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Vertex Set Partitions Preserving Conservativeness

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Let G be an undirected graph and $\mathscr{P}=\{X_1, ..., X_n\}$ be a partition of $V(G)$. Denote by $G/\mathscr{P}$ the graph which has vertex set $\{X_1, ..., X_n\}$, edge set E, and is obtained from G by identifying vertices in each class X_i of the partition $\mathscr{P}$. Given a conservative graph $(G, \mathbf{w})$, we study vertex set partitions preserving conservativeness, i.e., those for which $(G/\mathscr{P}, \mathbf{w})$ is also a conservative graph. We characterize the conservative graphs $(G/\mathscr{P}, \mathbf{w})$, where $\mathscr{P}$ is a terminal partition of $V(G)$ (a partition preserving conservativeness which is not a refinement of any other partition of this kind). We prove that many conservative graphs admit terminal partitions with some additional properties. The results obtained are then used in new unified short proofs for a co-NP characterization of Seymour graphs by A. A. Ageev, A. V. Kostochka, and Z. Szigeti (1997, *J. Graph Theory* **34**, 357–364), a theorem of E. Korach and M. Penn (1992, *Math. Programming* **55**, 183–191), a theorem of E. Korach (1994, *J. Combin. Theory Ser. B* **62**, 1–10), and a theorem of A. V. Kostochka (1994, *in* "Discrete Analysis and Operations Research. Mathematics and its Applications (A. D. Korshunov, Ed.), Vol. 355, pp. 109–123, Kluwer Academic, Dordrecht).

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1. INTRODUCTION

A ± 1 edge weighting on a graph G is called *conservative* if each circuit of G has nonnegative total weight. A pair $(G, \mathbf{w})$ where G is a graph and $\mathbf{w}$ is a conservative weighting of G is called a *conservative graph.*

Identifying any two vertices x and y of G, we obtain a new graph $G/\{x, y\}$. Clearly, if $\mathbf{w}$ is a conservative weighting of $G/\{x, y\}$, then it is that of G. On the other hand, if $(G, \mathbf{w})$ is a conservative graph then $(G/\{x, y\}, \mathbf{w})$ is not necessarily conservative. In this paper we study those identifications for which $(G/\{x, y\}, \mathbf{w})$ remains conservative (identifications preserving conservativeness). Starting from an arbitrary conservative graph $(G, \mathbf{w})$, a sequence of successive identifications preserving conservativeness gives rise to a new conservative graph $(G/\mathscr{P}, \mathbf{w})$, which is uniquely defined by a partition $\mathscr{P}$ of $V(G)$ (a partition preserving conservativeness). We investigate the structure of graphs corresponding to the terminal partitions, i.e., to those which are not refinements of any other partition preserving conservativeness. It turns out that these graphs have remarkable properties which can be used to provide unified short proofs for a number of known results in T-joins theory.

The paper is organized as follows. Section 2 contains basic definitions and notation. Section 3 presents the main tools of the paper. We characterize the conservative graphs $(G/\mathscr{P}, \mathbf{w})$ where $\mathscr{P}$ is a terminal partition of $V(G)$. We show then (Theorem 2) that many conservative graphs admit terminal partitions with some additional useful properties. The subsequent sections are devoted to applications of these results in new unified short proofs for a co-NP characterization of Seymour graphs [1] (Section 4), a theorem of Korach and Penn [5], a theorem of Korach [4] (Section 5), and a theorem of Kostochka [6] (Section 6). In Section 7, to make the paper self-contained, we present an alternative direct proof of the basic Lemma 1.

2. BASIC DEFINITIONS AND NOTATION

Let G be an undirected graph.

A set of edges $J \subseteq E(G)$ is called a *join* if for any circuit C, $|E(C) \cap J| \leqslant |E(C) \backslash J|$. Let $\mathbf{w}$ be a ± 1 valued weighting defined on edges of G. Denote by $E^{-}(\mathbf{w})$ the set of edges with weight -1. We will also refer to the edges with weight -1 as *negative edges*. By $T(\mathbf{w})$ denote the subset of vertices v having the property that v is incident with an odd number of negative edges. For a set of edges $F \subseteq E(G)$, we denote by $\mathbf{w}_F$ the ± 1 valued weighting with $E^-(\mathbf{w}) = F$.

A graph G is *bicritical* if $|E(G)| \geqslant 1$ and $G - \{x, y\}$ has a perfect matching for each pair of vertices $x, y \in V(G)$.

A ± 1 edge weighting $\mathbf{w}$ is called *conservative* if $E^-(\mathbf{w})$ is a join or, equivalently, if G has no circuit of negative total weight (a loop is treated as a circuit). For any two vertices x and y of G, let $\lambda_{\mathbf{w}}(x, y)$ denote the length of a $\mathbf{w}$-shortest path between x and y (from the definition of conservative weighting it follows that such a path does exist). We will also speak of $\mathbf{w}$-*distance* between x and y.

A *conservative graph* $(G, \mathbf{w})$ is a pair consisting of a graph G and a conservative weighting $\mathbf{w}$. Conservative graphs considered in this paper are assumed to be connected.

Let G be a graph and T be a vertex subset of G of even cardinality. A set F of edges of G is called a *T-join* if T coincides with the set of vertices having odd degree in the subgraph spanned by F. There exists a one-to-one correspondence between minimum T-joins (T-joins of the smallest cardinality), joins and conservative weightings. More precisely, by Guan's lemma [10], a T-join $F \subseteq E(G)$ has minimum cardinality if and only if the weighting $\mathbf{w}_F$ is conservative (or, equivalently, if F is a join).

In this paper we deal with partitions of vertex sets of graphs. Let G be a graph. Let $\mathscr{P}$ and $\mathscr{Q}$ be partitions of $V(G)$. We say that $\mathscr{P}$ is *a refinement of* $\mathscr{Q}$ and write $\mathscr{P} \prec \mathscr{Q}$ if for any $Y \in \mathscr{Q}$, Y is the union of some $X \in \mathscr{P}$. Now let $\mathscr{P}$ be a partition of $V(G)$ and $\mathscr{Q}$ be a partition of $V(G/\mathscr{P})$. Then $\mathscr{Q}$ can be treated as a partition of the set $\mathscr{P}$. Denote by $\mathscr{Q} \circ \mathscr{P}$ the partition of $V(G)$ each class of which is the union of the classes of $\mathscr{P}$ that are elements of some class of $\mathscr{Q}$.

