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ON THE NUMBER OF EDGES IN COLOUR-CRITICAL GRAPHS AND HYPERGRAPHS

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A (hyper)graph G is called k-critical if it has chromatic number k, but every proper sub(hyper)graph of it is $(k-1)$-colourable. We prove that for sufficiently large k, every k-critical triangle-free graph on n vertices has at least $(k-o(k))n$ edges. Furthermore, we show that every $(k+1)$-critical hypergraph on n vertices and without graph edges has at least $(k-3/\sqrt[3]{k})n$ edges. Both bounds differ from the best possible bounds by $o(kn)$ even for graphs or hypergraphs of arbitrary girth.

1. Introduction

In this paper, we continue studying colour-critical graphs and hypergraphs with few edges (cf. [10–12]).

A *hypergraph* $G=(V,E)$ consists of a finite set $V=V(G)$ of *vertices* and a set $E=E(G)$ of subsets of V, called *edges*, each having cardinality at least two. An edge e with $|e|=2$ is called an *ordinary edge*. A *graph* is a hypergraph in which each edge is ordinary. The *degree* $d_G(x)$ of a vertex x in G is the number of the edges in G containing x. The subhypergraph of G induced by $X\subseteq V(G)$ is denoted by $G[X]$, i.e. $V(G[X])=X$ and $E(G[X])=\{e\in E(G)\,|\,e\subseteq X\}$; further, $G-X=G[V(G)-X]$.

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Consider a hypergraph G and assign to each vertex x of G a set $L(x)$ of colours (positive integers). Such an assignment L of sets to vertices in G is referred to as a *colour scheme* (or briefly, a *list*) for G. An L*-colouring* of G is a mapping φ of $V(G)$ into the set of colours such that $\varphi(x)\in L(x)$ for all $x\in V(G)$ and $|\{\varphi(x)\,|\,x\in e\}|\geq 2$ for all $e\in E(G)$. If G admits an L-colouring, then G is said to be L*-colourable*. In case of $L(x)=\{1,\dots,k\}$ for all $x\in V(G)$, we also use the terms k*-colouring* and k*-colourable*, respectively. G is said to be k*-choosable* or k*-list-colourable* if G is L-colourable for every list L of G satisfying $|L(x)| = k$ for all $x \in V(G)$. The *chromatic number* $\chi(G)$ (*choice number* $\chi_l(G)$) of G is the least integer k such that G is k-colourable (k-choosable).

We say that a hypergraph G is L*-critical* where L is a given list for G if G is not L-colourable but every proper subhypergraph of G is L-colourable. In case of $L(x)=\{1,\dots,k-1\}$ for all $x\in V(G)$, we also use the term k*-critical.* A hypergraph G is said to be k*-list-critical* if G is L-critical for some list L of G where $|L(x)|=k-1$ for all $x\in V(G)$. Clearly, every k-critical hypergraph is k-list-critical.

It is known (see, e.g. [6]) that for every integer $k\geq 3$, there are infinitely many k-critical graphs with average degree less than k. Our first result is that this is not the case for triangle-free k-critical graphs provided that k is large.

Theorem 1. *Let G be a triangle-free graph on n vertices. If G is k-list-critical, then G has at least $(k-o(k))n$ edges. In particular, the average degree of G is at least $2k-o(k)$.*

The value $2k$–$o(k)$ is asymptotically tight (in k), since there are k-critical graphs of arbitrary girth and with average degree at most $2(k$–$2)$ (see, e.g. [2, 3, 9]).

For hypergraphs and large k, there is a large gap between lower and upper bounds on the number of edges in a uniform k-critical hypergraph. It was proved in [5, 14–16] that, for given integers $k\geq 3$, $r\geq 3$ and $n>k$, every k-critical r-uniform hypergraph on n vertices has at least $\max\{1,(k-1)/r\}n$ edges. The hypergraphs obtained by the best known constructions (see [1, 3]) have about $(k-2)n$ edges. We will prove that these constructions are close to the truth for large k.

