# Hypercube Subgraphs with Local Detours

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**Abstract:** A *minimal detour subgraph* of the $n$-dimensional cube is a spanning subgraph $G$ of $Q_n$ having the property that, for vertices $x, y$ of $Q_n$, distances are related by $d_G(x,y) \leq d_{Q_n}(x,y)+2$. For a spanning subgraph $G$ of $Q_n$ to be a *local detour subgraph*, we require only that the above inequality be satisfied whenever $x$ and $y$ are adjacent in $Q_n$. Let $f(n)$ (respectively, $f_l(n)$) denote the minimum

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*Dedicated to the memory of Paul Erdős

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number of edges in any minimal detour (respectively, local detour) subgraph of $Q_n$ (cf. Erdős et al. [1]). In this article, we find the asymptotics of $f_l(n)$ by showing that $3 \cdot 2^n(1 - \mathcal{O}(n^{-1/2})) < f_l(n) < 3 \cdot 2^n(1 + o(1))$. We also show that $f(n) > 3.00001 \cdot 2^n$ (*for* $n > n_0$), thus eventually $f_l(n) < f(n)$, answering a question of [1] in the negative. We find the order of magnitude of $F_l(n)$, the minimum possible maximum degree in a local detour subgraph of $Q_n$: $\sqrt{2n + 0.25} - 0.5 \leq F_l(n) \leq 1.5\sqrt{2n} - 1$.  J Graph Theory 30: 101–111, 1999

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## 1. INTRODUCTION

Parameters such as Hamming distances, distance sums, connectedness, maximum degree, diameter, and the size of a minimum dominating set in the hypercube and its spanning subgraphs have been investigated in many articles. Erdős, Hamburger, Pippert, and Weakley [1] introduced and studied a combination of these types of parameters. Some problems, questions, and conjectures were raised.

We say that a subgraph $G$ of the $n$-dimensional hypercube $Q_n$ has the $k$-*detour property* if any two vertices at distance $d \leq k$ in $Q_n$ are at distance at most $d + 2$ in $G$. Note that if $k = 1$, the definition gives the notion of a local detour subgraph, and if $k = n$, it yields the notion of a minimal detour subgraph of [1]. Let $f(n)$ and $f_l(n)$ denote the minimum number of edges of any minimal detour subgraph or any local detour subgraph of $Q_n$, respectively. Further, let $F_l(n)$ denote the minimum possible maximum degree in any local detour subgraph of $Q_n$.

Clearly every minimal detour subgraph of $Q_n$ is also a local detour subgraph, so $f_l(n) \leq f(n)$ for each $n$.

In [1], the following theorems were shown, among others:

**Theorem A.** *For each positive integer* $n$, $f(n)/|E(Q_n)| < 3/\sqrt{2n}$.

**Theorem B.** *For each positive integer* $n$, $f_l(n)/|E(Q_n)| < \sqrt{3/2n}$.

**Theorem C.** *If* $G$ *is a local detour subgraph of* $Q_n$, *then* $|E(G)| \geq 2 \cdot 2^n(1 - o(1))$. *In other words*, $f_l(n) \geq 2 \cdot 2^n(1 - o(1))$.

**Theorem D.** *Each local detour subgraph of* $Q_n$ *contains a vertex of degree at least* $\sqrt{n}$. *In other words*, $F_l(n) \geq \sqrt{n}$.

Also Erdős et al. [1] asked the following question.

**Question.** *Is* $f_l(n) = f$?

In this article, we improve Theorems B and C by finding the asymptotics of $f_l(n)$ and show that $f_l(n) < f(n)$. We also improve Theorem D and find a matching upper bound, which gives the right order of magnitude of $F_l(n)$.

Namely, in the next section we construct a local detour subgraph of $Q_n$ with $3 \cdot 2^n(1 + o(1))$ edges and a local detour subgraph of $Q_n$ with maximum degree at most $1.5\sqrt{2n}$. In the last section, we prove that (*i*) any local detour subgraph has a vertex of degree $\sqrt{2n} - 0.5$ or more, (*ii*) any local detour subgraph of $Q_n$ has at least $3 \cdot 2^n(1 - \sqrt{112/n})$ edges, and (*iii*) any subgraph of $Q_n$ with the 2-detour property has at least $(3.000013 - o(1)) \cdot 2^n$ edges.

This answers the Question in the negative.