The partition consisting of the singletons is called *trivial* (otherwise *non-trivial*). We denote by $G/\mathscr{P}$ the graph whose vertices are the classes of $\mathscr{P}$, edge set is E, and that is obtained from G by identifying vertices in each class of $\mathscr{P}$. For any set $X \subseteq V(G)$, $\langle X \rangle$ denotes the partition that consists of X and the singletons. We will write G/X instead of $G/\langle X \rangle$.

It is an easy observation that if $\mathbf{w}$ is a conservative weighting of $G/\mathscr{P}$ for some partition $\mathscr{P}$, and $\mathscr{Q}$ is a refinement of $\mathscr{P}$, then $\mathbf{w}$ is a conservative weighting of $G/\mathscr{Q}$. This motivates the following definitions. Let $(G, \mathbf{w})$ be a conservative graph. We say that a partition $\mathscr{P}$ of $V(G)$ *preserves conservativeness* if $\mathbf{w}$ is a conservative weighting of $G/\mathscr{P}$. We denote by $\Pi(G, \mathbf{w})$ the set of all partitions preserving conservativeness. We call a partition $\mathscr{P} \in \Pi(G, \mathbf{w})$ *terminal* if it is not a refinement of any other partition in $\Pi(G, \mathbf{w})$. Note that a partition $\mathscr{P} \in \Pi(G, \mathbf{w})$ is terminal if and only if the $\mathbf{w}$-distance between any two vertices in $G/\mathscr{P}$ is negative. We denote by $\Pi^*(G, \mathbf{w})$ the set of all terminal partitions of $(G, \mathbf{w})$.

3. PARTITIONS PRESERVING CONSERVATIVENESS

We call a conservative graph $(G, \mathbf{w})$ *prime* if

(1) each block of G is bicritical;

(2) $E^-(\mathbf{w})$ forms a perfect matching in each block of G.

THEOREM 1. *Let $(G, \mathbf{w})$ be a conservative graph and $\mathscr{P} \in \Pi(G, \mathbf{w})$. Then $\mathscr{P}$ is terminal if and only if $(G/\mathscr{P}, \mathbf{w})$ is prime.*

[After the paper was completed Zoltan Szigeti informed us that an equivalent statement was established by A. Sebő in his Ph.D. thesis [9].]

The following lemma is crucial in our proofs of Theorem 1 and other results of this section.

LEMMA 1. *Let $(G, \mathbf{w})$ be a 2-connected conservative graph. Let $x \in V(G)$ be incident with at least two negative edges. Then there exists another vertex z such that $\lambda_{\mathbf{w}}(x, z) \geqslant 0$.*

Because of its importance we present two—direct and indirect—proofs of this lemma. This direct proof is contained in Section 7. Here we give a very short indirect proof which relies on the following very special case of a fundamental theorem due to Sebő (see Theorem 4.4 in [10]).

LEMMA 2 (Sebő). *Let $(G, \mathbf{w})$ be a conservative graph and $x_0 \in V(G)$. Let X be the vertex set of a component of the subgraph of G induced by the set $\{x \in V(G) : \lambda_{\mathbf{w}}(x_0, x) \leqslant -1\}$. Then among the edges entering X at most one is negative.*

Proof of Lemma 1. Assume to the contrary that for any vertex t distinct from x, $\lambda_{\mathbf{w}}(x, t) \leqslant -1$. Since $G - x$ is connected, by Lemma 2, $V(G) \backslash \{x\}$ can be entered by at most one negative edge while by the assumptions of the lemma it is entered by at least two adjacent negative edges; a contradiction. ∎

In the subsequent arguments we actually use the following apparently equivalent statement.

LEMMA 3. *Let $(G, \mathbf{w})$ be a 2-connected conservative graph. Let v be a vertex of G incident with at least two negative edges. Then G has a cut set $X \subseteq V(G)$ such that $v \in X$ and $(G/X, \mathbf{w})$ is a conservative graph.*

Proof. By Lemma 1 the family $\mathscr{F} = \{Y \subseteq V(G) : |Y| \geqslant 2,\ x \in Y$ and $\langle Y \rangle \in \Pi(G, \mathbf{w})\}$ is non-empty. Let X be an inclusion-wise maximal member of $\mathscr{F}$. Assume that $G - X$ is connected. Then $(G/X, \mathbf{w})$ satisfies the assumptions of Lemma 1, and hence there is $Y \in \mathcal{F}$ such that $X \subset Y$, contradicting the choice of X. Thus X is a cut set of G, as desired. ∎

Further we need the following two easy observations concerning the special case when the set of negative edges is a matching in the underlying graph.

LEMMA 4. *Let $(G, \mathbf{w})$ be a conservative graph and $E^-(\mathbf{w})$ form a matching in G. Let H be a subgraph of G induced by $T(\mathbf{w})$.*

(a) *If $H - \{x, y\}$ has no perfect matching for some $x, y \in V(H)$, then $(G/\{x, y\}, \mathbf{w})$ is a conservative graph.*

(b) *If $x \in V(G)$ is incident with no negative edge, then $(G/\{x, y\}, \mathbf{w})$ is a conservative graph for any $y \in V(G)$.*

Proof. (a) We have to show that $\lambda_{\mathbf{w}}(x, y) \geqslant 0$. Assume not. Then G has an x, y path of negative weight. Denote by R the set of edges of this path. Since $E^-(\mathbf{w})$ is a matching, the path is alternating and has weight -1. It follows that the symmetric difference $E^-(\mathbf{w}) \oplus R$ is a perfect matching of $H - \{x, y\}$, a contradiction.

