stable*) and the event ((u_1, v_1) *stable*) behave monotonically with respect to the h-values on the intersection $N(u_1) \cap N(u_2)$, in case this intersection is non-empty. That is, if one of these events occurs, then by changing the value of some $h(w)$ for w in this intersection from $-h(u_1) = -h(u_2)$ to $h(u_1)$, this event still occurs. Thus, it follows from the FKG Inequality (cf. e.g., [3], Chapter 6) that

$$\text{Prob}[(u_2, v_2) \text{ stable } | h(u_1) = h(u_2), (u_1, v_1) \text{ stable }] \geq \frac{3}{4} + \frac{1}{2}\epsilon(u_2, e), \tag{10}$$

where $\epsilon(u_2, e)$ is defined just like $\epsilon(u_1, e)$ before. Combining (7), (9) and (10),

$$\begin{aligned}\text{Prob}[(u_1, v_1), (u_2, v_2) \text{ stable } | \; h(u_1) = h(u_2)] \geq \frac{9}{16} + \frac{3}{8}\epsilon(u_1, e) + \frac{3}{8}\epsilon(u_2, e) \\ + \frac{1}{4}\epsilon(u_1, e)\epsilon(u_2, e),\end{aligned}$$

and therefore, by (6),

$$\begin{aligned}&\text{Prob}[h'(u_1) \neq h'(u_2) | h(u_1) = h(u_2)] \\ &\quad \leq \frac{1}{2}\left(\frac{7}{16} - \frac{3}{8}\epsilon(u_1, e) - \frac{3}{8}\epsilon(u_2, e) - \frac{1}{4}\epsilon(u_1, e)\epsilon(u_2, e)\right).\end{aligned} \tag{11}$$

Similar arguments can be used to estimate the conditional probability

$$\text{Prob}[h'(u_1) \neq h'(u_2) | h(u_1) \neq h(u_2)].$$

Here are the details. Note, first, that

$$\begin{aligned}&\text{Prob}[h'(u_1) \neq h'(u_2) | h(u_1) \neq h(u_2)] \\ &\quad = \frac{1}{2} + \frac{1}{2}\text{Prob}[(u_1, v_1), (u_2, v_2) \text{ stable } | h(u_1) \neq h(u_2)].\end{aligned} \tag{12}$$

Next, observe that

$$\begin{aligned}&\text{Prob}[(u_1, v_1), (u_2, v_2) \text{ stable } | h(u_1) \neq h(u_2)] = \text{Prob}[(u_1, v_1) \text{ stable } | h(u_1) \neq h(u_2)] \\ &\cdot \text{Prob}[(u_2, v_2) \text{ stable } | h(u_1) \neq h(u_2), (u_1, v_1) \text{ stable}].\end{aligned} \tag{13}$$

Furthermore,

$$\begin{aligned}&\text{Prob}[(u_1, v_1) \text{ stable } | h(u_1) \neq h(u_2)] = \text{Prob}[v_1 \text{ stable } | \; h(u_1) \neq h(u_2)] \\ &\quad + \text{Prob}[v_1 \text{ active } | \; h(u_1) \neq h(u_2)] \cdot \text{Prob}[u_1 \text{ stable } | \; h(u_1) \neq h(u_2), v_1 \text{ active}].\end{aligned}$$

As before, by the choice of M,

$$\text{Prob}[v_1 \text{ stable } | \; h(u_1) \neq h(u_2)] = \text{Prob}[v_1 \text{ stable}] = \frac{1}{2},$$

and

$$\begin{aligned}&\text{Prob}[u_1 \text{ stable } | \; h(u_1) \neq h(u_2), v_1 \text{ active}] \\ &\quad = \text{Prob}[u_1 \text{ stable } | \; h(u_1) \neq h(u_2)] = \frac{1}{2} - \epsilon(u_1, e),\end{aligned}$$

since if $h(u_1) \neq h(u_2)$ then u_1 is stable if and only if $h(u_2) \neq \text{sign}\left(\sum_{u \in N(u_1)} w(u_1 u)h(u)\right)$.

Substituting, we conclude that

$$\text{Prob}[(u_1, v_1) \text{ stable} \mid h(u_1) \neq h(u_2)] = \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2} - \epsilon(u_1, e)\right) = \frac{3}{4} - \frac{1}{2}\epsilon(u_1, e). \quad (14)$$

By a similar computation, and using the FKG Inequality it follows next, that

$$\text{Prob}[(u_2, v_2) \text{ stable} \mid h(u_1) \neq h(u_2), (u_1, v_1) \text{ stable}] \leq \frac{3}{4} - \frac{1}{2}\epsilon(u_2, e), \quad (15)$$

since when $h(u_1) \neq h(u_2)$ then the event $((u_1, v_1)$ *stable*) is monotone increasing with respect to changing the values of some $h(w)$ for $w \in N(u_1) \cap N(u_2)$ from $h(u_2)$ to $h(u_1)$, whereas the event $((u_2, v_2)$ *stable*) is monotone decreasing with respect to such a change.

By (13)–(15),

$$\begin{aligned}&\text{Prob}[(u_1, v_1), (u_2, v_2) \text{ stable} \mid h(u_1) \neq h(u_2)]\\ &\quad\leq \frac{9}{16} - \frac{3}{8}\epsilon(u_1, e) - \frac{3}{8}\epsilon(u_2, e) + \frac{1}{4}\epsilon(u_1, e)\epsilon(u_2, e),\end{aligned}$$

and therefore, by (12),

$$\begin{aligned}&\text{Prob}[h'(u_1) \neq h'(u_2) | h(u_1) \neq h(u_2)]\\ &\quad\leq \frac{1}{2}\left(\frac{25}{16} - \frac{3}{8}\epsilon(u_1, e) - \frac{3}{8}\epsilon(u_2, e) + \frac{1}{4}\epsilon(u_1, e)\epsilon(u_2, e)\right). \quad (16)\end{aligned}$$

Combining (11) and (16) we finally conclude that

$$\begin{aligned}\text{Prob}[h'(u_1) \neq h'(u_2)] &= \text{Prob}[h(u_1) = h(u_2)] \cdot \text{Prob}[h'(u_1) \neq h'(u_2) | h(u_1) = h(u_2)]\\ &\quad + \text{Prob}[h(u_1) \neq h(u_2)] \cdot \text{Prob}[h'(u_1) \neq h'(u_2) | h(u_1) \neq h(u_2)]\\ &\leq \frac{1}{2} - \frac{3}{16}\epsilon(u_1, e) - \frac{3}{16}\epsilon(u_2, e).\end{aligned}$$

Since (u_1, u_2) was a typical edge, by linearity of expectation and by (8), the expected value of $w(V_-, V_+)$ satisfies

$$\begin{aligned}w(V_-, V_+) &\leq \sum_{i=1}^{n} \sum_{j \in N(i)} \left(\frac{w(ij)}{4} - \frac{3}{16}w(ij)\epsilon(i, ij)\right)\\ &\leq \sum_{i=1}^{n} \sum_{j \in N(i)} \frac{w(ij)}{4}\left(1 - \frac{3}{4}\epsilon_{d_i}\right) = \sum_{ij \in E} \frac{w(ij)}{2}\left(1 - \frac{3}{8}\epsilon_{d_i} - \frac{3}{8}\epsilon_{d_j}\right).\end{aligned}$$

This completes the proof. □

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Received 21 January 1998 and accepted 15 March 1999

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