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NOTE

Local and Mean Ramsey Numbers for Trees

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In this note we find the local and mean k-Ramsey numbers for many trees for which the Erdős–Sós tree conjecture holds. © 2000 Academic Press

The usual Ramsey number $R(G, k)$ is the smallest positive integer n such that any coloring of the edges of K_n by at most k colors contains a monochromatic copy of G. Over the past years several papers have been written in which the number of different colors used has no longer been restricted to k, but a restriction is placed on the number of colored edges incident to the vertices. To be precise, let H be a fixed graph of order n and let f be a coloring on the edges of H. For each $v \in V(H)$ define $k_f(v)$ as the number of distinct colors that appear on the edges of H incident to v. The coloring f is called a *local k-coloring* if $k_f(v) \leqslant k$ for all $v \in V(H)$, and is called a *mean k-coloring* if $(1/n) \sum_v k_f(v) \leqslant k$. Further, the *local k-Ramsey number* $R(G, k\text{–loc})$ is defined as the smallest positive integer n such that

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any local k-coloring of K_n contains a monochromatic copy of G. The *mean k-Ramsey number* $R(G, k\text{–mean})$ is defined analogously.

Proofs of the existence of these Ramsey numbers as well as other related results can be found in [2–7]. Since every k-coloring of the edges of K_n is a local k-coloring and every local k-coloring is a mean k-coloring, it is clear that $R(G, k) \leqslant R(G, k\text{–loc}) \leqslant R(G, k\text{–mean})$. In [5] it is shown that the first of these inequalities may be strict for certain trees. It is not known at this tune whether the second of these inequalities is ever strict. Caro and Tuza [3] have proved several results concerning this inequality and Schelp [6] has shown $R(K_m, k\text{–loc}) = R(K_m, k\text{–mean})$ for $m \geqslant 3$ and $k \geqslant 2$.

In 1963 Erdős and Sós conjectured that every graph with average degree greater than $n-2$ contains every tree T on n vertices. The purpose of this note is to give upper bounds for $R(T, k\text{–loc})$ and lower bounds for $R(T, k\text{–mean})$ for trees. In particular, it appears that $R(T, k\text{–loc}) = R(T, k\text{–mean})$ for many trees where the Erdős–Sós conjecture holds. It is expected that the k-local and k-mean Ramsey numbers are identical for most graphs. In fact, it would be interesting to determine whether this pair of numbers can ever differ for some graph G.

Lemma 1. *If every graph H with average degree $a(H) > a$ contains a subgraph isomorphic to G, then $R(G, k\text{–mean}) \leqslant ak + 2$.*

Proof. Suppose that for $N > ak + 1$, there exists a mean k-coloring f of edges of K_N such that $K_n N$ contains no monochromatic copy of G. Assume that m colors were used, there are e_i edges of color i, these e_i edges are incident with v_i vertices. From the definition of a mean k-coloring,

$$\sum_{i=1}^{m} v_i \leqslant kN. \tag{1}$$

Under conditions of the lemma, for every i, $2e_i \leqslant av_i$. Therefore,

$$\frac{N(N-1)}{2} = \sum_{i=1}^{m} e_i \leqslant \frac{a}{2} \sum_{i=1}^{m} v_i \leqslant \frac{akN}{2}.$$

It follows that $N - 1 \leqslant ak$, a contradiction to the choice of N. ∎

Let us recall the following well-known result of Wilson [8].

Lemma 2. *For every integer $d \geqslant 3$, there exists $N_0 = N_0(d)$ such that for every $N > N_0$ with the property that $(N-1)/(d-2)$ and $N(N-1)/(d-1)(d-2)$ are integers, the edge set of K_N can be partitioned into complete graphs on $d-1$ vertices.*

We shall need the following easy consequence of Lemma 2.

Corollary 3. *For every graph G on d vertices without isolated vertices, and for every sufficiently large k such that* $k(k-1)/(d-1)$ *is an integer,*

$$R(G, k\text{–}loc) \geqslant (d-2)\,k+2. \tag{2}$$

Proof. Under the conditions of the corollary, for $N=k(d-2)+1$,

$$\frac{N(N-1)}{(d-1)(d-2)}=\frac{(k(d-1)-k+1)\,k}{d-1}$$

is an integer, and so the conditions of Lemma 2 are satisfied. ▮

Call a tree $T=(V, E)$ *an ES-tree* if every graph with average degree greater than $|V|-2$ contains T. In 1963, Erdős and Sós conjectured that every tree is an ES-tree. Many trees are known to be ES-trees, and there are numerous partial results concerning the conjecture (see [1, 9]). It was announced recently that M. Ajtai, J. Komlós, and E. Szemerédi confirmed the Erdős–Sós conjecture for sufficiently large trees.

Theorem 4. *For every ES-tree T on d vertices and for every sufficiently large k such that* $k(k-1)/(d-1)$ *is an integer,*

$$R(T, k\text{–loc})=R(T, k\text{–mean})=(d-2)\,k+2. \tag{3}$$

Proof. Immediate from Lemma 1 and Corollary 3. ▮

It is likely that $R(T, k\text{–loc})=R(T, k\text{–mean})$ for all trees T and for all k. In fact, we believe that k-local and k-mean Ramsey numbers are the same for sparse graphs of large order.

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