(b) For any vertex $y \in V(G)$, $\lambda_{\mathbf{w}}(x, y)$ is nonnegative, for at most half of the edges in any path connecting x and y can have weight -1. ∎

Proof of Theorem 1. $(\Rightarrow)$. Let $H = G/\mathcal{P}$ be a prime graph. We have to show that $\lambda_{\mathbf{w}}(x, y) \leqslant -1$ for any two vertices x and y of H. We may assume that H is 2-connected. Choose any two vertices $x, y \in V(H)$. Since H is bicritical, the graph $H - \{x, y\}$ has a perfect matching M. Note that $M \oplus E^-(\mathbf{w})$ contains an alternating path connecting x and y. This path has weight -1 and therefore $\lambda_{\mathbf{w}}(x, y) \leqslant -1$, as desired.

$(\Leftarrow)$. Let $\mathcal{P} \in \Pi^*(G, \mathbf{w})$ and let B be a block of $G/\mathcal{P}$. By Lemmas 3 and 4(b), the negative edges form a perfect matching M in B. By Lemma 4(a), B is bicritical. Thus B satisfies properties (1) and (2) in the definition of the prime graph, as desired. ∎

We call a conservative graph $(G, \mathbf{w})$ *rigid* (otherwise *non-rigid*) if $E^-(\mathbf{w})$ is a matching in G and $T(\mathbf{w})$ induces a bicritical subgraph of G. Note that by definition a rigid conservative graph $(G, \mathbf{w})$ is prime if and only if it is 2-connected and $T(\mathbf{w}) = V(G)$.

Let $\mathcal{P}$ be a terminal partition of a rigid conservative graph $(G, \mathbf{w})$. By Lemma 4(b), for any $X \in \mathcal{P}$, $|X \cap T(\mathbf{w})| = 1$. It follows that $G/\mathcal{P}$ has a spanning subgraph isomorphic to the subgraph of G induced by $T(\mathbf{w})$. Thus, we arrive at

Remark 1. If a conservative graph $(G, \mathbf{w})$ is rigid, then $G/\mathcal{P}$ is 2-connected for each $\mathcal{P} \in \Pi^*(G, \mathbf{w})$.

Non-rigid conservative graphs admit terminal partitions with special properties which turn out to be useful in many applications.

Let $(G, \mathbf{w})$ be a conservative graph. We say that a partition $\mathcal{P} \in \Pi(G, \mathbf{w})$ is *regular* if, for any $X \in \mathcal{P}$, $|X| \geqslant 2$ implies that $G - X$ is disconnected (or, equivalently, that X is a cut vertex of $G/\mathcal{P}$). Note that by the definition if a regular partition $\mathcal{P}$ is non-trivial, then $G/\mathcal{P}$ has a cut vertex.

Given a conservative graph $(G, \mathbf{w})$, let $\Pi_r^*(G, \mathbf{w})$ denote the set of all regular terminal partitions of $(G, \mathbf{w})$.

By Remark 1, a rigid conservative graph admits a regular terminal partition (the trivial one) only if it is prime. Thus the following holds.

Remark 2. If a conservative graph $(G, \mathbf{w})$ is rigid and non-prime, then no terminal partition of $(G, \mathbf{w})$ is regular.

THEOREM 2. *Every non-rigid conservative graph admits a regular terminal partition.*

LEMMA 5. *Let $(G, \mathbf{w})$ be a 2-connected non-rigid conservative graph. Then G has a cut set $X \subseteq V(G)$ such that $(G/X, \mathbf{w})$ is a conservative graph.*

Proof. Let $(G, \mathbf{w})$ be a counterexample to the statement with the minimum number of vertices. Note that $|V(G)| \geqslant 4$. By Lemma 3 the negative edges form a matching in G. Since $(G, \mathbf{w})$ is non-rigid, the subgraph of G induces by $T(\mathbf{w})$ is not bicritical. By Lemma 4(a), it follows that $\langle \{v_1, v_2\} \rangle \in \Pi(G, \mathbf{w})$ for some distinct v_1 and $v_2 \in V(G)$. Let $H = G/\{v_1, v_2\}$. Since G is a counterexample, $G - \{v_1, v_2\}$ is connected and consequently, H is 2-connected. Moreover, H has two negative edges incident with the vertex $\{v_1, v_2\}$. By Lemma 3, G has a set $X \supset \{v_1, v_2\}$ such that $\langle X \rangle \in \Pi(G, \mathbf{w})$, and $G - X$ is disconnected; a contradiction. ▮

Proof of Theorem 2. Let $(G, \mathbf{w})$ be a non-rigid conservative graph. Let $\mathcal{P} \in \Pi(g, \mathbf{w})$ be a regular partition which is not a refinement of any other regular partition. By Theorem 1 it suffices to show that $(G/\mathcal{P}, \mathbf{w})$ is prime. Assume not. Let $H = G/\mathcal{P}$ and let B be a non-prime block of H. Let $\mathbf{w}'$ be the restriction of $\mathbf{w}$ on $E(B)$.

Claim 1. $(B, \mathbf{w}')$ is rigid.

Assume not. Let $X \subseteq V(B)$ be a set guaranteed by Lemma 5. Then $\mathcal{R} = \langle X \rangle \circ \mathcal{P}$ is a regular partition of $(G, \mathbf{w})$ and, moreover, $\mathcal{P} \prec \mathcal{R}$, contradicting the choice of $\mathcal{P}$.

Claim 2. H has a cut vertex.

Otherwise $\mathcal{P}$ is trivial, $G = H = B$, and by Claim 1, $(G, \mathbf{w})$ is rigid, contradicting the assumptions of the theorem.

Now let x_1 be a cut vertex of H lying in B. Since B is rigid and non-prime, B has a vertex x_2 not covered by $E^{-}(\mathbf{w}')$. By Lemma 4(b), $\lambda_{\mathbf{w}}(x_1, x_2) \geqslant 0$. Set $X = \{x_1, x_2\}$ and $\mathscr{R} = \langle X \rangle \circ \mathscr{P}$. The partition $\mathscr{R}$ is regular and $\mathscr{P} \prec \mathscr{R}$, a contradiction. ∎

4. SEYMOUR GRAPHS

Let G be a graph and let $T \subseteq V(G)$ be a vertex subset of even cardinality. Given a set $X \subseteq V(G)$, the cut $\delta(X)$ is called a T-*cut* if $|X \cap T|$ is odd. Let $\nu(G, T)$ and $\tau(G, T)$ denote respectively the maximum number of edge disjoint T-cuts and the cardinality of a minimum T-join in G.