## 2. CONSTRUCTIONS

In this section, we establish upper bounds for $f_l(n)$ and $F_l(n)$. To do that we need the following result by Kabatianski and Panchenko [2].

**Lemma 1** [2]. *The $m$-dimensional hypercube $Q_m$ has a dominating set $D \subset V(Q_m)$ with $|D| = (1 + o(1))2^m/m$.*

**Theorem 1.** *For every $n$ there is a subgraph $G$ of $Q_n$ with the 1-detour property, which has $3 \cdot 2^n(1 + o(1))$ edges.*

***Proof.*** In this proof we write vertices of $Q_j$ as binary $j$-tuples. First, we define two subsets $S_1$ and $S_2$ of $V(Q_n)$. Let $m = \lfloor n/2 \rfloor$. Represent $Q_n$ as $Q_m \times Q_{n-m}$, where the $Q_m$ is induced by the first $m$ coordinates and the $Q_{n-m}$ by the last $n - m$ coordinates of elements of our $Q_n$. Each vector $w \in Q_n$ can be written as $w = vu$, where $v \in V(Q_m)$ is an $m$-tuple, and $u \in V(Q_{n-m})$ is an $(n-m)$-tuple. By Lemma 1, the $m$-dimensional hypercube $Q_m$ has a dominating set $D_1$ of size $(1+o(1))2^m/m$, and the $(n-m)$-dimensional hypercube $Q_{n-m}$ has a dominating set $D_2$ of size $(1 + o(1))2^{n-m}/(n - m)$. Let

$$S_1 = \{w = vu \in V(Q_n) | v \in D_1, u \in V(Q_{n-m})\},$$

$$S_2 = \{w = vu \in V(Q_n) | v \in V(Q_m), u \in D_2\}.$$

Then $|S_1| = 2^{n-m} \cdot (1 + o(1))2^m/m = (1 + o(1))2^{n+1}/n$, and, similarly, $|S_2| = (1 + o(1))2^{n+1}/n$. Let $S = S_1 \cup S_2$. Clearly, $|S| = (1 + o(1))2^{n+2}/n$.

Now, let $E(G)$ consist of those edges of $Q_n$ that are incident with $S$, and let $G$ be the subgraph of $Q_n$ spanned by the edge set $E(G)$. We need to show that $G$ has the 1-detour property, and $|E(G)| = 3 \cdot 2^n \cdot (1 + o(1))$. To prove the latter statement, observe that $|E(G)| = \sum_{v \in S} \deg_{Q_n}(v) - |E(Q_n[S])|$, where $Q_n[S]$ is the subgraph of $Q_n$ induced by $S$. Since each $w \in S$ is adjacent in $Q_n$ to at least $m$ vertices in $S$, we get $|E(G)| \leq n|S| - 0.5m|S| = (3 + o(1))2^n$.

Finally, let $w_1 = v_1u_1$ and $w_2 = v_2u_2$ be two vertices of $G$ that are adjacent in $Q_n$ but are not adjacent in $G$. Since the degree in $G$ of every vertex in $S$ is $n$, neither $w_1$ nor $w_2$ can be in $S$. The vectors $w_1$ and $w_2$ differ in exactly one coordinate, say $i$. Without loss of generality, we can presume that $i \leq m$. Thus, $w_1 = v_1u$ and $w_2 = v_2u$, where $v_1$ and $v_2$ differ only in coordinate $i$. By the definition, there is a vertex $x \in D_2$ that is adjacent to $u$. Then the vertices $y = v_1x$ and $z = v_2x$ are in $S_2$ and differ only in coordinate $i$. Hence, path $(w_1, y, z, w_2)$ is present in $G$. ∎

**Remark 1.** *Since for $m = 2^k - 1$ the graph $Q_m$ has a perfect dominating set of vertices, our construction for $n = 2(2^k - 1)$ has strictly fewer than $3 \cdot 2^n(1 - 1/n)$ edges.*

**Remark 2.** *The construction of Theorem 1 can be generalized to a $k$-detour subgraph with $\frac{1}{2}(k+1)(k+2)2^n(1+o(1))$ edges; when $k$ is close to $n$; this gives little useful information, but we think that when $k$ is small relative to $n$ the result is optimal. To do so, we represent $Q_n$ as the direct product of $(k+1)$ smaller hypercubes of approximately equal sizes and select a minimum size dominating set in each of them. A vertex of $Q_n$ is included in set $S$, if its projection onto at least one of the small hypercubes belongs to its dominating set. Let graph $G$ include all edges incident with at least one vertex from $S$. Then $G$ has the $k$-detour property and $|E(G)| = \frac{1}{2}(k+1)(k+2)2^n(1+o(1))$.*

