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Acyclic k-Strong Coloring of Maps on Surfaces

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ABSTRACT. A coloring of graph vertices is called acyclic if the ends of each edge are colored in distinct colors and there are no two-colored cycles. Suppose each face of rank not greater than k, $k \geq 4$, on a surface S^N is replaced by the clique on the same set of vertices. Then the pseudograph obtained in this way can be colored acyclically in a set of colors whose cardinality depends linearly on N and on k. Results of this kind were known before only for $1 \leq N \leq 2$ and $3 \leq k \leq 4$.

KEY WORDS: embedded graphs, map coloring, acyclic coloring, acyclic graphs, cliques.

1. Introduction

Graph coloring problems for graphs embedded in surfaces play an important role in graph theory. The famous four-color problem is one of them.

Let $V(G)$ denote the set of vertices of a graph G and let $E(G)$ denote the set of its edges. A (regular) k-*coloring* of the graph G is a function $f : V(G) \to \{1, 2, \dots, k\}$ such that $f(x) \neq f(y)$ for any pair of adjacent vertices x and y of G.

A vertex coloring of a graph is said to be a k-*cyclic* or a k-*strong coloring* if any two vertices on the boundary of any face of rank not greater than k have distinct colors. (Below we use both terms.) Such a coloring is equivalent to a regular coloring of the pseudograph obtained by replacing each face of rank not greater than k by the clique having the same number of vertices. Let $\chi_k(G)$ denote the minimal number of colors sufficient for a k-cyclic coloring of a graph G, and let $\chi_k(S^N)$ be the minimal number of colors sufficient for a k-cyclic coloring of each map on the surface S^N of Euler characteristic N. (Sometimes the arguments will be omitted.)

The case $k = 3$ corresponds to the usual regular coloring; the four-color-map theorem due to Appel and Haken [1] and the Heawood Theorem [2] give the sharp upper bound for $\chi_k(S^N)$ for the plane ($N = 2$) and for all other surfaces, respectively.

The case $k = 4$ admits various formulations, in particular, in terms of combined vertex-face coloring and vertex coloring of so-called 1-embeddable graphs. For the plane, Borodin [3] proved the sharp estimate $\chi_4 \leq 6$, confirming Ringel's conjecture [4], Schumacher [5] proved the sharp estimate $\chi_4 \leq 7$ for the projective plane, and Ringel [6] proved that

$$\chi_4(S^N) \leq 2H(N)/\sqrt{3}, \qquad \text{where} \quad H(N) = \left[(7 + \sqrt{49 - 24N})/2\right]$$

is the Heawood number.

In [7], Ore and Plummer proved that for any $k \geq 3$ each plane graph G has a k-cyclic $2k$-coloring, and recently Borodin, Sanders, and Zhao [8] improved this estimate, proving that $\chi_k(G) \leq 9k/5$ for any $k \geq 3$.

A vertex coloring of a graph is called *acyclic* if it is regular, i.e., the ends of each edge are colored in distinct colors, and there are no two-colored cycles. Note that a loop is always a one-colored edge, and any double edge provides a two-colored cycle. We regard a coloring acyclic if the ends of any nonloop edge e are colored differently and there are no two-colored cycles of length greater than 2. Borodin [9] proved that each plane graph admits an acyclic 5-coloring. This estimate is sharp. In [10] Albertson and Berman proved that each graph embeddable in a surface S^N with $N < 0$ is acyclically $(8-2N)$-colorable. Alon, Mohar and Sanders [11] proved, using the acyclic 5-colorability of plane graphs, that any graph on the projective plane is acyclic 7-colorable and that this estimate is sharp. They showed as well that

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each graph embeddable in arbitrary surface S^N is acyclic $\mathcal{O}(N^{4/7})$-colorable and that this estimate is not more than $(\log N)^{1/7}$ times worse than the sharp one.

Acyclic colorings have a number of applications to other coloring problems [12–16]. Suppose $a(G) \leq a$. Then the star chromatic number of the graph G is not greater than $a2^{a-1}$ (Grünbaum [13]) and the oriented chromatic number also is not greater than $a2^{a-1}$ (Raspaud and Sopena [16]); any graph G with an edge m-coloring admits a homomorphic mapping on a graph with not more than am^{a-1} vertices (Alon and Marshall [12]); any composite graph with a vertex m-coloring and an edge n-coloring admits a homomorphic mapping on a graph with $a(2n+m)^{a-1}$ vertices (Nešhetřil and Raspaud [15]).