Since each T-join has at least one edge in common with each T-cut, $\nu(G, T) \leqslant \tau(G, T)$. The example $G = K_4$, $T = V(G)$ shows that this inequality can be strict. Nevertheless, it is known that several families of connected graphs (bipartite graphs (Seymour [11])), series-parallel graphs (Seymour [12]), graphs containing neither an odd K_4 nor an odd prism (Gerards [3]) satisfy $\nu(G, T) = \tau(G, T)$ for any even $T \subseteq V(G)$. A graph G is called a *Seymour graph* if $\nu(G, T) = \tau(G, T)$ for every even subset $T \subseteq V(G)$.

Given a conservative graph $(G, \mathbf{w})$, a circuit C of G is called a $\mathbf{w}$-*zero circuit* if the total weight of the edges of C is equal to zero.

In [1] it is proved that a graph G is not a Seymour graph if and only if there exist a conservative weighting $\mathbf{w}$ and $\mathbf{w}$-zero circuits C_1, C_2 such that the graph $C_1 \cup C_2$ is either an odd K_4 or an odd prism. Since the problem of deciding if a weighting is conservative is polynomially solvable [8, p. 241], this implies that the class of Seymour graphs belongs to co-NP.

Our goal in this section is to present a short proof of a weaker version of this theorem (in fact, it is equivalent to the original one, see [1]) also providing a co-NP characterization of Seymour graphs.

THEOREM 3. *A graph G is not a Seymour graph if and only if there exist a conservative weighting* $\mathbf{w}$ *and* $\mathbf{w}$*-zero circuits C_1 and C_2 such that $C_1 \cup C_2$ is non-bipartite and has the maximum degree* 3.

The “if” part of this theorem is due to Sebő; a simple proof (even under weaker assumptions) can be found in [1].

In the proof of the “only if” part as in [1] we will make use of a theorem of Lovász. A connected graph G is called 1-*extendable* if every edge of G lies in a perfect matching. A subdivision of a graph G is said to be *even* if the number of new vertices inserted in every edge of G is even.

The following is an easy consequence of Theorem 5.4.11 in [8].

LEMMA 6 (Lovász). *Let G be a 1-extendable non-bipartite graph. Then G contains an even subdivision of either K_4 or triangular prism.*

We say that a partition $\mathscr{P}$ is *tree-like* if $G/\mathscr{P}$ has no circuits of length more than two. Note that if $\mathscr{P} \in \Pi^*(G, \mathbf{w})$ is tree-like then $E^-(\mathbf{w})$ induces a spanning tree of $G/\mathscr{P}$. Moreover, it is clear that if $\Pi(G, \mathbf{w})$ contains a tree-like partition then so does $\Pi^*(G, \mathbf{w})$.

We will use as an intermediate step the following well-known observation: if a conservative graph $(G, \mathbf{w})$ admits a tree-like terminal partition then $(G, T(\mathbf{w})) = \tau(G, T(\mathbf{w}))$. Thus it remains to prove

LEMMA 7. *Let G be a connected graph. Then*

(a) *G admits a conservative weighting $\mathbf{w}'$ such that $\Pi^*(G, \mathbf{w}')$ contains no tree-like partition implies*

(b) *G admits a conservative weighting $\mathbf{w}''$ with $\mathbf{w}''$-zero circuits C_1 and C_2 such that $C_1 \cup C_2$ is non-bipartite and has the maximum degree* 3.

Proof. Assume that the implication does not hold and G is a counterexample with the smallest number of vertices. Let $\mathbf{w}$ be a conservative weighting of G such that $\Pi^*(G, \mathbf{w})$ contains no tree-like partition. It follows that G has at least two negative edges, i.e., $|E^-(\mathbf{w})| \geqslant 2$. Furthermore, by minimality, G is 2-connected.

Claim 1. $(G, \mathbf{w})$ is non-rigid.

Otherwise the subgraph of G induced by $T(\mathbf{w})$ is bicritical (and, consequently, 1-extendable) and non-bipartite. Applying Lemma 6 we obtain that G contains an even subdivision of K_4 or triangular prism. It follows that G satisfies (b), contradicting the choice of $(G, \mathbf{w})$.

By Claim 1 and Theorem 2, the set of regular terminal partitions $\Pi_r^*(G, \mathbf{w})$ is nonempty. Since $(G, \mathbf{w})$ is non-rigid, $G/\mathscr{P}$ has a cut vertex for any $\mathscr{P} \in \Pi_r^*(G, \mathbf{w})$. Now among all partitions in $\Pi_r^*(G, \mathbf{w})$ choose a partition $\mathscr{P}$ with the minimum number of vertices in a smallest leaf block B of $G/\mathscr{P}$. Let $X \in \mathscr{P}$ be the cut vertex of $G/\mathscr{P}$ belonging to B. Set $Z = \{z \in V(G) : \{z\} \in V(B) \backslash X\}$, $Y = N_G(Z)$, $G_1 = G/Y$, $G_2 = G_1[Z \cup Y]$, $G_3 = G_1 - Z$. Note that by Theorem 1 the graph G_2 is bicritical.

Consider the conservative graph $(G_3, \mathbf{w}')$ where $\mathbf{w}'$ is the restriction of $\mathbf{w}$ to the edge set of G_3.

Claim 2. $\Pi^*(G_3, \mathbf{w}')$ contains no tree-like partition.

Observe first that $|E^-(\mathbf{w}')| \geqslant 2$, for otherwise by the choice of B and $\mathscr{P}$, $|Z| = 1$ and $\mathscr{P}$ must be tree-like, contradicting the choice of $\mathbf{w}$. Assume that the claim is false. Then, because of $|E^-(\mathbf{w}')| \geqslant 2$ and by Remark 1, $(G_3, \mathbf{w}')$

is non-rigid. Therefore, $(G_3, \mathbf{w}')$ admits a tree-like regular terminal partition $\{X_1, ..., X_m\}$. Hence again, by the choice of $\mathscr{P}$ and B, $|Z| = 1$. And then $\{X_1, ..., X_m, Y, Z\}$ is a tree-like partition of $(G, \mathbf{w})$, contradicting the choice of $\mathbf{w}$.