Theorem 2. *Let G be a hypergraph on n vertices and without ordinary edges. If G is k-list-critical, then G has at least $k(1-3/\sqrt[3]{k})n$ edges.*

Theorem 2 implies, in particular, that every $(k+1)$-critical hypergraph on n vertices and without ordinary edges has at least $k(1-3/\sqrt[3]{k})n$ edges.

Recently, it was proved in [9] that, for every $k \geq 3$, $r \geq 3$, $g \geq 3$ and infinitely many n, there are $(k+1)$-critical r-uniform hypergraphs on n vertices having girth g and fewer than kn edges.

2. Proof of Theorem 1

The proof of Theorem 1 is mainly based on the following recent result of A. Johansson [7].

Theorem 3. *If G is a triangle-free graph with maximum degree at most Δ, then $\chi_l(G) \leq o(\Delta)$.* ∎

By a *hereditary graph property* we mean a class $\mathcal{P}$ of graphs such that if G is a member of $\mathcal{P}$, then every graph isomorphic to some induced subgraph of G is a member of $\mathcal{P}$, too. Theorem 1 is an immediate consequence of Theorem 3 and the following result.

Theorem 4. *Let $\mathcal{P}$ be a hereditary graph property such that $\chi_l(G) \leq f(\Delta)$ for every graph $G \in \mathcal{P}$ with maximum degree at most Δ where $f(k) = o(k)$. Then every k-list-critical graph $G \in \mathcal{P}$ on n vertices has at least $(k - o(k))n$ edges.*

Proof. We may assume that f is a continuous and monotonically increasing function where $f(0) \geq 1$. Consequently, there is function g such that $g(k)f(g(k)) = k^2$ for every integer $k \geq 1$. Then it follows by an easy calculation that

$$(k - f(g(k)))(1 - \frac{k}{g(k)}) \geq k - 2f(g(k)) \tag{1}$$

for every integer $k \geq 1$. Furthermore, because of $f(k) = o(k)$, we conclude that

$$f(g(k)) = o(k).$$

Next, consider a k-list-critical graph $G \in \mathcal{P}$ on n vertices and m edges. We show that if k is sufficiently large, then $m \geq (k - 2f(g(k)))n = (k - o(k))n$.

Since G is k-list-critical, there is a list L for G such that G is L-critical and $|L(x)| = k - 1$ for every $x \in V = V(G)$. For $x \in V$ and $U \subseteq V$, let $d(x : U)$ denote the number of vertices in U that are adjacent to x in G. Let $X = \{x \in V \mid d_G(x) \geq g(k)\}$, $Y = V - X$. We distinguish two cases.

Case 1. There exists a non-empty subset A of Y such that, for all $a \in A$,

$$d(a : V - A) \leq k - 1 - \lfloor f(g(k)) \rfloor. \tag{2}$$

Since G is L-critical, there is an L-colouring φ of $G-A$. For the induced subgraph $G'=G[A]$ of G, define the list L' by $L'(a)=L(a)-\{\varphi(v) \mid av\in E(G)\,\&\,v\in V-A\}$ for every $a\in A$. From (2) it then follows that, for all $a\in A$,

$$|L'(a)| \geq k-1-d(a:V-A) \geq \lfloor f(g(k))\rfloor.$$

Furthermore, for all $a\in A$, we have

$$d_{G'}(a) \leq d_G(a) \leq g(k).$$

Since $G'\in\mathcal{P}$, this implies that G' is L'-colourable and, therefore, G is L-colourable, a contradiction.

Case 2. For every non-empty subset A of Y there exists an $a\in A$ such that

$$d(a:V-A) \geq k-\lfloor f(g(k))\rfloor \geq k-f(g(k)).$$

This implies, in particular, that there is an orientation of G such that for the indegree of every vertex $y\in Y$ we have $d^-(y)\geq k-f(g(k))$. Clearly, $d^+(x)+d^-(x)=d_G(x)\geq g(k)$ for every $x\in X$. Because of (1) and $\frac{k^2}{g(k)}=f(g(k))=o(k)$, we now conclude that if k is sufficiently large, then $(1-\frac{k}{g(k)})\geq\frac{k}{g(k)}$ and, moreover,

$$\begin{aligned}
m &= \sum_{v\in V}\frac{k}{g(k)}d^+(v)+\sum_{v\in V}\left(1-\frac{k}{g(k)}\right)d^-(y)\\
&\geq \sum_{x\in X}\frac{k}{g(k)}(d^+(x)+d^-(x))+\sum_{y\in Y}\left(1-\frac{k}{g(k)}\right)d^-(y)\\
&\geq k|X|+(k-2f(g(k)))|Y|\\
&\geq (k-2f(g(k)))n
\end{aligned}$$