Besides, Hakimi, Mitchem and Schmeichel [14, pp. 38–39] proved that $E(G)$ can be split into $a(G)$ star (i.e., with star connected components) forests. As an immediate corollary, together with the results of [9] this confirms the Algor and Alon conjecture [17] that the set of edges of any planar graph is decomposable into five star forests.

In the present paper we study colorings that are both acyclic and k-cyclic. Namely, we consider acyclic colorings of the pseudograph obtained by replacing each face of the map of rank not greater than k by the k-clique. This means that each face of rank k is endowed with all "invisible diagonals." For $k = 3$ such a coloring coincides with an acyclic coloring.

The following statement is the main result of the paper.

Theorem 1. *Each map on the surface S^N admits an acyclic k-strong coloring in $c_N k + d_N$ colors for any $k \geq 4$ and $N \leq 0$, where $c_N = \max\{999, 117 - 471N\}$ and $d_N = 39 - 156N$.*

We deliberately use the simplest scheme of argument; a more sophisticated argument allows one to diminish c_N and d_N.

Corollary 1. *Any map on the plane $(N = 2)$ or on the projective plane $(N = 1)$ admits an acyclic k-strong coloring in $c_0 k + d_0$ colors for any $k \geq 4$.*

Proof. Indeed, each map on the plane or on the projective plane is a map on the torus or on the Klein bottle, respectively. □

In [18] we proved that any projective-plane graph (and therefore, any plane graph) admits an acyclic 4-strong 20-coloring, i.e., that any graph 1-embeddable into the projective plane is acyclic 20-colorable. Thus, Theorem 1 and Corollary 1 extend the results of [9] $(N = 2,\ k = 3)$, [11] $(N \leq 1,\ k = 3)$ and [18] $(1 \leq N \leq 2,\ k = 4)$.

2. Proof of Theorem 1

The sets of vertices, edges and faces of the graph under consideration will be denoted by V, E and F respectively. The *rank* $s(f)$ of a face f is the number of edges in its boundary $\partial(f)$ taking the multiplicities into account. For example, a bridge enters the boundary of the face twice. For the sake of simplicity of the argument, we restrict ourselves to the case of a connected boundary $\partial(f)$ for any face $f \in F$. The *degree* of a vertex v, i.e., the number of edges incident to this vertex (loops counted twice), is denoted by $d(v)$. A $\geq k$*-vertex* is a vertex of degree at least k, and so on.

For given S^N and k, let P''' be a counterexample with the minimal number of vertices. An acyclic k-strong coloring in $c_N k + d_N$ colors we are looking for will be called *good*, for brevity.

Erasing in P''' each loop e forming a 1-face we diminish the rank of the other face incident to e by 1. Similarly, erasing one of the two boundary edges, say e_1, from each 2-face $f = e_1 e_2$ in P''', we obtain a face of the same rank as the one incident to e_1 and different from f.

Whenever the pseudograph P'' thus obtained admits a good coloring, restoring the erased loops and double edges preserves the coloring. Hence, P'' also is a minimal counterexample.

Triangulating all $> k$-faces P'' by adding diagonals we obtain one more counterexample P' with the minimal number of vertices. (A good coloring of P' would be a good coloring for P'' as well.)

Now erase in P' the common edge of two adjacent 3-faces, whenever the two exist, and repeat this operation until we obtain a counterexample P without adjacent 3-faces. Hence, we have proved the following statement.

Lemma 1. *If f is a face in P, then $3 \le s(f) \le k$; there are no adjacent 3-faces in P.*

Speaking unprecisely, only adding nontrivial adjacency represented by "visible" edges and "invisible diagonals" can spoil a good coloring. Therefore, we shall take care of adjacencies when transforming the pseudograph P into a smaller pseudograph admitting a good coloring.

Below, the following two remarks will be useful.

Remark 1. Contracting an edge $e = vz$ into the vertex $v * z$ diminishes the rank of each face f in P by 0, 1 or 2 depending on the multiplicity with which e occurs in $\partial(f)$.

The map R obtained in this way admits a good coloring since P is minimal. Let us pull back this coloring to P assigning the color of the vertex $v * z$ to z and leaving v uncolored. Then, in order to obtain a good coloring of the map P, it is sufficient to find a color for v so that no one-colored edges that are not loops and no two-colored cycles that are not 2-cycles appear.