**Theorem 2.** *For every $n$ there is a subgraph $G$ of $Q_n$ with the 1-detour property of maximum degree at most $1.5\sqrt{2n} - 1$.*

***Proof.*** For $n < 5$ the statement is trivial. Let $n \geq 5$ and $m$ be the power of 2 lying in the half-open interval $[\sqrt{n/2}, \sqrt{2n})$, say, $m = 2^r$. Denote $s = \lceil (n - m + 1)/m \rceil$ and partition the set $\{1, \ldots, n\}$ into $m + 1$ parts $P_0, \ldots, P_m$ so that $|P_0| = m - 1$ and $|P_i| \in \{s-1, s\}$ for $i = 1, \cdots m$. Let $H$ be the subgraph of $Q_n$ spanned by the edges along the coordinates in $P_0$. Clearly, $H$ is the disjoint union of $2^{n-m+1}$ copies of $Q_{m-1}$. Since $m - 1 = 2^r - 1$, we can partition $V(Q_{m-1})$ into $m$ perfect dominating sets $S_1, \ldots, S_m$. For $i = 1, \ldots, m$, let $S_i'$ denote the union of translates of $S_i$ over all copies of $Q_{m-1}$ in $H$. Then every $S_i'$ is a dominating set in $H$ and induces in $Q_n$ the disjoint union of $2^{m-1}/m$ copies of $Q_{n-m+1}$. Now, for $i = 1, \ldots, m$, add to $H$ the edges along the coordinates in $P_i$ that are adjacent to the vertices in $S_i'$. This is the desired graph $G$.

In order to see that $G$ is a local detour subgraph of $Q_n$, consider an arbitrary edge $(u, v) \in E(Q_n) \setminus E(G)$. The vertices $u$ and $v$ differ in some coordinate $i \in \{1, \ldots, n\} \setminus P_0$, say, $i \in P_1$. That means that they belong to different components, say $C_u$ and $C_v$, of $H$, which are adjacent in $Q_n$. Then the common projection of $u$ and $v$ into a copy of $Q_{m-1}$ has a neighbor $z$ in $S_1$. Let $z_u$ and $z_v$ be the images of $z$ in $C_u$ and $C_v$, respectively. By the definition, $z_u$ and $z_v$ are adjacent. Then $(u, z_u, z_v, v)$ is a path in $G$.

The maximum degree in $G$ is

$$m - 1 + s \leq m - 1 + n/m \leq \sqrt{n/2} + \sqrt{2n} - 1 = 1.5\sqrt{2n} - 1.$$

∎

Theorem 2 together with Theorem 3 below gives the order of magnitude of $F_l(n)$.

**Remark 3.** *For each fixed $k$, the construction of Theorem 2 can be generalized to a $k$-detour subgraph with the maximum degree of order $n^{k/(k+1)}$. To do so, we choose $m$ as a power of 2, which is a bit more than $n^{k/(k+1)}$, choose $s$ just large enough so that if we partition $\{m, m+1, \ldots, n\}$ into parts of size close to $s$, say $j$ parts $P_1, \ldots, P_j$, then $\binom{j}{k} \leq m$. Then we assign to each $k$-element subset of*

$\{P_1, \ldots, P_j\}$ *one of the perfect dominating sets* $S_i$ *of* $Q_{m-1}$ *and can argue as in the proof of the theorem.*

**Remark 4.** *It follows from the proof of Theorem* 11 *in* [1] *that any subgraph of* $Q_n$ *with the* $k$*-detour property has a vertex of degree of order* $n^{k/(k+1)}$. *This together with Remark* 3 *gives for each fixed* k *the order of magnitude of the minimum possible maximum degree in the subgraphs of* $Q_n$ *with the* $k$*-detour property.*

## 3. LOWER BOUNDS

First, we are going to improve the lower bound of Theorem D.