This proves Theorem 4. ∎

Remark. Recently, Johansson [8] proved that for every positive integer r there is a constant c_r such that $\chi_l(G)\leq(c_r\Delta\log\log\Delta)/\log\Delta$ for every K_r-free graph G with maximum degree at most $\Delta\geq 2$. Using this result, Theorem 4 implies that if r is an positive integer, then every k-list-critical K_r-free graph on n vertices has at least $(k-o(k))n$ edges.

3. Proof of Theorem 2

We need the Lovász Local Lemma in general form (see e.g. [4, p.53-54]):

Lemma 1. *Let $A_1,\dots,A_n$ be events in an arbitrary probability space. A directed graph $D=(V,E)$ on the set of vertices $V=\{1,\dots,n\}$ is called a dependency digraph for the events $A_1,\dots,A_n$ if for each i, $1\le i\le n$, the event A_i is mutually independent of all the events A_j such that $(i,j)\notin E$. Suppose that $D=(V,E)$ is a dependency digraph for the above events and suppose there are real numbers $x_1,\dots,x_n$ such that $0\le x_i<1$ and*

$$\mathbf{P}(A_i)\le x_i\prod_{(i,j)\in E}(1-x_j)$$

for all $1\le i\le n$. Then

$$\mathbf{P}\Big(\bigwedge_{i=1}^{n}\overline{A_i}\Big)\ge\prod_{i=1}^{n}(1-x_i).$$

In particular, with positive probability no event A_i holds. ∎

The following technical observation will be also used.

Claim 1. *Let $k>0$, $0<b\le 1/k$ and $f(y)=e^{a-by}/(k-y)$. Then f is a monotonically increasing function on the interval $(0,k)$.*

Proof. For $y\in(0,k)$ we have

$$f'(y)=\frac{-b\,e^{a-by}(k-y)+e^{a-by}}{(k-y)^2}=\frac{e^{a-by}(1-bk+by)}{(k-y)^2}>0.$$ ∎

Proof of Theorem 2. Let $G=(V,E)$ be a hypergraph on n vertices and without ordinary edges, and let L be a list for G such that $|L(x)|=k$ for all $x\in V$. Assume that G is L-critical. Let $z=\sqrt[3]{k}$. We have to show that $|E|\ge k(1-3/z)|V|$. For $z\le 3$, this is evident. Now, asssume $z>3$. Define the function g from the set of positive integers into the set of real numbers by

$$g(m)=\begin{cases}1-1/z \text{ if } m=1,\\ 2^{1-m}/z \text{ if } m\ge 2.\end{cases}\tag{3}$$

In order to count the number of edges in G, consider the following *Procedure*:

<u>*Step* 0</u>: Let $V_0=V$, $E_0=E$. If we have

$$w_0(v):=\sum_{\{e\in E_0|v\in e\}}g(|e|)<k(1-3/z)\tag{4}$$

for every $v\in V_0$, then stop. Otherwise, choose a vertex $v_1\in V_0$ for which (4) does not hold and go to Step 1.

<u>*Step t*</u> $(t \geq 1)$: If $t = n$, then stop. Otherwise, let $V_t = V_{t-1} - \{v_t\}$ and let E_t denote the family of all non-empty sets $e \cap V_t$ where $e \in E$. If we have

$$w_t(v) := \sum_{\{e \in E_t | v \in e\}} g(|e|) < k(1 - 3/z) \tag{5}$$

for every $v \in V_t$, then stop. Otherwise, choose a vertex $v_{t+1} \in V_t$ for which (5) does not hold and go to Step $t+1$.

