Europ. J. Combinatorics (2000) **21**, 249–255
Article No. eujc.1999.0330
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On the Number of Edges in Hypergraphs Critical with Respect to Strong Colourings

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A colouring of the vertices of a hypergraph G is called *strong* if, for every edge A, the colours of all vertices in A are distinct. It corresponds to a colouring of the *generated graph* $\Gamma(G)$ obtained from G by replacing every edge by a clique. We estimate the minimum number of edges possible in a k-critical t-uniform hypergraph with a given number of vertices. In particular we show that, for $k \geq t+2$, the problem reduces in a way to the corresponding problem for graphs. In the case when the generated graph of the hypergraph has bounded clique number, we give a lower bound that is valid for sufficiently large k and is asymptotically tight in k; this bound also holds for list strong colourings.

1. INTRODUCTION

A graph G is *k-vertex-critical*, or simply *k-critical*, if G is k-chromatic but $G-v$ is $(k-1)$-colourable, for every vertex v of G; and G is *k-edge-critical* if G is k-chromatic but $G-e$ is $(k-1)$-colourable, for every edge e of G.

Since the minimum degree of every k-critical graph is at least $k-1$, the number of edges in such a graph G is at least $\frac{k-1}{2}|V(G)|$; and the complete graph K_k has exactly $\frac{k(k-1)}{2} = \frac{k-1}{2}|V(K_k)|$ edges. In 1957, G. A. Dirac [5] started studying the minimum possible number $f(k,n)$ of edges in a k-critical graph on n vertices for $k \geq 4$ and $n > k$. Results of this type proved to be useful in estimating chromatic numbers of graphs embedded in surfaces. Nice applications of such results are given by Krivelevich [14].

The extended notion of $f(k,t,n)$, defined to be the minimum possible number of edges in a k-critical t-uniform hypergraph on n vertices, was studied by Abbott *et al*. [1–3] and by Kostochka *et al*. [9, 11]. The fact that $f(3,t,n) = n$ for every t and infinitely many n was proved in the early seventies independently by Lovász [15], Woodall [17], Seymour [16] and Burstein [4].

In this paper, we address the same problem for the different notion of strong hypergraph colouring. We say that a colouring of a hypergraph G is *strong* if no two vertices sharing an edge have the same colour. A strong colouring of a hypergraph G corresponds to an edge-colouring of the hypergraph dual to G and to a (usual) colouring of the *generated graph* $\Gamma(G)$ with the same vertex set, where two vertices are adjacent in $\Gamma(G)$ if they share an edge in G.

Let $f_s(k,t,n)$ denote the minimum number of edges in a t-uniform hypergraph on n vertices that is k-critical with respect to strong colouring, and let $F_s(k,t) = \inf_{n>k} \frac{f_s(k,t,n)}{n}$. If G is a k-edge-critical graph on n vertices ($k \geq t+1$) and G' is obtained from G by adding $t-2$ new vertices into every edge of G, then G' is a k-critical t-uniform hypergraph on $n+(t-2)|E(G)|$ vertices. This suggests that it would be more natural to study

$$F'_s(k,t) = \inf_{n>k} \frac{f_s(k,t,n)}{n-(t-2)\,f_s(k,t,n)} = \inf_{G=(V,E)} \frac{|E|}{|V|-(t-2)|E|},$$

where the infimum is taken over all t-uniform hypergraphs G that are k-critical with respect to strong colouring. This also shows that $F'_s(k,t) \leq F(k)$ for every $k > t$, where $F(k) =$

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0195–6698/00/020249 + 07 $35.00/0

$\inf_{n>k} \frac{f(k,n)}{n}$, since it is easy to see that the minimum value of $f(k, n)$ for fixed n is attained by a k-edge-critical graph. We will prove that

$$F'_s(k, t) = F(k) \tag{1}$$

for every $k \geq t+2$, and observe that $F'_s(t+1, t) < F(t+1)$ for every $t \geq 4$. The case $k = t+1$ is studied more carefully (and in the more general setting of *panchromatic colourings*) in [13].

It is known (see, e.g., [7]) that for every $k \geq 4$,

$$\frac{1}{2}\left(k - 1 + \frac{k-3}{k^2-3}\right) \leq F(k) < \tfrac{1}{2}k. \tag{2}$$

(The lower bound is due to Gallai [6] and the upper bound follows from the Hajós construction.) Together, (1) and (2) imply that $F'_s(k, t)$ is between $\frac{1}{2}(k-1)$ and $\frac{1}{2}k$ for every $k \geq t+2$. Our second result is that if the clique number of the generated graph $\Gamma(G)$ of a k-critical t-uniform hypergraph G is bounded and k is sufficiently large, then the number of edges of G is about twice as large as is guaranteed by $F'_s(k, t)$. This new bound is asymptotically tight in k. It will also be proved for list critical hypergraphs.