Remark 2. Suppose a vertex v is incident to the edges $e_i = vz_i$ enumerated in cyclic order, where $0 \le i \le d(v) \le k-1$. (Of course, the vertices z_i must not all be distinct.) Let us split each nontriangular face $f_i = \dots z_i v z_{i+1}$ (indices taken modulo $d(v)$) into the triangle $z_i v z_{i+1}$ and the face f_i'. Then $s(f_i') < s(f_i)$. Erasing v and all the e_i, we obtain a face of rank $d(v) \le k$, whence the map obtained in this way admits a good coloring. Here all distinct vertices z_i have distinct colors. Choosing the color α not contained in the set $\bigcup_{0\le i\le k-1} \partial(f_i)$ for the vertex v, we obtain a two-colored α, β-cycle (i.e., a cycle consisting of vertices colored alternately in α and β) passing through v and a vertex u such that the color of u is β and $u \in \partial(f_i) \setminus \{z_i, z_{i+1}\}$ for some f_i as the unique obstruction for a good coloring of P.

Lemma 2. *If $v \in V(P)$, then $d(v) \ge 2$.*

Proof. If $d(v) = 1$, then contract the edge vz, pull back the good coloring of the pseudograph thus obtained to P and for v choose a color distinct from that of the vertices that belong to the same faces as v does (there is not more than k of them). □

Lemma 3. *Any face in P is incident to not more than three vertices (not necessarily distinct) of degree at least three.*

Proof. If the boundary $\partial(f)$ of a face f in P contains precisely two (not necessarily distinct) vertices u, w of degree greater than two, then erase the longer of the two chains constituting $\partial(f)$. Then we obtain a face f' of rank $\le k$. A good coloring of the resulting pseudograph can be easily extended to a good coloring of P since vertices of $\partial(f')$ have pairwise distinct colors.

The case with $\partial(f)$ having only one vertex of degree greater than two can be easily reduced to the previous one. And if the degree of all vertices of $\partial(f)$ is two, then P is a cycle, and this is a contradiction. □

If each face in P is homeomorphic to an open 2-disk, then the Euler formula for P gives $|V| - |E| + |F| = N$; otherwise $|V| - |E| + |F| \ge N$.

Then the obvious relations $2|E| = \sum_{v\in V} d(v) = \sum_{f\in F} s(f)$ give

$$\sum_{v\in V}(d(v)-4) + \sum_{f\in F}(s(f)-4) \le -4N, \tag{1}$$

or

$$-2n_2 + \sum_{v\in V_{3+}}(d(v)-4) + \sum_{f\in F}(s(f)-4) < -4N+1, \tag{2}$$

where n_i is the number of i-vertices in P, and V_{i+} is the set of $\ge i$-vertices in P.

Denote by $n_2(f)$ the number of 2-vertices on the boundary of a face f (taking multiplicities into account). The *reduced rank* $s(f)$ of f is the difference $s(f) - n_2(f)$. Essentially, $s^*(f)$ is the number of > 2-vertices in $\partial(f)$ taking multiplicities into account. Then (2) reads

$$\sum_{v\in V_{3+}}(d(v)-4) + \sum_{f\in F}(s^*(f)-4) < -4N+1. \tag{3}$$

By Lemma 3, $s^*(f) \geq 3$ for all $f \in F$. Let $f_i^*(v)$ be the number of i^*-faces at v, i.e., faces with $s^*(f) = i$. Then

$$\sum_{v \in V_{3+}} \left(d(v) - 4 - \frac{f_3^*(v)}{3} \right) + \sum_{f \in F_{4+}^*} (s^*(f) - 4) < -4N + 1, \tag{4}$$

where F_{i+}^* is the set of all $\geq i^*$-faces in P, or

$$\sum_{v \in V_{3+}} \left(d(v) - 4 - \frac{f_3^*(v)}{3} + \frac{4N-1}{n^*} \right) + \sum_{f \in F_{4+}^*} (s^*(f) - 4) < 0, \tag{5}$$

where $n^* = |V_{3+}| = |V| - n_2$.