**Theorem 3.** $F_l(n) \geq \sqrt{2n + \frac{1}{4}} - \frac{1}{2}$.

***Proof.*** Let $G$ be a local detour subgraph of $Q_n$ with maximum degree $F = F_l(n)$. For a vertex $v$, let $d(v)$ denote the degree of $v$ in $G$. Then the total number of 2-edge paths in $G$ is exactly $\sum_{v \in V(G)} \frac{1}{2} d(v)(d(v)-1)$. A detour in $G$ is a 3-edge path, whose edges are coplanar and the end-vertices are not adjacent in $G$. There exists a detour for every edge of $Q_n$ missed in $G$; thus the total number of detours is at least $n2^{n-1} - |E(G)|$. Every detour contains two 2-edge paths, and any 2-edge path in $G$ can belong to at most one detour. Thus, the total number of 2-edge paths in $G$ is at least $n2^n - 2|E(G)| = n2^n - \sum_{v \in V(G)} d(v)$, so we have

$$n2^n \leq \sum_{v \in V(G)} \frac{1}{2} d(v)(d(v)+1) \leq \frac{1}{2} 2^n F(F+1),$$

which implies $(F + \frac{1}{2})^2 \geq 2n + \frac{1}{4}$. ∎

In the rest of this section, we show that the construction of Theorem 1 is asymptotically optimal, and that for large $n$ any subgraph of the hypercube $Q_n$ with the 2-detour property has more edges than this construction. To this end, it is enough to consider the subgraphs $G$ of $Q_n$ such that

$$|E(G)| \leq 4 \cdot 2^n. \tag{1}$$

**Lemma 2.** *The number of vertices of degree* 1 *in a subgraph* $G$ *of* $Q_n$ *with the* 1*-detour property satisfying* (1) *is at most* $8 \cdot 2^n/n$.

***Proof.*** Assume indirectly that this number is greater than $8 \cdot 2^n/n$. As was observed in the proof of Theorem 12 of [1], the neighbor of any vertex of degree 1 in $G$ has degree $n$ and is not adjacent to any other vertex of degree 1. It follows that the number of vertices of degree $n$ in $G$ also is larger than $8 \cdot 2^n/n$. Therefore, $|E(G)| > 0.5n \cdot (8 \cdot 2^n/n) = 4 \cdot 2^n$, a contradiction to (1). ∎

From now on, we assume the following notation. For a subgraph $G$ of $Q_n$ and an arbitrary number $\alpha$, let $L = L(G, \alpha) = \{v \in V(Q^n) | d_G(v) \leq \alpha\}$, and $H = H(G, \alpha) = \{v \in V(Q^n) | d_G(v) > \alpha\} = V(Q^n) \setminus L$. For a vertex $v$, let $d_L(v)$ be the number of edges of $G$ incident with $v$ having the second end in $L$, and $d_H(v)$ be the number of edges incident with $v$ having the second end in $H$.

**Lemma 3.** *Let $G$ be a subgraph of $Q_n$ with the 1-detour property. Then*

$$n|L| \le \alpha 2^n + 2\alpha|E(G)| + \sum_{v\in H} d_L(v)d_H(v). \tag{2}$$

***Proof.*** Let $d(v) = d_L(v) + d_H(v)$ denote the degree of $v$ in $G$. We are going to count oriented 3-paths $(v_0, v_1, v_2, v_3)$ in $G$, where $v_0 \in L$ and $\{v_0, v_3\} \in E(Q_n)$. Since any missing edge has such a detour, the number of paths with $v_0 = v$ as their starting vertex is at least $n - d(v)$. On the other hand, the number of oriented 3-paths where $v_1$ or $v_2$ belongs to $L$ does not exceed $2\alpha|E(G)|$. Indeed, there are at most $2|E(G)|$ choices for the edge $\{v_1, v_2\}$. If one of its ends belongs to $L$, there are at most $\alpha$ choices for the other two edges of the path (they have to be parallel in $Q_n$). The number of oriented 3-paths in which $v_0 \in L$ while $v_1, v_2 \in H$, is at most $\sum_{v\in H} d_L(v)d_H(v)$. Indeed, let $v_1 = v$; there are $d_L(v)$ choices for $v_0$ and $d_H(v)$ choices for $v_2$. The last vertex, $v_3$, is determined uniquely, because the two ending edges have to be parallel. Therefore,

$$\sum_{v\in L}(n - d(v)) \le 2\alpha|E(G)| + \sum_{v\in H} d_L(v)d_H(v),$$

which implies (2). ∎

**Theorem 4.** *Any subgraph of $Q_n$ with the 1-detour property has at least $3 \cdot 2^n(1 - \sqrt{112/n})$ edges.*

***Proof.*** Set $\alpha = \alpha(n) = \sqrt{\frac{9}{7}n}$. Since

$$d_L(v)d_H(v) \le \frac{1}{4}(d_L(v) + d_H(v))^2 = \frac{1}{4}d(v)^2 \le \frac{n}{4}d(v),$$