2. Some Notation

When we try to colour the vertices of a t-uniform hypergraph G strongly with $k \geq t$ colours, we can always colour the vertices of degree one last and without difficulty. In view of this, we call the vertices of degree one *inessential vertices* and the vertices of degree greater than one *essential*. Also, in view of this, we consider the *skeleton* $S(G)$, which is the subgraph of $\Gamma(G)$ induced by the essential vertices of G.

As we have already remarked, although the class of k-vertex-critical graphs is wider than the class of (connected) k-edge-critical graphs, the value of $f(k, n)$ is always attained by a k-edge-critical graph. In a similar way, to study $F'_s(k, t)$, it is enough to consider the subclass of k-critical t-uniform hypergraphs introduced in the next paragraph.

If G is a hypergraph, $v \in A \in E(G)$ and $\deg_G(v) \geq 2$, then the (v, A)-*splitting* of G is obtained by replacing the edge A by the edge $A - v + v'$, where v' is a new vertex. A *splitting* of G is the (v, A)-splitting for some essential vertex $v \in A \in E(G)$. A hypergraph G is called *k-splitting-critical* if G needs at least k colours for its strong colouring but every splitting of G has a strong colouring with $k - 1$ colours.

Clearly, (v, A)-splitting is a weaker operation than deleting the edge A. Hence every k-splitting-critical hypergraph is k-critical. On the other hand, a t-uniform hypergraph G that needs at least $k > t$ colours for its strong colouring can be reduced by a series of splittings to a k-splitting-critical hypergraph with the same number of edges and at least the same number of vertices. Thus, in order to obtain bounds on $F'_s(k, t)$, it is enough to estimate the numbers of edges in k-splitting-critical t-uniform hypergraphs.

Since some of our results also hold for list colouring, we give the relevant definitions. A *list* L for a hypergraph G is a mapping that assigns a set $L(v)$ of (admissible) colours to every vertex v of G. We say that a (strong) colouring φ of G is an *L-colouring* if $\varphi(v) \in L(v)$ for every $v \in V(G)$. A hypergraph G is *L-splitting-critical* if it has no L-colouring but every edge-deletion and every splitting (giving the new vertex the list of the original vertex) produces an L-colourable hypergraph. In these terms, a k-splitting-critical hypergraph is L-splitting-critical, where $L(v) = \{1, \ldots, k-1\}$ for every vertex v. We will say that a hypergraph G is *list-k-splitting-critical* if it is L-splitting-critical for at least one list L with $|L(v)| = k - 1$ for every $v \in V(G)$.

We will call an edge *full* if all its vertices are essential, and *hollow* if it has exactly two essential vertices.

3. Reduction to Graphs

THEOREM 1. *For every $k \geq t+2$ and every k-splitting-critical t-uniform hypergraph G,*

$$|E(G)| \geq \frac{|V(G)|}{t-2+1/F(k)}.$$

In other words,

$$\frac{|E(G)|}{|V(G)|-(t-2)|E(G)|} \geq F(k).$$

PROOF. Let G be a counterexample to the theorem with the smallest possible number of non-hollow edges, and let $S(G)$ be the skeleton of G. Since G is k-splitting-critical, $S(G)$ is k-vertex-critical.

If every edge of G is hollow then $|E(G)| = |E(S(G))|$ and $|V(G)| = (t-2)|E(G)| + |V(S(G))|$. Since $S(G)$ is k-vertex-critical, therefore $|E(S(G))| \geq F(k)|V(S(G))|$, giving

$$\frac{|E(G)|}{|V(G)|} \geq \frac{|E(G)|}{(t-2)|E(G)|+|E(G)|/F(k)} = \frac{1}{t-2+1/F(k)}.$$