We set $\mathrm{ch}(v) = d(v) - 4 - f_3^*(v)/3 + (4N-1)/n^*$ if $v \in V_{3+}$; $\mathrm{ch}(f) = 0$ if f is a 3^*-face, and $\mathrm{ch}(f) = s^*(f) - 4$ if $f \in F_{4+}^*$. Then (5) gives

$$\sum_{v \in V_{3+}} \mathrm{ch}(v) + \sum_{f \in F} \mathrm{ch}(f) < 0. \tag{6}$$

The remaining part of the proof consists in redistributing the *charge* $\mathrm{ch}(x)$ on $x \in V_{3+} \cup F$ preserving the sum of charges in such a way that the *new charge* $\mathrm{ch}^*(x)$ becomes positive for any $x \in V_{3+} \cup F$. The contradiction with (6) will complete the proof.

We start by proving the following statement.

Lemma 4. *We have* $n^* > 39(-4N+1)$.

Proof. The Euler formula (2) for the map P^* obtained from P by contacting each chain $uv_1 \dots v_s w$, where $d(u) \geq 3$, $d(v_1) = \dots = d(v_s) = 2$, $d(w) \geq 3$, to the edge uw can be easily rewritten in the form

$$\sum_{v \in V(P^*)} (d_{P^*}(v) - 6) + \sum_{f \in F(P^*)} (2s_{P^*}(f) - 6) \leq -6N,$$

whence $\sum_{v \in V(P^*)} (d_{P^*}(v) - 6) \leq -6N$ or $|E(P^*)| \leq 3n^* - 3N$.

By Lemma 1, $n \leq n^* + k|E(P^*)|$.

But $n > c_N k + d_N$ since otherwise P would admit a trivial good coloring (with all vertices colored distinctly), and therefore,

$$\begin{aligned} n^*(1+3k) - 3kN > c_N k + d_N &\geq (117 - 471N)k + 39 - 156N \\ &= 39(-4N+1)(1+3k) - 3kN, \end{aligned}$$

and the required assertion (compare the left most and the right most expressions). □

Lemma 5. *If* $d(v) \geq 7$, *then* $\mathrm{ch}(v) \geq 0$.

Proof. Indeed,

$$\mathrm{ch}(v) \geq d(v) - 4 - \frac{f_3^*(v)}{3} + \frac{4N-1}{n^*} \geq d(v) - 4 - \frac{d(v)}{3} + \frac{4N-1}{n^*} \geq \frac{2}{3} + \frac{4N-1}{n^*},$$

and we apply Lemma 4.

If $v \in V$ and $\mathrm{ch}(v) < 0$, then we call the vertex v *poor*. Let us set $\xi = (-4N+1)/n^*$ and $\varepsilon = \frac{1}{39}$. By Lemmas 4 and 5, $\xi \leq \varepsilon$ and for each poor vertex v we have $3 \leq d(v) \leq 6$.

The charge redistribution rules are as follows.

R1. Each ≥ 14-vertex gives $2/3 + \varepsilon$ to each $\geq 4^*$-face and $1/3 + \varepsilon$ to each 3^*-face incident to it.

R2. Each poor vertex v gets $2/3 + \varepsilon$

a) from each $\geq 4^*$-face f incident to v under the assumption that there are no ≥ 14-vertices z in $\partial(f)$ not connected with v along $\partial(f)$ by a chain of 2-vertices, and

b) from each $\geq 13^*$-face f incident to v.

R3. If a 3^*-face f is incident to a ≥ 14-vertex, then f gives $1/6 + \varepsilon/2$ to each poor vertex incident to f. □

Lemma 6. *By rule* R3, *each poor vertex* v *gets the charge* $1/6+\varepsilon/2$ *from each of the two* 3^**-faces cyclically adjacent at* v.

Proof. We prove first that if a 3^*-face f is incident to a 2-vertex u, then the ≥ 3-vertex opposite to u in $\partial(f)$ is, in fact, a ≥ 14-vertex.

Let $\partial(f)=xx_1x_2\dots x_{k(x)}yy_1y_2\dots y_{k(y)}zz_1z_2\dots z_{k(z)}$, where all x_i, y_i and z_i are 2-vertices, and x, y and z are ≥ 3-vertices. Suppose $k(x)\geq 1$, i.e., x_1 exists. Denote by f_{xy} the face neighboring f along the chain $xx_1x_2\dots x_{k(x)}xy$ (which can coincide with f). The faces f_{yz} and f_{zx} are defined similarly.

Contract the edge xx_1 and pull back a good coloring of the map thus obtained to $P-x_1$. Choosing for v the color not entering the face f and the faces incident to z we obtain a good coloring. This cannot be done only if $d(z)\geq 14$.