By Claim 2, G_3 satisfies (a). Consequently, by the minimality of G, G_3 admits a conservative weighting $\mathbf{w}''$ with $\mathbf{w}''$-zero circuits $\tilde{C}_1$ and $\tilde{C}_2$ such that the graph $\tilde{H} = \tilde{C}_1 \cup \tilde{C}_2$ is non-bipartite and $\Delta(\tilde{H}) = 3$. Denote by H (respectively, C_k, $k = 1, 2$) the subgraph of G spanned by the edges of $\tilde{H}$ (respectively, $\tilde{C}_k$). Let $Y \cap V(H) = \{v_0, ..., v_l\}$. It follows from $\Delta(\tilde{H}) = 3$ that $l \leqslant 2$. If $|Y \cap V(H)| \leqslant 1$, then $H = \tilde{H}$ and we are done, so we may assume that $l \in \{1, 2\}$. Since $N_G(Z) = Y$, for every $k \in \{0, ..., l\}$, there is an edge $f_k \in E(G)$ connecting v_k with Z. Since G_2 is bicritical, for each $k \in \{0, ..., l\}$, it has a perfect matching $F_k \subset E(G_2)$ such that $f_k \in F_k$. Denote by $\tilde{D}_i$, $i \in \{1, ..., l\}$, the circuit in $F_0 \cup F_i$ containing f_0 and f_i. Set $\tilde{S} = \tilde{D}_1 \cup \tilde{D}_2$. Let S (respectively, D_k) denote the subgraph of G spanned by $E(\tilde{S})$ (respectively, $\tilde{L}_k$). By construction, each L_k is a path, and hence S is a connected subgraph of G.

Case 1. $l = 1$. Then $S = L_1$ and S is an even path whose ends are v_0 and v_1. Let $M = F_0 \cap E(S)$. Set

$$\mathbf{w}^*(e) := \begin{cases} \mathbf{w}''(e) & \text{if} \quad e \in E(G_3), \\ -1 & \text{if} \quad e \in M, \\ +1 & \text{otherwise.} \end{cases} \tag{1}$$

Since M is a matching, $\mathbf{w}^*$ is a conservative weighting of G_1 and, consequently, that of G. Clearly, $\Delta(H \cup S) = 3$. Recall that $H \cup S = C_1 \cup C_2 \cup S$. We may have that either C_1, C_2 are both paths or exactly one of them, say C_1, is a path while C_2 is a circuit. In the former case $C_1 \cup S$ and $C_2 \cup S$ are the desired $\mathbf{w}^*$-zero circuits, otherwise $C_1 \cup S$ and C_2 are those.

Case 2. $l = 2$. In this case, exactly one vertex v_i, say v_0, is incident with an edge which is contained in both circuits $\tilde{C}_1$ and $\tilde{C}_2$. In other words, $Y \cap V(C_k) = \{v_0, v_k\}$, $k = 1, 2$. Let $M = F_0 \cap E(S)$. Set

$$\mathbf{w}^*(e) := \begin{cases} \mathbf{w}''(e) & \text{if} \quad e \in E(G_3), \\ -1 & \text{if} \quad e \in M, \\ +1 & \text{otherwise.} \end{cases} \tag{2}$$

As in Case 1, $\mathbf{w}^*$ is a conservative weighting of G and $\Delta(H \cup S) = 3$. By construction, $C_k \cup L_k$ is a $\mathbf{w}^*$-zero circuit for each $k = 1, 2$. Since $\tilde{H}$ is non-bipartite, $H \cup S$ is non-bipartite too. ∎

5. JOINS CONSISTING OF k COMPONENTS

Let J be a join of a graph G. We say that J consists of k components $J_1, ..., J_k$ if the subgraph of G spanned by J consists of k components $H_1, ..., H_k$ and $J_l = J \cap E(H_l)$, $l = 1, ..., k$.

Let G be a connected graph and $T \subseteq V(G)$ be even. Let J be a minimum T-join of G. Note that $\nu(G, T)$ is uniquely determined by the pair (G, J), or by the conservative graph $(G, \mathbf{w})$ with $\mathbf{w} = \mathbf{w}_J$. By Guan's lemma, given a join J or a conservative weighting $\mathbf{w}$ in a graph G, there will be no confusion to write $\nu(G, J)$ or $\nu(G, \mathbf{w})$ instead of $\nu(G, T)$.

We first present a short proof of the following theorem due to Korach and Penn (for different proofs see also [10] and [2]).

THEOREM 4 (Korach and Penn [5]). *Let J be a connected graph G, consisting of k components. Then*

$$|J| - (k-1) \leqslant \nu(G, J) \leqslant |J|. \tag{3}$$

Proof. The inequality $\nu(G, J) \leqslant |J|$ is obvious, so we have to prove only the left inequality in (3). Let (G, J) be a counterexample to the theorem with the minimum number of vertices. Set $\mathbf{w} = \mathbf{w}_J$.

Claim. $(G, \mathbf{w})$ is non-prime.

Assume to the contrary that $(G, \mathbf{w})$ is a prime conservative graph whose blocks are $B_1, ..., B_m$ and $|B_i| = b_i$, $i = 1, ..., m$. By the definition of prime graph, $|J| = \sum_{i=1}^{m} b_i/2$. By induction on the number of blocks, we conclude that $k = 1 + \sum_{i=1}^{m} (b_i/2 - 1)$, and hence $k = 1 + |J| - m$. But $\nu(G, J) \geqslant m$ and whence (3) holds, contradicting the assumption that G is a counterexample.

By Claim and Theorem 1, G has vertices x and y with $\lambda_{\mathbf{w}}(x, y) \geqslant 0$. Consider $G' = G/\{x, y\}$. Then, by the choice of x and y, J is a join of G'. The number of components of J in G' does not exceed that in G i.e., k. By the minimality of G, $\nu(G', J) \geqslant |J| - (k-1)$. But, clearly, $\nu(G', J) \leqslant \nu(G, J)$. ∎

We say that a conservative weighting $\mathbf{w}$ (a join $J = E^-(\mathbf{w})$) of a graph G is *attainable* (otherwise *non-attainable*) if $\nu(G, \mathbf{w}) = |E^-(\mathbf{w})|$ (or, which is the same, $\nu(G, J) = |J|$). In these terms a graph G is a Seymour graph if and only if every conservative weighting (every join) of G is attainable.