So suppose not every edge of G is hollow. Let A be any edge of G with $m \geq 3$ essential vertices, and let G' be obtained from G by the following operation:
(a) for every pair $\{v, w\}$ of distinct essential vertices in A, we add a hollow edge A_{vw} (call it a 'new' edge) whose vertices apart from v and w will all be inessential;
(b) we delete A and all inessential vertices belonging to A.
G' has the same skeleton as G, and hence is k-chromatic. Maybe it is not k-splitting-critical, but after deleting some new edges it must become k-splitting-critical. Let G'' be a k-splitting-critical subhypergraph of G' and let s be the number of 'new' edges in G''. G'' has fewer non-hollow edges than G and so, by the choice of G, $|E(G'')| \geq |V(G'')|/(t-2+1/F(k))$. Note that $|E(G)| = |E(G'')| - s + 1$ and $|V(G)| = |V(G'')| - (t-2)s + (t-m)$. Since $s \leq \binom{m}{2}$ and, by (2), $F(k) \geq F(t+2) > (t+1)/2$,

$$\begin{aligned}
|E(G)| - \frac{|V(G)|}{t-2+1/F(k)} &\geq \frac{|V(G'')|-|V(G)|}{t-2+1/F(k)} - s + 1 \\
&= \frac{(t-2)s-(t-m)-(s-1)(t-2+1/F(k))}{t-2+1/F(k)} \\
&= \frac{m-2-(s-1)/F(k)}{t-2+1/F(k)} \\
&\geq \frac{m-2-(m^2-m-2)/(t+1)}{t-2+2/(t+1)} \\
&= \frac{(m-2)[(t+1)-(m+1)]}{t^2-t},
\end{aligned}$$

which is non-negative for $2 \leq m \leq t$. This proves the theorem. □

Since, by definition, there exists a sequence $\{G_n\}_{n=1}^{\infty}$ of k-splitting-critical t-uniform hypergraphs with

$$\frac{|E(G_n)|}{|V(G_n)|-(t-2)|E(G_n)|} \to F'_{\mathrm{s}}(k,t) \qquad \text{as } n \to \infty,$$

we obtain the following consequence.

COROLLARY 1. *$F'_s(k,t) = F(k)$ for every $k \geq t+2$.*

REMARK 1. The lower bound in (2) was proved by Gallai [6] for the usual chromatic number, but the proof works for list colouring as well (as was observed, for example, in [10]). Thus the proof of our Theorem 1 also works for list-k-splitting-critical t-uniform hypergraphs.

REMARK 2. If an edge A of a k-splitting-critical hypergraph G has fewer than t vertices, where $k > t$, then adding an inessential vertex to A creates a new k-splitting-critical hypergraph with the same number of edges and a greater number of vertices. Hence the value of $F'_s(k,t)$ will not change if in the definition of $f_s(k,t,n)$ we replace the words 't-uniform hypergraph' by the words 'hypergraph with no edge having more than t vertices'.

To show that $F'_s(t+1,t) < F(t+1)$, we use the following construction (also described in [13]). For $t \geq 3$, let $H_0(t)$ be the t-uniform hypergraph with vertices v_0, v_1 and $v_{i,j}$ $(1 \leq i \leq t-1,\ 2 \leq j \leq t)$, whose edges are $\{v_1, v_{i,2}, \ldots, v_{i,t}\}$ $(1 \leq i \leq t-1)$ and $\{v_{1,t}, v_{2,t}, \ldots, v_{t-1,t}, v_0\}$. Then $H_0(t)$ has $t^2 - 2t + 3$ vertices and t edges; and in any strong t-colouring of $H_0(t)$, v_0 must be given the same colour as v_1. Call $H_0(t)$ a *superlink joining* v_0 and v_1.

Now, for $m \geq 1$, form a t-uniform hypergraph $H_m(t)$ by taking vertices $v_0, v_1, \ldots, v_s$ and adding superlinks joining v_{i-1} and v_i $(1 \leq i \leq s)$ and a hollow edge containing v_s and v_0, these s superlinks and one edge having no vertices in common except for $v_0, v_1, \ldots, v_s$. Then $H_m(t)$ is a k-splitting-critical t-uniform hypergraph and it has $m(t^2-2t+2)+(t-1)$ vertices and $mt+1$ edges. So

$$|V(H_m(t))| - (t-2)|E(H_m(t))| = m(t^2 - 2t + 2) + (t-1) - (t-2)(mt+1) = 2m+1$$

and

$$\frac{|E(H_m(t))|}{|V(H_m(t))| - (t-2)|E(H_m(t))|} = \frac{mt+1}{2m+1} < \frac{t}{2}.$$

It follows that $F'_s(t+1,t) < \frac{t}{2}$. Recall from (2) that $F(t+1) > \frac{t}{2}$.

4. HYPERGRAPHS WITHOUT LARGE CLIQUES

To deal with hypergraphs whose skeletons do not have large cliques, we need the following theorem due to Johansson.