In order to complete the proof of the lemma, recall that, by the second assertion of Lemma 1, one of the two 3^*-faces f_1 or f_2 adjacent along the cycle at v is incident to a 2-vertex, z. If z is incident to both f_1 and f_2, then the required statement follows from R3 and the statement proved just now. Otherwise z is opposite to a ≥ 3-vertex w, which is incident to both f_1 and f_2, since by the statement above the vertex z cannot be opposite to a poor vertex v whose degree, by Lemma 5, is not greater than 6. Hence, $d(w)\geq 14$, and we can apply R3 once again. ☐

Lemma 7. *If* $d(v)=3$, *then* v *is incident to at least two faces giving* v *the charge* $2/3+\varepsilon$ *according to the rule* R2.

Proof. Suppose v is incident to chains $vx_1x_2\dots x_{k(x)}x$, $vy_1y_2\dots y_{k(y)}y$ and $vz_1z_2\dots z_{k(z)}z$, where all x_i, y_i and z_i are 2-vertices, while x, y and z are >2-vertices. Suppose that

$$\partial(f_1)=\dots xx_{k(x)}\dots x_2x_1vy_1y_2\dots y_{k(y)}y,\qquad \partial(f_2)=\dots yy_{k(y)}\dots y_2y_1vz_1z_2\dots z_{k(z)}z,$$
$$\partial(f_3)=\dots zz_{k(z)}\dots z_2z_1vx_1x_2\dots x_{k(x)}x.$$

Suppose the converse, namely, that neither f_1, nor, by symmetry, f_2 gives $2/3+\varepsilon$ to v. Then, by R2, both f_1 and f_2 are $\leq 12^*$-faces, and the degree of each vertex in $\partial(f_1)\setminus\{x,y\}$ and in $\partial(f_2)\setminus\{y,z\}$ is at most 13.

Contract the chain $vy_1y_2\dots y_{k(y)}y$ to the vertex $v*y$. Pull back a good coloring of the map obtained in this way to P and note that the vertices x and z have distinct colors. Choose pairwise distinct colors for $y_1,y_2,\dots,y_{k(y)}$ and v not entering the boundaries of the faces f_1, f_2, f_3 and those not more than $2\times 9\times(13-1)$ faces that are incident to >2-vertices from $\partial(f_1)\setminus\{x,y\}\cup\partial(f_2)\setminus\{y,z\}$. The number of restrictions is at most $3k+2\times 9\times(13-1)k<c_Nk+d_N$, and there arise neither one-colored nonloop edges, nor two-colored cycles of length greater than two. ☐

Lemma 8. *If* $d(v)=4$, *then* v *is incident to at least one face giving* v *the charge* $2/3+\varepsilon$ *according to the rule* R2.

Proof. Suppose v is incident to chains $vx_i^ix_2^i\dots x_{k(x^i)}^ix^i$ following in the cyclic order, where $0\leq i\leq 3$, all x_j^i are 2-vertices, and all x^i are >2-vertices.

Let
$$\partial(f^i)=x^ix_{k(x^i)}^i\dots x_2^ix_1^ivx_1^{i+1}x_2^{i+1}\dots x_{k(x^{i+1})}^{i+1}x^{i+1}\dots,$$
for a face f^i with superscripts taken modulo 4.

Suppose that neither of the faces f^i gives $2/3+\varepsilon$ to v. Then, by R2, each face f^i is a $\leq 12^*$-face, and the degree of each vertex from $\partial(f^i)\setminus\{x^i,x^{i+1}\}$ is at most 13.

Add the edge x^ix^{i+1} to f^i for all $0\leq i\leq 3$ if there is no such edge in $\partial(f^i)$ yet. Erase v and pull back a good coloring of the map thus obtained to P. Choose for v a color not entering the boundaries of the faces incident to the vertices from $\partial(f^i)\setminus\{x^i,x^{i+1}\}$ for all $0\leq i\leq 3$. The number of restrictions is less than $4\times 9\times 13k\leq c_Nk+d_N$, and it is easy to see that neither one-colored edges, nor two-colored cycles arise. ☐

Lemma 9. *If* $v\in V_{3+}$, *then* $\operatorname{ch}^*(v)\geq 0$.