The following simple consequence of Theorem 2 turns out to be very useful when dealing with joins consisting of two components.

LEMMA 8. *Let G be a connected graph. Let J be a join of G consisting of two components J_1 and J_2. Let $|J| \geqslant 3$. Then there exists a vertex $v \in V(G)$ incident with exactly one edge in J and such that J is a join of $G/N_G(v)$.*

Proof. Consider the conservative graph $(G, \mathbf{w})$ with $\mathbf{w} = \mathbf{w}_J$. Since J consists of two components and $|J| \geqslant 3$, at least two edges in J are adjacent. Therefore $(G, \mathbf{w})$ is non-rigid and by Theorem 2, $\Pi_r^*(G, \mathbf{w})$ is nonempty. Let $\mathscr{P} \in \Pi_r^*(G, \mathbf{w})$. Then $G/\mathscr{P}$ has at least two blocks and, since J consists of two components, at most one of these blocks has more than two vertices. Therefore there is a leaf block of $G/\mathscr{P}$ consisting of two vertices X_1 and X_2 connected by exactly one negative edge e. Since $\mathscr{P}$ is regular, one of them, say X_2, is a cut vertex of $G/\mathscr{P}$ while X_1 consists of one vertex v incident with exactly one edge e in J. Since $N_G(v) \subseteq X_2$, J is a joint of $G/N_G(v)$. ∎

We now present a short proof of a characterization of attainable joins consisting of two components due to Korach [4] (for a different proof see also [2]). This theorem can be considered as a refinement of Theorem 3 in the special case of conservative weightings whose set of negative edges consists of two components.

Let $(G, \mathbf{w})$ be a conservative graph in which G is a subdivision of K_4. Following Frank [2], call $(G, \mathbf{w})$ a *bad-K_4 graph* if G is non-bipartite and is the union of two $\mathbf{w}$-zero circuits.

THEOREM 5 (Korach [4]). *Let G be a connected graph and J be a join of G consisting of two components. The following statements are equivalent:*

(a) *J is attainable;*

(b) *the union of any two $\mathbf{w}_J$-zero circuits is bipartite;*

(c) *$(G, \mathbf{w})$ contains no bad-K_4 graph.*

Proof. The (easy) proof of $(a) \Rightarrow (b)$ coincides with the proof of the "if part" of Theorem 3 (see, e.g., [1]). Since $(b) \Rightarrow (c)$ is obvious, it remains to prove $(c) \Rightarrow (a)$. We may assume that G is a 2-connected graph. We prove by induction on $|J|$ the following somewhat stronger statement: if J is non-attainable, then there exist two $\mathbf{w}_J$-zero circuits C_1 and C_2 such that

(1) $H := C_1 \cup C_2$ is a subdivision of K_4;

(2) if s is a vertex of degree 3 in H incident with exactly on edge f in J, then f lies in both circuits C_1 and C_2.

(3) if s is a vertex of H incident with no edge in $J \cap E(H)$, then s is not covered by J.

If $|J| = 2$, then $H = K_4$ and the statement is obviously true. Suppose that the statement is true if $|J| < k$, $k > 2$.

By Lemma 8 G has a vertex v incident with exactly one edge e in J and such that J is a join of $G/N_G(v)$. Set $X = N_G(v)$. We may assume that $e \in J_1$. Then $|J_1| \geqslant 2$. Set $J' = J \setminus \{e\}$. Set $\tilde{G} = G/X$. Note that $\tilde{G}$ has exactly one block $\tilde{G}_1$ with $|V(\tilde{G}_1)| > 2$ and $E(\tilde{G}_1) \cap J_i \neq \emptyset$, $i = 1, 2$. Otherwise we would have that either J is attainable in G or J has more than two components. It follows that J' is a non-attainable join in $\tilde{G}_1$ and, by the induction hypothesis, $\tilde{G}_1$ has two J'-zero circuits $\tilde{L}_1$ and $\tilde{L}_2$ satisfying (1)–(3). Let $H' = L_1 \cup L_2$ correspond to $\tilde{H}' = \tilde{L}_1 \cup \tilde{L}_2$ in G. Let $V(H') \cap X = \{u_1, ..., u_l\}$. If $l \leqslant 1$ we are done with $H = H'$. Therefore we may assume that $l \in \{2, 3\}$. Since $e \in J_1$ and $E(\tilde{G}_1) \cap J_1 \neq \emptyset$, by (3) we have that $e = u_i v$ for some i, say $i = 1$, and, moreover, u_1 is incident with some edge $f \in J \cap E(\tilde{G}_1)$.

Case 1. $l = 3$. Observe that u_2 and u_3 are not covered by J. Otherwise J would consist of one component in $\tilde{G}$ and therefore it would be attainable in $\tilde{G}$ and thereby in G. Note that L_1 and L_2 are paths with the common endpoint u_1. We may assume that u_{i+1} is the second endpoint of the path L_i, $i = 1, 2$. Set $C_i = L_i \cup P_i$, where P_i is the 2-path on vertices (u_1, v, u_{i+1}), $i = 1, 2$. By construction C_1 and C_2 are $\mathbf{w}_J$-zero circuits, and their union $H = C_1 \cup C_2$ satisfies properties (1)–(3).

Case 2. $l = 2$. As in the previous case u_2 is not covered by J. Denote by P the 2-path $u_1 u_2 v$. It is easy to see that L_1 and L_2 are either both paths with the endpoints u_1 and u_2 or one of them, say, L_1 is a circuit whereas the other is a path. Set $C_i = L_i \cup P$, $i = 1, 2$ if the former case holds, and $C_1 = L_1$ and $C_2 = L_2 \cup P$ otherwise. Again, by construction, C_1 and C_2 are $\mathbf{w}_J$-zero circuits and $H = C_1 \cup C_2$ satisfies properties (1)–(3). ∎

Remark 3. It follows from the proof above that if G is a graph and J is a non-attainable join of G consisting of two components, then G contains a bad-K_4 graph H such that $E(H) \cap J$ also consists of two components.