First, suppose that the Procedure terminates in Step n. Then $V = \{v_1, \ldots, v_n\}$ and $w_{i-1}(v_i) \geq k(1 - 3/z)$ for $i = 1, \ldots, n$. Let

$$S = \sum_{i=1}^{n} w_{i-1}(v_i) = \sum_{e \in E_0, v_1 \in e} g(|e|) + \cdots + \sum_{e \in E_{n-1}, v_n \in e} g(|e|).$$

On the one hand, we have $S \geq k(1 - 3/z)|V|$. On the other hand, we infer that

$$S = \sum_{e \in E} (1 - 1/z + \sum_{i=2}^{|e|} 2^{1-i}/z) < \sum_{e \in E} 1 = |E|.$$

Consequently, $|E| > k(1 - 3/z)|V|$.

Now, suppose that the Procedure terminates in Step h, where $h < n$. In the sequel, let $\widetilde{V} = V_h$, $\widetilde{E} = E_h$ and $\widetilde{e} = e \cap \widetilde{V}$ for every $e \in E$. Note that $\widetilde{E}$ is the family of all non-empty sets $\widetilde{e}$ where $e \in E$. For every vertex $v \in \widetilde{V}$, let F_v denote the set of all edges $e \in E$ such that $\widetilde{e} = \{v\}$, and let $a_v = |F_v|$. Let $F = \{e \in E \mid |\widetilde{e}| \geq 2\}$. Since the Procedure stopped in Step h, for every $v \in \widetilde{V}$, we have

$$w_h(v) = \sum_{\widetilde{e} \in \widetilde{E}, v \in \widetilde{e}} g(|\widetilde{e}|) = a_v(1 - 1/z) + \sum_{e \in F, v \in e} g(|\widetilde{e}|) < k(1 - 3/z) \tag{6}$$

and, therefore,

$$a_v < k(1 - 3/z)/(1 - 1/z) < k(1 - 2/z). \tag{7}$$

Since G is L-critical, there is an L-colouring φ of $G - \widetilde{V}$. To arrive at a contradiction we shall show that φ can be extended to some L-colouring of G.

For every edge $e \in E$ such that $e \neq \widetilde{e}$, let $v(e)$ denote an arbitrary vertex of $e - \widetilde{e}$ and let $\varphi(e) = \varphi(v(e))$. Define a list $\widetilde{L}$ for $\widetilde{V}$ by

$$\widetilde{L}(v) = L(v) \setminus \{\varphi(e) \mid e \in F_v\}$$

for every $v \in \widetilde{V}$. From (7) it then follows that

$$|\widetilde{L}(v)| \geq |L(v)| - a_v = k - a_v > k - k(1 - 2/z) = 2k/z \geq 1 \tag{8}$$

for every $v \in \widetilde{V}$. Consider a random $\widetilde{L}$-colouring of $\widetilde{V}$, that is, each vertex $v \in \widetilde{V}$ is coloured independently of all other vertices with a colour $c_v \in \widetilde{L}(v)$ and with equal probability $1/|\widetilde{L}(v)|$. We say that such a random colouring γ is e-*bad* for some $e \in F$ if all vertices of $\widetilde{e}$ receive the same colour c and, in case of $\widetilde{e} \neq e$, we have $c = \varphi(e)$. Clearly, if γ is not e-bad for all $e \in F$, then $\varphi \cup \gamma$ is an L-colouring of G.

Let $Y_e = \bigcap_{v \in \widetilde{e}} \widetilde{L}(v)$ and $y_e = |Y_e|$. For every $e \in F$, denote by A_e the event that our random colouring is e-bad. Then it follows immediately that, for $\widetilde{e} \neq e$, we have

$$\mathbf{P}(A_e) \leq \begin{cases} \prod_{v \in \widetilde{e}} (k - a_v)^{-1} & \text{if } \varphi(e) \in Y_e, \\ 0 & \text{otherwise,} \end{cases} \tag{9}$$

and, for $\widetilde{e} = e$, we have

$$\mathbf{P}(A_e) \leq y_e \prod_{v \in \widetilde{e}} (k - a_v)^{-1}. \tag{10}$$