THEOREM 2 (JOHANSSON [8]). *For every positive integer r, there exist constants c_r and D_r such that for every $D > D_r$ the list chromatic number of every graph with maximum degree D with no complete subgraphs on $r+1$ vertices is at most $c_r D \frac{\ln\ln D}{\ln D}$.*

We will prove our bound in a more general setting. Let us say that a colouring of vertices of a hypergraph G is *t-strong* if every edge of G contains vertices of at least t different colours. Note that a 2-strong colouring is a usual colouring and any t-strong colouring of a t-uniform hypergraph G is a strong colouring of G. Critical hypergraphs can now be defined in an obvious way.

THEOREM 3. *Let $r \geq t$ be positive integers and let k be sufficiently large with respect to r. Let G be a hypergraph that is list-k-splitting-critical with respect to t-strong colouring, whose skeleton $S(G)$ does not contain a complete subgraph on $r+1$ vertices. (The criticality implies that every edge has at least t vertices). Then*

$$|E(G)| \geq k(1 - 6(\ln k)^{-1/3})(|V(G)| - (t-2)|E(G)|).$$

PROOF. Let k be sufficiently large and let G be a counterexample to the theorem. Let L be a list for G for which the theorem fails, so that $|L(v)| = k - 1$ for each vertex v. Define $F = \{v \in V(G) \mid d_G(v) \geq k\sqrt{\ln k}\}$ and $M = V(G) \setminus F$.

We shall use the following two claims. Claim 1 is evident, and Claim 2 follows from Claim 1 and the absence of K_{r+1} in the skeleton $S(G)$.

CLAIM 1. *Every edge in G has at most $t - 2$ vertices of degree* 1.

CLAIM 2. *Every edge in G has at most $r + t - 2$ vertices in total.*

There are now two cases to consider.

CASE 1. There exists $B \subseteq M$ such that:

(i) at least one $a \in B$ belongs to the skeleton $S(G)$;
(ii) each edge intersecting B has at least $t - 1$ vertices in common with B;
(iii) for each $a \in B$, the number of edges containing a and having exactly $t - 1$ vertices in common with B is at most $k - \lceil k/\sqrt[3]{\ln k}\rceil - 1$.

Let G' be obtained from G by splitting all vertices in B into vertices of degree one. By (i), $G' \neq G$, and hence G' has a t-strong L-colouring. Let f be the restriction to $V(G) \setminus B$ of a t-strong L-colouring of G'. We will prove that f can be extended to a t-strong L-colouring of G. To this end, we construct auxiliary lists for $a \in B$ as follows. First, for each $A \in E(G)$ having exactly $t - 1$ vertices in common with B, we choose a colour c_A used by f on $A \setminus B$. Then, for each $a \in B$, we define the list $L'(a)$ by deleting from the original list $L(a)$ the colour c_A for each $A \in E$ having exactly $t - 1$ vertices in common with B and such that $a \in A$. By (iii),

$$|L'(a)| \geq \lceil k/\sqrt[3]{\ln k}\rceil \qquad \text{for each } a \in B. \tag{3}$$

Let $S[B]$ be the subgraph of the skeleton $S(G)$ that is induced by B. Observe that, by the definition of M and Claim 2, the maximum degree of $S[B]$ is less than $D = (r+t-3)k\sqrt{\ln k}$. If k is sufficiently large in comparison with r, then

$$k/\sqrt[3]{\ln k} > c_r D \frac{\ln \ln D}{\ln D}.$$

Thus, by (3), we can apply Theorem 2 to $S[B]$ to deduce that $S[B]$ is L'-colourable. Let us fix an L'-colouring g of $S[B]$.

We now use g to deduce that f can be extended to B. Indeed, let A be an arbitrary edge intersecting B. All the vertices in $A \cap B$ belonging to $S(G)$ have different colours. For vertices of degree 1 in $A \cap B$, by Claim 2 (since we may assume $k - 1 > r + t - 2$) we can choose different colours not used on $S(G) \cap A$. So if $|A \cap B| \geq t$, we are done. If $|A \cap B| = t - 1$, then by the definition of L', the $t - 1$ colours used on $A \cap B$ differ from c_A, and so A contains vertices of at least t different colours.

This shows that f can be extended to B, and so G has a t-strong L-colouring. This contradiction shows that Case 1 is impossible.

CASE 2. For every $B \subseteq M$, at least one of (i), (ii) and (iii) is false. We construct a growing sequence $F_0, F_1, F_2, \ldots$ of subsets of $V(G)$ as follows. First we put $F_0 = F$.

Suppose that F_{i-1} is constructed. Let $B_{i-1} = V(G) \setminus F_{i-1} \subseteq M$. Under the conditions of Case 2, there are the following possibilities.

(i′) F_{i-1} contains all the vertices of $S(G)$. Then we stop.