Proof. Suppose first that v is poor i.e., $\mathrm{ch}(v) < 0$. Then, by Lemma 5, $d(v) < 7$. If $d(v) = 3$, then $\mathrm{ch}(v) = -1-\xi-f_3^*(v)/3$, and we obtain the required statement by Lemma 7, since $2(2/3+\varepsilon) > 1+\xi+1/3$.

If $d(v) = 4$, then $\mathrm{ch}(v) = -\xi - f_3^*(v)/3$. By Lemma 8, the vertex v gets the charge $2/3+\varepsilon$ from at least one $\geq 4^*$-face incident to it. If v is incident to at most two 3^*-faces, then $\mathrm{ch}^*(v) \geq \varepsilon - \xi \geq 0$. Otherwise, by Lemma 6, v would get at least $2(1/6+\varepsilon/2)$ from three 3^*-faces incident to v, whence $\mathrm{ch}^*(v) \geq 0$.

If $d(v) = 5$, then $\mathrm{ch}(v) = 1 - \xi - f_3^*(v)/3$ and we obtain the required statement if v is incident to at most two 3^*-faces. If there are r such faces at v, $3 \leq r \leq 5$, then, by Lemma 6, v gets at least $2(1/6+\varepsilon/2)$ if $r = 3$ and at least $4(1/6+\varepsilon/2)$ if $r \geq 4$, whence $\mathrm{ch}^*(v) \geq 0$.

If $d(v) = 6$, then $\mathrm{ch}(v) = 2 - \xi - f_3^*(v)/3$, and we obtain the required statement if v is incident to at most five 3^*-faces. Otherwise, by Lemma 6, v gets $6 \times (1/6+\varepsilon/2)$ from the six 3^*-faces incident to v, whence $\mathrm{ch}^*(v) \geq 0$.

Now suppose v is not poor. If it gives nothing to neighboring vertices according to rules R1 and R2, then $\mathrm{ch}^*(v) = \mathrm{ch}(v) \geq 0$. Otherwise, $d(v) \geq 14$ and v makes not more than $d(v)$ transfers according to rule R1. Therefore,

$$\mathrm{ch}^*(v) = d(v) - 4 - \xi - d(v)\left(\frac{2}{3}+\varepsilon\right),$$

whence $\mathrm{ch}^*(v) \geq 0$ since $\xi \leq \varepsilon$. □

Lemma 10. *If* $f \in F$, *then* $\mathrm{ch}^*(f) \geq 0$.

Proof. If the degree of any vertex incident to a 3^*-face is at most 13, then this face does not affect the charge. Otherwise it gets at least $1/3+\varepsilon$ by rule R1 and it gives, by rule R3, not more than $2(1/6+\varepsilon/2)$ to the poor vertices incident to this face. In both cases $\mathrm{ch}^*(f) \geq 0$.

Now suppose $f \in F_{4+}^*$. If at least two vertices of degree ≥ 14 are incident to f, then

$$\mathrm{ch}^*(f) \geq s^*(f) - 4 + 2\left(\frac{2}{3}+\varepsilon\right) - (s^*(f)-2)\left(\frac{2}{3}+\varepsilon\right) = (s^*(f)-4)\left(\frac{1}{3}-\varepsilon\right) \geq 0.$$

If f contains only one ≥ 14-vertex z, then

$$\mathrm{ch}^*(f) \geq s^*(f) - 4 + \frac{2}{3} + \varepsilon - (s^*(f)-3)\left(\frac{2}{3}+\varepsilon\right) \geq 0,$$

since f gives the charge $2/3+\varepsilon$ to not more than $s^*(f)-3$ poor vertices by R3: neither z, nor the ≥ 3-vertices closest to z on the left and on the right along $\partial(f)$ get anything from f.

Now suppose there are no vertices of degree ≥ 14 in f. If $s^*(f) \leq 12$, then

$$\mathrm{ch}^*(f) = \mathrm{ch}(f) = s^*(f) - 4 \geq 0$$

since f does not participate in the redistribution of charges. Finally, if $s^*(f) \geq 13$, then, by R2,

$$\mathrm{ch}^*(f) \geq s^*(f) - 4 - s^*(f)\left(\frac{2}{3}+\varepsilon\right) = s^*(f)\left(\frac{1}{3}-\varepsilon\right) - 4 \geq 0,$$

since $\varepsilon = 1/39$.

The lemmas above imply that $\mathrm{ch}^*(x) \geq 0$ for all $x \in V_{3+} \cup F$. This contradicts (6), which completes the proof of the theorem. □

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