In order to show that $\mathbf{P}(\bigwedge_{e \in F} \overline{A_e}) > 0$, we apply the Local Lemma. For every $e \in F$, let $x_e = 2^{1-|\widetilde{e}|}/kz$ and $F(e) = \{e' \in F \mid e' \cap e \neq \emptyset\}$. Clearly, for each $e \in F$, $x_e < 1$ and the event A_e is mutually independent of all the events $A_{e'}$ such that $e' \notin F(e)$. In what follows, consider some edge $e \in F$ with $|\widetilde{e}| = m$. Then $m \geq 2$ and

$$\begin{aligned} X(e) := x_e \prod_{e' \in F(e)} (1 - x_{e'}) &\geq \frac{2^{1-m}}{kz} \prod_{v \in \widetilde{e}} \prod_{(e' \in F, v \in e')} (1 - x_{e'}) \geq \\ &\geq \frac{2^{1-m}}{kz} \prod_{v \in \widetilde{e}} \exp\left\{ - \sum_{(e' \in F, v \in e')} \frac{1}{zk 2^{|\widetilde{e}'|-1} - 1} \right\} \\ &\geq \frac{2^{1-m}}{kz} \prod_{v \in \widetilde{e}} \exp\left\{ - \sum_{(e' \in F, v \in e')} \frac{2^{1-|\widetilde{e}'|}}{zk - 1/2} \right\} \\ &= \frac{2^{1-m}}{kz} \exp\left\{ - \sum_{v \in \widetilde{e}} \frac{2zk}{2zk - 1} \sum_{(e' \in F, v \in e')} \frac{2^{1-|\widetilde{e}'|}}{zk} \right\} \\ &= \frac{2^{1-m}}{kz} \exp\left\{ - \sum_{v \in \widetilde{e}} \frac{2zk}{2zk - 1} \sum_{(e' \in F, v \in e')} \frac{g(|\widetilde{e}'|)}{k} \right\}. \end{aligned}$$

From (6) it then follows that

$$X(e) \geq \frac{2^{1-m}}{kz} \exp\left\{ -\frac{2zk}{2zk-1} \sum_{v\in\widetilde{e}} \left[1 - \frac{3}{z} - \frac{a_v}{k}(1-1/z) \right] \right\}. \tag{11}$$

Let $p(e)=\mathbf{P}(A_e)/X(e)$. We want to show that $p(e)\leq 1$. If $\widetilde{e}\neq e$, then from (9) and (11) we obtain that

$$p(e) \leq \frac{2^{m-1}kz}{\prod_{v\in\widetilde{e}}(k-a_v)} \exp\left\{ \frac{2zk}{2zk-1} \sum_{v\in\widetilde{e}} \left[1 - \frac{3}{z} - \frac{a_v(z-1)}{kz} \right] \right\}.$$

This implies, using (7) and Claim 1 with $y=a_v$ and $b=\frac{2zk}{2zk-1}\frac{z-1}{kz}=\frac{2(z-1)}{2zk-1}\leq\frac{1}{k}$, that

$$\begin{aligned} p(e) &< \frac{2^{m-1}kz}{(2k/z)^m} \exp\left\{ \frac{2zk}{2zk-1}\, m \left[1 - \frac{3}{z} - \frac{(k-2k/z)(z-1)}{kz} \right] \right\} \\ &\leq \frac{z^{m+1}}{2\,k^{m-1}} \exp\left\{ \frac{2zk}{2zk-1}\, m \left[-\frac{3}{z} + \frac{3z-2}{z^2} \right] \right\} \leq \frac{z^{m+1}}{2\,k^{m-1}}. \end{aligned}$$

Consequently, because of $m\geq 2$ and $z=\sqrt[3]{k}$, we obtain $p(e)\leq 1$. Now, consider the case $\widetilde{e}=e$. Since G does not contain ordinary edges, we then have $|\widetilde{e}|=|e|=m\geq 3$, Furthermore, from (10) and (11) it follows that

$$p(e) \leq y_e \frac{2^{m-1}kz}{\prod_{v\in\widetilde{e}}(k-a_v)} \exp\left\{ \frac{2zk}{2zk-1} \sum_{v\in\widetilde{e}} \left[1 - \frac{3}{z} - \frac{a_v(z-1)}{kz} \right] \right\}. \tag{12}$$