Lemma 3 yields

$$\sum_{v\in H} d(v) \ge 4|L| - \frac{4\alpha}{n}2^n - \frac{8\alpha}{n}|E(G)|.$$

By Lemma 2,

$$\sum_{v\in L} d(v) \ge 2|L| - 8 \cdot 2^n/n,$$

and, thus,

$$2|E(G)| = \sum_{v\in V(Q_n)} d(v) \ge 6|L| - \frac{4\alpha}{n}2^n - \frac{8\alpha}{n}|E(G)| - 8 \cdot 2^n/n.$$

Because $2|E(G)| \ge \alpha|H|$, we get

$$2|E(G)|\left(1 + \frac{6}{\alpha} + \frac{4\alpha}{n}\right) \ge 6|H| + 6|L| - \frac{4\alpha}{n}2^n - 8 \cdot 2^n/n.$$

It is easily verified that

$$\left(1-\frac{2\alpha}{3n}-\frac{4}{3n}\right)\Bigg/\left(1+\frac{6}{\alpha}+\frac{4\alpha}{n}\right)\geq 1-\frac{4\alpha}{6n}-\frac{4\alpha}{n}-\frac{6}{\alpha}.$$

Since $|H|+|L|=2^n$, we finally get

$$|E(G)|\geq 3\cdot 2^n\left(1-\frac{4\alpha}{6n}-\frac{4\alpha}{n}-\frac{6}{\alpha}\right)=3\cdot 2^n(1-\sqrt{112/n}).$$

∎

**Theorem 5.** *Any subgraph of $Q_n$ with the 2-detour property has at least $(3.000013-o(1))\cdot 2^n$ edges.*

***Proof.*** We break the proof of Theorem 5 into a series of observations and lemmas. Let $S$ be a subset of $V(Q_n)$. We say that a pair of vertices $\{v',v''\}$ is an *$S$-bridge* if:

(1) $v',v''\in S$, and
(2) $v',v''$ are at a distance of 2 in both $Q_n$ and $G$, and
(3) The middle vertex of every 2-path from $v'$ to $v''$ in $G$ also belongs to $S$.

Notice that any pair of vertices at a distance of 2 in $G$ is a $V(Q_n)$-bridge.

We say that an ordered quadruple of vertices $(v_0,v_1,v_3,v_4)$ in $G$ is an *$S$-detour* if:

(1) $v_0,v_4$ are at a distance of 2 in $Q_n$, and
(2) $\{v_1,v_3\}$ is an $S$-bridge, and
(3) $\{v_0,v_1\}$ and $\{v_3,v_4\}$ are edges of $G$ and are parallel in $Q_n$.

Every $S$-detour induces either one or two oriented 4-paths $(v_0,v_1,v_2,v_3,v_4)$ in $G$, where $v_2\in S$ is called a *middle vertex* of the detour. Thus, every $S$-detour has either one or two middle vertices.

We name a quadruple of vertices of $Q_n$ an *empty square* in $G$, if they induce a $Q_2$-subgraph in $Q_n$ and induce no edges in $G$.

We will use the notation of Lemma 3, where we set $\alpha=n^{1/3}$. We also denote $d(v)=d_L(v)+d_H(v)$ the degree of $v$ in $G$.

**Lemma 4.** *The number of empty squares in $G$, where all four vertices belong to $L$, is at least $\binom{n}{2}2^{n-2}-\binom{n}{2}|H|-(n-1)|E(G)|$.*

***Proof.*** The total number of $Q_2$ in $Q_n$ is $\binom{n}{2}2^{n-2}$. The number of $Q_2$ with at least one vertex in $H$ does not exceed $\binom{n}{2}|H|$. The number of $Q_2$ that are not empty in $G$ does not exceed $(n-1)|E(G)|$, because every edge of $G$ belongs to $(n-1)$ squares in $Q_n$. ∎