6. CONNECTED PARTITIONS

In this section we show that the set of terminal partitions of any conservative graph always contains a partition satisfying special connectivity properties. This result [6] was a starting point of the present research. The proof in [6] is long and sophisticated. Here we present a short proof based on ideas developed in Section 2. Several applications of this theorem can be found in [6] and [7].

Let $(G, \mathbf{w})$ be a conservative graph. We call a partition $\mathscr{P} \in \Pi(G, \mathbf{w})$ *connected* if

(1) for any two distinct classes X and Y of $\mathscr{P}$, the set X is contained in a component of $G - Y$;

(2) for any $X \in \mathscr{P}$ and any component C of $G - X$, the graph $G - C$ is connected.

If $\mathscr{P}$ is a partition of $V(G)$ and $\mathscr{Q}$ is a partition of $V(G/\mathscr{P}) = \mathscr{P}$, then $\mathscr{Q} \circ \mathscr{P}$ will denote the partition $\mathscr{R} = \{Z \subseteq V(G) : Z = \bigcup_{X \in Y} X,\ Y \in \mathscr{Q}\}$.

Theorem 6 [6]. *For any conservative graph* $(G, \mathbf{w})$, *there exists a connected partition* $\mathscr{P}$ *such that* $(G/\mathscr{P}, \mathbf{w})$ *is prime.*

By Theorem 1 the partition $\mathscr{P}$ is terminal.

Lemma 9. *Let* $(G, \mathbf{w})$ *be a conservative graph. If* $\mathscr{P}$ *is a connected partition of* $(G, \mathbf{w})$ *and* $\mathscr{Q}$ *is a connected partition of* $(G/\mathscr{P}, \mathbf{w})$, *then* $\mathscr{R} = \mathscr{Q} \circ \mathscr{P}$ *is a connected partition of* G.

Proof. We have to check properties (1) and (2) in the definition of connected partition.

(1) Let $Y_1, Y_2 \in \mathscr{R}$, $Y_1 \neq Y_2$. Take $X \in \mathscr{P}$ such that $X \subseteq Y_1$. Since $\mathscr{P}$ is connected, for any $Z \subseteq Y_2$ such that $Z \in \mathscr{P}$, Z lies in a component of $G - X$. As $\mathscr{Q}$ is connected, it follows that Y_2 lies in a component C of $G - X$. But $G - C$ is connected and thus X lies in a component of $G - Y_2$. Since X was chosen arbitrarily and Q is connected, Y_1 lies in a component of $G - Y_2$.

(2) Let C be a component of $G - Y$, $Y \in \mathscr{R}$. It suffices to show that Y lies in a component of $G - C$. Take $X \in \mathscr{P}$ such that $X \subseteq Y$. Then C lies in a component C' of $G - X$. Since $\mathscr{P}$ is connected, the graph $G - C'$ is connected. It follows that X is contained in a component of $G - C$. Since X was chosen arbitrarily and Q is connected, Y lies in a component of $G - C$. ∎

Lemma 10. *Let* $(G, \mathbf{w})$ *be a 2-connected conservative graph. If* $(G, \mathbf{w})$ *is non-prime, then there exists a set* $X \subseteq V(G)$ *such that* $|X| \geqslant 2$, $(G/X, \mathbf{w})$ *is conservative, and for any component* C *of* $G - X$ *and any* $x \in X$, *the graph* $G - C$ *is connected and* $N_G(x) \cap V(C) \neq \varnothing$.

Proof. We may assume that $\lambda_{\mathbf{w}}(v_1, v_2) < 0$ if $v_1 v_2 \in E(G)$, $v_1 \neq v_2$, for otherwise we are done with $X = \{v_1, v_2\}$. By Lemma 4 it follows that $(G, \mathbf{w})$ is non-rigid. Hence, by Lemma 5, there exists a set $Y \subseteq V(G)$ such that $(G/Y, \mathbf{w})$ is conservative, and $G - Y$ is disconnected. Let X be an inclusion-wise minimal set among all sets of this kind. Since G is 2-connected and $G - X$ has at least two components, the conclusion follows. ∎

Proof of Theorem 6. Let $(G, \mathbf{w})$ be a conservative graph. Since the partition consisting of singletons is connected, the set of connected partitions of $(G, \mathbf{w})$ is nonempty. Let $\mathscr{P}$ be a connected partition of $(G, \mathbf{w})$ which is not a refinement of any other connected partition. We show that $H = G/\mathscr{P}$ is prime. Assume not. Let B be a non-prime block of H and let $X \in V(B)$ be a set guaranteed by Lemma 10. As B is 2-connected, the partition $\mathscr{Q}$ consisting of X and the singletons is a connected partition of $(H, \mathbf{w})$. Then by Lemma 9, $\mathscr{P}$ is a refinement of the connected partition $\mathscr{Q} \circ \mathscr{P}$, a contradiction. ▮

A slight modification of the above argument yields a strengthening of Theorem 6 in the case of planar graphs.

THEOREM 7. *For any planar conservative graph* $(G, \mathbf{w})$, *there exists a connected partition* $\mathscr{P}$ *such that* $(G/\mathscr{P}, \mathbf{w})$ *is prime and planar.*

Proof. Let $(G, \mathbf{w})$ be a planar conservative graph. Let $\mathscr{P}$ be a connected partition of $(G, \mathbf{w})$ with $G/\mathscr{P}$ planar, which is not a refinement of any other connected partition with this property. Assume that $H = G/\mathscr{P}$ is non-prime. Let B be a non-prime block of H and let $X \in V(B)$ be a set guaranteed by Lemma 10. Denote by r the number of components of $H - X$. Since a planar graph cannot contain $K_{3,3}$-minor, from the properties of X stated in Lemma 10 we obtain that only the following three cases are possible: $r = 1$; $r = 2$ and $|X| \geqslant 2$; $r \geqslant 3$ and $|X| = 2$. Since $H[X]$ is connected if $r = 1$, all these cases imply that H/X is planar. Consider the partition $\mathscr{R} = \langle \mathscr{X} \rangle \circ \mathscr{P}$. By construction and in view of Lemma 10, $\mathscr{R}$ is connected, $\mathscr{P} \prec \mathscr{R}$, and $G/\mathscr{R}$ is planar; a contradiction. ▮