Therefore, as in case $\widetilde{e}\neq e$, we infer from (12), (7) and Claim 1 that

$$p(e) < y_e \frac{z^{m+1}}{2\,k^{m-1}}.$$

If $m\geq 4$, then, since $y_e\leq k$ and $z=\sqrt[3]{k}$, this implies $p(e)<z^{m+1}/(2\,k^{m-2})\leq 1$. Now, assume $m=3$. If $y_e\leq 2k/z$, then we obtain

$$p(e) < \frac{2k}{z}\frac{z^{m+1}}{2\,k^{m-1}} = \frac{z^m}{k^{m-2}} = \frac{z^3}{k} = 1.$$

If $y_e > 2k/z$, then we argue as follows. Since $y_e\leq k-a_v$ for each $v\in\widetilde{e}$, we infer from (12) and Claim 1 with $y=a_v$ and $b=\frac{2(z-1)}{2zk-1}$ that

$$p(e) \leq \frac{2^{m-1}kz\,y_e}{y_e^m} \exp\left\{ \frac{2zk}{2zk-1}\, m \left[1 - \frac{3}{z} - \frac{(k-y_e)(z-1)}{kz} \right] \right\}$$

$$
\begin{aligned}
&= \frac{4kz}{y_e^2} \exp\left\{ \frac{6zk}{2zk-1} \left[1 - \frac{3}{z} - \frac{(k-y_e)(z-1)}{kz} \right] \right\} \\
&= \frac{4kz}{y_e^2} \exp\left\{ \frac{6(zy_e - y_e - 2k)}{2zk-1} \right\} =: h(y_e).
\end{aligned}
$$

The function h is convex on the interval $I = [2k/z, k]$, since, for all $y \in I$, we have

$$
(\ln h(y))' = -2/y + \frac{6(z-1)}{2zk-1} \quad \text{and} \quad (\ln h(y))'' = \frac{2}{y^2} > 0.
$$

Therefore, in order to prove that $p(e) \leq 1$ for the case $y_e > 2k/z$ it is sufficient to show that $h(y) \leq 1$ holds for $y = 2k/z$ as well as $y = k$. Since $z = \sqrt[3]{k} > 3$, we have, on the one hand,

$$
h(2k/z) = \frac{z^3}{k} \exp\left\{ -\frac{12k}{(2zk-1)z} \right\} = \exp\left\{ -\frac{12k}{(2zk-1)z} \right\} \leq 1.
$$

On the other hand, we have

$$
\begin{aligned}
h(k) = \frac{4kz}{k^2} \exp\left\{ \frac{6k(z-3)}{2zk-1} \right\} &= \frac{4}{z^2} \exp\left\{ \frac{3}{1 - 1/2z^4} \frac{z-3}{z} \right\} \\
&\leq \frac{4}{z^2} \exp\left\{ 3.04 \frac{z-3}{z} \right\}.
\end{aligned}
$$

The function $\widetilde{h}(z) = \frac{4}{z^2} \exp\left\{ 3.04 \frac{z-3}{z} \right\}$ reaches its maximum at $z_0 = 4.56$, since $(\ln \widetilde{h}(z))' = \frac{9.12 - 2z}{z^2}$ is positive on $(0, z_0)$ and negative for all $z > z_0$. Since

$$
\widetilde{h}(4.56) = \frac{4}{4.56^2} \exp\left\{ 3.04 \frac{1.56}{4.56} \right\} < \frac{1}{4.56} \exp\{1.04\} < 1,
$$

we also have $h(k) \leq 1$. This proves $p(e) \leq 1$ provided that $y_e > 2k/z$.

Therefore, $p(e) \leq 1$ for all $e \in F$. Consequently, by Lemma 1, $\mathbf{P}(\bigwedge_{e \in F} \overline{A_e}) > 0$ implying that there is an $\widetilde{L}$-colouring γ of $\widetilde{V}$ such γ is not e-bad for every edge $e \in F$. Hence there is an L-colouring of G. This contradiction proves Theorem 2. ∎

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