**Lemma 5.** *The number of $V(Q_n)$-detours $(v_0,v_1,v_3,v_4)$ in $G$ such that $v_0,v_4\in L$ is at least $n(n-1)2^{n-1}-2n(n-1)|H|-4(n-1)|E(G)|$.*

***Proof.*** Consider an empty square in $G$ in which all four vertices belong to $L$. There are 4 choices of an ordered pair of its opposite vertices. Let $(v_0, v_4)$ be such a pair. Since $G$ has a 2-detour property, there should exist an oriented path $(v_0, v_1, v_2, v_3, v_4)$ in $G$. It is easy to see that $(v_0, v_1, v_3, v_4)$ is a $V(Q_n)$-detour. Thus, Lemma 5 follows from Lemma 4. ∎

**Lemma 6.** *The number of $V(Q_n)$-detours in $G$ that are not $H$-detours does not exceed* $3(n-1)\alpha^2 2^n$.

***Proof.*** Let $(v_0, v_1, v_3, v_4)$ be a $V(Q_n)$-detour, which is not an $H$-detour. Then at least one of the following conditions is true:

(i) $v_1 \in L$, or
(ii) $v_3 \in L$, or
(iii) There is a middle vertex $v_2$ of this detour that belongs to $L$.

For each of these conditions, we claim that the number of $V(Q_n)$-detours that satisfies this particular condition does not exceed $(n-1)\alpha^2 2^n$. Indeed, consider detours with $v_1 \in L$. There are at most $2^n$ choices for $v_1 \in L$. Since $d(v_1) \leq \alpha$, there are at most $\alpha$ choices for $v_0$ and at most $\alpha$ choices for $v_2$. There are at most $(n-1)$ choices for $v_3$. The remaining vertex, $v_4$, is determined uniquely, because the edges $\{v_0, v_1\}$ and $\{v_3, v_4\}$ are parallel. Thus, $(n-1)\alpha^2 2^n$ is an upper bound of the number of such detours. Similarly, the same bound holds for the number of detours with $v_3 \in L$. Finally, consider detours that satisfy (*iii*). There are at most $2^n$ choices for a middle vertex $v_2 \in L$. Since $d(v_2) \leq \alpha$, there are at most $\alpha$ choices for $v_1$ and at most $\alpha$ choices for $v_3$. There are at most $(n-1)$ choices for $v_0$, and the remaining vertex, $v_4$, is determined uniquely, because the edges $\{v_0, v_1\}$ and $\{v_3, v_4\}$ are parallel. ∎

Let $\epsilon$ be a positive constant that we will choose later. We set $H'_\epsilon = \{v \in H : |2d_L(v) - n| \leq \epsilon n, |2d_H(v) - n| \leq \epsilon n\}$ and $H''_\epsilon = H \setminus H'_\epsilon$. In particular, $v \in H'_\epsilon$ implies $d(v) \geq (1-\epsilon)n$.

**Lemma 7.** *The number of $H$-detours in $G$ that are not $H'_\epsilon$-detours does not exceed*

$$3(n-1) \sum_{v \in H''_\epsilon} d(v)^2.$$

***Proof.*** Similarly to the proof of Lemma 6, if an $H$-detour $(v_0, v_1, v_3, v_4)$ is not an $H'_\epsilon$-detour, then either $v_1 \in H''_\epsilon$, or $v_3 \in H''_\epsilon$, or the detour possesses a middle vertex $v_2 \in H''_\epsilon$. The number of detours which satisfy just one of these conditions does not exceed $(n-1)\sum_{v \in H''_\epsilon} d(v)^2$. For instance, for every choice of a middle vertex $v_2 \in H''_\epsilon$, the number of choices of $(v_1, v_3)$ as its neighbors does not exceed $d(v_2)^2$. The number of choices of $v_0$ does not exceed $(n-1)$, and $v_4$ is determined uniquely. ∎

Let $A_4$ be the family of $Q_2$-subgraphs of $Q_n$, where all 4 vertices belong to $H'_\epsilon$ and all 4 edges appear in $G$. Let $A_3$ be the family of $Q_2$-subgraphs of $Q_n$, where all 4 vertices belong to $H'_\epsilon$ and only 3 of the 4 edges appear in $G$. Notice that each $Q_2$ from $A_3 \cup A_4$ induces exactly two unoriented $H'_\epsilon$-bridges in $G$. Let $A_2$ be the family of $Q_2$-subgraphs of $Q_n$ that induce exactly one $H'_\epsilon$-bridge in $G$. Such a subgraph contains a 2-path with both edges present in $G$ and all 3 vertices belonging to $H'_\epsilon$, and at least one of the other two edges is missing from $G$.