(ii′) There exists an edge $A \in E(G)$ such that $0 < |A \setminus F_{i-1}| = |A \cap B_{i-1}| \leq t-2$. Then we put $F_i = F_{i-1} \cup A$. In this case, we say that i is *of type* (ii′) and A is *responsible* for i.

(iii′) For some vertex $a \in B_{i-1} = V(G) \setminus F_{i-1}$, there are $s = k - \lceil k/\sqrt[3]{\ln k}\rceil$ edges $A_1, \ldots, A_s$ containing a such that each A_j has exactly $t-1$ vertices in common with B_{i-1}. Then, we put $F_i = F_{i-1} \cup \bigcup_{j=1}^{s} A_j$. In this case, we say that i is *of type* (iii′) and a and $A_1, \ldots, A_s$ are *responsible* for i.

Let the construction terminate with the set F_m. It follows from the construction that F_m contains all the vertices in $S(G)$.

Our aim is to show that $|E(G)| \geq k(1 - 6(\ln k)^{-1/3})(|V(G)| - (t-2)|E(G)|)$, i.e.,

$$|V(G)| \leq |E(G)| \left(t - 2 + \frac{1}{k(1 - 6(\ln k)^{-1/3})}\right).$$

We will prove the slightly stronger inequality

$$|V(G)| \leq |E(G)| \left(t - 2 + \frac{1}{k} + \frac{6}{k\sqrt[3]{\ln k}}\right).$$

To do this, we use discharging. First, let every edge have charge $t - 2 + \frac{1}{k} + \frac{6}{k\sqrt[3]{\ln k}}$. We will prove that it is then possible for the edges to distribute their charges among the vertices in such a way that every vertex gets a charge of at least 1; this will prove the theorem.

STEP 0. Let every edge A give charge $\frac{1}{k\sqrt{\ln k}}$ to every $v \in A$. After this step, each vertex v has charge $\frac{d(v)}{k\sqrt{\ln k}}$; in particular, every vertex in $F_0 = F$ has charge at least 1. By Claim 2, every edge A has at most $r + t < (\ln k)^{1/6}$ vertices and hence still has charge at least

$$t - 2 + \frac{1}{k} + \frac{6}{k\sqrt[3]{\ln k}} - \frac{|A|}{k\sqrt{\ln k}} > t - 2 + \frac{1}{k} + \frac{5}{k\sqrt[3]{\ln k}}.$$

STEP i $(1 \leq i \leq m)$. If i is of type (ii′) and A is responsible for i, let A give 1 to each $v \in A \setminus F_{i-1}$. In this case, A sends out at most $t-2$ and already each vertex in F_i has charge at least 1. So, suppose i is of type (iii′) and a and $A_1, \ldots, A_s$ are responsible for i. In this case, let each A_j give 1 to each of the $t-2$ vertices in $(A_j \setminus F_{i-1}) \setminus \{a\}$ and give $\frac{1}{k} + \frac{5}{k\sqrt[3]{\ln k}}$ to a. Then a gets at least

$$(k - \lceil k/\sqrt[3]{\ln k}\rceil)\left(\frac{1}{k} + \frac{5}{k\sqrt[3]{\ln k}}\right),$$

which is greater than 1 for large k. Thus again each vertex in F_i has charge at least 1.

STEP $m+1$. By this stage, only vertices of degree 1 belonging to $V(G) \setminus F_m$ have charge less than one. Observe that each of them belongs to an edge which is not responsible for any $i \leq m$, and that (by Claim 1) each such edge contains at most $t-2$ such vertices. Thus each such edge can give 1 to each such vertex contained in it. Now all vertices have charge at least 1, and the theorem is proved. □

Although Theorem 3 is stated and proved above for the general case of list-colourings, the bound is already asymptotically tight in k for the particular case of usual strong colourings (which are list-colourings with constant lists): in ([9], Theorem 1) it is proved that there are k-critical graphs H with arbitrarily large girth such that $|E(H)|/|V(H)| < k$, and by inserting $t-2$ inessential vertices into every edge of such a graph we get a t-uniform hypergraph G that is k-critical with respect to t-strong colourings. Thus, even imposing the stronger condition of a large girth would not enable the bound in Theorem 3 to be improved asymptotically in k.

ACKNOWEDGEMENTS

This work was carried out while A. V. Kostochka was visiting Nottingham, funded by Visiting Fellowship Research Grant GR/L54585 from the Engineering and Physical Sciences Research Council. The work of this author was also partly supported by grants 96-01-01614 and 97-01-01075 of the Russian Foundation for Fundamental Research.

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Received 13 January 1999 and accepted 22 March 1999

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