Remark 4. By Theorems 2 and 6, each non-rigid conservative graph has a regular terminal partition and a connected terminal partition. It is worth noting that there are non-rigid conservative graphs having no terminal partition which is simultaneously regular and connected. As an example consider the conservative graph $(G, \mathbf{w})$, where G is a 3-path (x, y, z, u) and $E^-(\mathbf{w}) = \{yz, zu\}$. It has exactly two terminal partitions: $\{\{x, z\}, \{y\}, \{u\}\}$ and $\{\{x, y\}, \{z\}, \{u\}\}$. The first partition is regular but not connected; the second one is connected but not regular. By contrast, conservative graphs $(G, \mathbf{w})$ satisfying $\lambda_{\mathbf{w}}(v_1, v_2) < 0$ for each $v_1 v_2 \in E(G)$, $v_1 \neq v_2$, possess a remarkable property: it can be shown that every terminal partition of $(G, \mathbf{w})$ is regular and connected.

7. APPENDIX: A DIRECT PROOF OF LEMMA 1

In this section we give a direct short proof of Lemma 1. Though the proof does not refer to the fundamental theorem of Sebő, it exploits key points of Sebő's proof including the following observation.

LEMMA 11 ("Switching Lemma" [10]). *Let $(G, \mathbf{w})$ be a conservative graph. Let C be a $\mathbf{w}$-zero circuit of G. Let $\mathbf{w}'$ be the ± 1 edge weighting with $E^-(\mathbf{w}') = E^-(\mathbf{w}) \oplus E(C)$. Then $\mathbf{w}'$ is conservative and, moreover, the distance functions $\lambda_{\mathbf{w}'}$ and $\lambda_{\mathbf{w}}$ coincide.*

Proof. The conservativeness of $\mathbf{w}'$ follows from Guan's lemma. By symmetry it suffices to show that $\lambda_{\mathbf{w}'} \leqslant \lambda_{\mathbf{w}}$. Let P be an x, y path in G. Set $F = E(P) \oplus E(C)$. Since $\mathbf{w}'(E(C) \backslash E(P)) = \mathbf{w}(E(C) \cap E(P))$,

$$\mathbf{w}'(F) = \mathbf{w}(E(P) \backslash E(C)) + \mathbf{w}(E(P) \cap E(C)) = \mathbf{w}(E(P)).$$

Note that F is an $\{x, y\}$-join of G. Hence the subgraph spanned by F contains an x, y path Q. Since $F \backslash E(Q)$ spans disjoint circuits in G and $\mathbf{w}'$ is conservative, $\mathbf{w}'(E(Q)) \leqslant \mathbf{w}(E(P))$, as desired. ∎

LEMMA 12. *Let $(G, \mathbf{w})$ be a 2-connected bipartite conservative graph. Let $x \in V(G)$ be incident with at least two negative edges. Then there exists another vertex z such that $\lambda_{\mathbf{w}}(x, z) \geqslant 0$ and z lies in the same color class as x.*

Proof. We proceed by induction on the number of vertices. Note that the lemma is true if $|V(G)| \leqslant 4$. Assume now that $|V(G)| > 4$ and the lemma is true for all conservative graphs with smaller numbers of vertices. Set $\lambda^* = \min\{\lambda_{\mathbf{w}}(x, x') \colon x' \in V(G)\}$. Note that $\lambda^* \leqslant -1$. Let y be a vertex of G with $\lambda_{\mathbf{w}}(x, y) = \lambda^*$. Observe (as in [10]) that

$$y \text{ is incident with exactly one negative edge.} \tag{$*$}$$

Otherwise, for some $y' \in N_G(y)$, $\lambda_{\mathbf{w}}(x, y')$ would be less than $\lambda_{\mathbf{w}}(x, y)$, contradicting the choice of y.

Let $U = N_G(y)$. Note that $|U| \geqslant 2$. We claim that $\mathbf{w}$ is a conservative weighting of G/U. Assume to the contrary that G/U has a circuit $\tilde{L}$ of negative weight. Then the edges of $\tilde{L}$ span a y', y'' path L in G with y', $y'' \in N_G(y)$ and yy', $yy'' \notin E^-(\mathbf{w})$. Note that the weight of L cannot be less than -2. In fact, since y' and y'' lie in the same color class of G, it is equal to -2. It follows that $C := (y, y', L, y'', y)$ is a $\mathbf{w}$-zero circuit of G. Consider the conservative weighting $\mathbf{w}'$ with $E^-(\mathbf{w}') = E^-(\mathbf{w}) \oplus E(C)$. By construction y is incident with three edges in $E^-(\mathbf{w}')$. But, by Lemma 11, y must satisfy property $(*)$ with respect to the weight $\mathbf{w}'$, a contradiction.

Note that G/U is bipartite. We may assume that $x \notin U$, for otherwise we are done taking z to be any vertex in $U \backslash \{x\}$. Consider the block B of G/U that contains x. Note that B and the restriction of $\mathbf{w}$ to $E(B)$ satisfy the assumptions of the lemma. Hence, by the induction hypothesis, B has a vertex z' lying at nonnegative $\mathbf{w}$-distance from x. Now if $z' \neq \{U\}$ we take $z = z'$; otherwise we let z be an arbitrary vertex of U. ∎

Proof of Lemma 1. Consider the graph H obtained from G by replacing each edge $e = uv$ with a path (u, z_e, v) of length 2. Furthermore, for any $e \in F(G)$, set $\mathbf{w}'(uz_e) = \mathbf{w}'(vz_e) = \mathbf{w}(e)$. Now $(H, \mathbf{w}')$ is a 2-connected bipartite conservative graph containing the vertex x incident with at least two $\mathbf{w}'$-negative edges. By Lemma 12, H has another vertex z lying in the same color class as x and such that $\lambda'_{\mathbf{w}}(x, z) \geqslant 0$. By construction it follows that $z \in V(G)$ and $\lambda_{\mathbf{w}}(x, z) = \lambda_{\mathbf{w}'}(x, z)/2 \geqslant 0$, as desired. ∎

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