**Lemma 8.**

$$|A_4| \le \frac{1}{32}(1+\epsilon)^2 n^2 |H'_\epsilon|.$$

***Proof.*** Indeed, $4|A_4| \le \sum_{v\in H'_\epsilon} \frac{1}{2} d_H(v)^2 \le |H'_\epsilon| \cdot \frac{1}{2}(\frac{1}{2}(n+\epsilon n))^2$. ∎

**Lemma 9.** $|A_3| \le \frac{1}{4}\epsilon(1+\epsilon)n^2|H'_\epsilon|$.

***Proof.*** Every $Q_2$-subgraph in $A_3$ has two vertices $v, u \in H'_\epsilon$, where $\{v,u\} \in E(Q_n)\setminus E(G)$. Given $\{v,u\}$, the other two vertices of this subgraph can be selected in at most $d_H(v) \le \frac{1}{2}(1+\epsilon)n$ ways. Thus,

$$2|A_3| \le \sum_{v\in H'_\epsilon} (n-d(v)) \cdot \frac{1}{2}(1+\epsilon)n \le |H'_\epsilon| \cdot \epsilon n \cdot \frac{1}{2}(1+\epsilon)n.$$

∎

**Lemma 10.** $|A_2| \le \frac{1}{2}\epsilon(1+\epsilon)n^2|H'_\epsilon|$.

***Proof.*** Every $Q_2$-subgraph in $A_2$ has a vertex $v \in H'_\epsilon$ and another vertex $u$ so that $\{v,u\} \in E(Q_n) \setminus E(G)$. Similarly to the proof of Lemma 9,

$$|A_2| \le \sum_{v\in H'_\epsilon} (n-d(v)) \cdot \frac{1}{2}(1+\epsilon)n \le |H'_\epsilon| \cdot \epsilon n \cdot \frac{1}{2}(1+\epsilon)n.$$

∎

**Lemma 11.**

$$\frac{1}{16}(1+\epsilon)^2(1+17\epsilon)n^3|H'_\epsilon| \ge n(n-1)2^{n-1} - 2n(n-1)|H| - 4(n-1)|E(G)| \\ -3(n-1)\alpha^2 2^n - 3(n-1)\sum_{v\in H''_\epsilon} d(v)^2. \quad (3)$$

***Proof.*** The total number of $H'_\epsilon$-bridges is $2|A_4| + 2|A_3| + |A_2|$. Because $2d_L(v) \le (1+\epsilon)n$ for any $v \in H'_\epsilon$, we notice that any unoriented $H'_\epsilon$-bridge supports at most $(1+\epsilon)n$ oriented $H'_\epsilon$-detours with both ends in $L$. Thus, $(1+\epsilon)n \cdot (2|A_4| + 2|A_3| + |A_2|)$ is an upper bound for the number of oriented $H'_\epsilon$-detours with both ends in $L$. The left-hand side of (3) is obtained by replacing $|A_4|, |A_3|$, and $|A_2|$ in the last expression by their upper bounds from Lemmas 8–10. The right-hand side of (3) is a lower bound of the number of oriented $H'_\epsilon$-detours with both ends in $L$ derived from Lemmas 5–7. ∎

**Lemma 12.** *For any vertex $v \in H''_\epsilon$,*

$$4d_L(v)d_H(v) \le nd(v) - \epsilon^2 d(v)^2.$$

***Proof.*** Let $d = d(v), d_{\min} = \min\{d_L(v), d_H(v)\}, d_{\max} = \max\{d_L(v), d_H(v)\}$. If $\epsilon \ge 1$, then $H''_\epsilon$ is empty, so we may assume that $0 < \epsilon < 1$. Let $\theta$ be defined by $d_{\min} = (1-\theta)d/2$; then $\theta \ge 0$.

The desired inequality is equivalent to $\epsilon^2 - \theta^2 \le (n/d) - 1$, so we prove this. Since $n \ge d$, the inequality is satisfied if $\theta \ge \epsilon$, so we may assume that $0 \le \theta < \epsilon < 1$.

For any real $\epsilon$, we have $\epsilon - \epsilon^2 < \frac{1}{2}$ and then $\epsilon < 1$ gives $2\epsilon < 1/(1-\epsilon)$. From this and $\theta < \epsilon$, it follows first that $\epsilon + \theta < 1/(1-\epsilon)$, then that $\epsilon^2 - \theta^2 < \frac{\epsilon-\theta}{1-\epsilon} = \frac{1-\theta}{1-\epsilon} - 1$. Now, if both $d_L(v)$ and $d_H(v)$ are greater than $(1-\epsilon)n/2$, then the fact that $v \in H''_\epsilon$, implies $d_{\max} > (1+\epsilon)n/2$, which results in $d = d_{\min} + d_{\max} > n$, and, therefore, $(1-\theta)d/2 = d_{\min} \le (1-\epsilon)n/2$, and so $(1-\theta)/(1-\epsilon) \le n/d$, which gives the desired conclusion. ∎

We are now ready to proceed with the proof of Theorem 5. Recall that by (1), $|E(G)| \le 4 \cdot 2^n$. Because $|E(G)| \ge \frac{1}{2}\alpha|H|$, we may assume that $|H| = \mathcal{O}(n^{-1/3}2^n)$. Let $a(\epsilon) = \frac{1}{16}(1+\epsilon)^2(1+17\epsilon)$. Since $\alpha = n^{1/3}$, Lemma 3 implies

$$\sum_{v \in H} d_L(v)d_H(v) \ge n2^n - \mathcal{O}(n^{2/3}2^n), \tag{4}$$

and Lemma 11 implies

$$a(\epsilon)n^2|H'_\epsilon| + 3\sum_{v \in H''_\epsilon} d(v)^2 \ge \frac{1}{2}n2^n - \mathcal{O}(n^{2/3}2^n).$$

For $v \in H'_\epsilon$, we have $d_L(v)d_H(v) \le \frac{n}{4}d(v)$. For $v \in H''_\epsilon$, we may apply Lemma 12. Thus,

$$\sum_{v \in H} d_L(v)d_H(v) \le \frac{n}{4}\sum_{v \in H} d(v) - \frac{\epsilon^2}{4}\sum_{v \in H''_\epsilon} d(v)^2.$$

By combining the last two inequalities, we get

$$\sum_{v \in H} d_L(v)d_H(v) \le \frac{n}{4}\sum_{v \in H} d(v) - \frac{\epsilon^2}{24}n2^n + \frac{a(\epsilon)\epsilon^2}{12}n^2|H'_\epsilon| + \mathcal{O}(n^{2/3}2^n).$$

We now estimate $(1-\epsilon)n|H'_\epsilon| \le \sum_{v \in H} d(v)$, and get

$$\sum_{v \in H} d_L(v)d_H(v) \le \left(\frac{1}{4} + \frac{a(\epsilon)\epsilon^2}{12(1-\epsilon)}\right) n \sum_{v \in H} d(v) - \frac{\epsilon^2}{24}n2^n + \mathcal{O}(n^{2/3}2^n). \tag{5}$$

Inequalities (4) and (5) yield

$$\sum_{v \in H} d(v) \ge 4 \cdot \frac{1 + \frac{\epsilon^2}{24}}{1 + \frac{a(\epsilon)\epsilon^2}{3(1-\epsilon)}} \cdot 2^n - \mathcal{O}(n^{-1/3}2^n).$$

By Lemma 2,

$$\sum_{v\in L} d(v) \geq 2|L| - 8\cdot 2^n/n = 2\cdot 2^n - \mathcal{O}(n^{-1/3}2^n),$$

and thus,

$$|E(G)| = \frac{1}{2}\sum_{v\in L} d(v) + \frac{1}{2}\sum_{v\in H} d(v) \geq \left(1 + 2\cdot\frac{1+\frac{\epsilon^2}{24}}{1+\frac{a(\epsilon)\epsilon^2}{3(1-\epsilon)}}\right)\cdot 2^n - \mathcal{O}(n^{-1/3}2^n).$$

When we select $\epsilon = 1/33$, the expression in the brackets in the last formula is equal to $3.00001306\cdots$. This completes the proof of Theorem 5. ∎

## 4. FINAL REMARKS

We do not have any knowledge how $f_l(n)/2^n$ changes as a function of $l$. Therefore, we cannot resolve the conjectures of [1] whether the order of magnitude of $f(n)$ is $\sqrt{n}2^{n-1}$ or the function $f(n)/2^n$ is unbounded.

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