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The Dimension of Neighboring Levels of the Boolean Lattice

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Abstract. The order dimension of suborders of the Boolean lattice $\mathcal{B}_n$ is considered. It is shown that the suborder of $\mathcal{B}_n$ consisting of levels s and $s+1$ has dimension $O(\log n/\log\log n)$. This improves a bound in [1].

We follow the notation and definitions of [4]. For any positive integers n, s, t with $s < t \leqslant n$, let $\mathcal{B}_n$ denote the Boolean lattice of subsets of $\{1, \ldots, n\}$ ordered by inclusion, and $\mathcal{B}_n(s, t)$ denote the restriction of $\mathcal{B}_n$ to the sets of cardinal ities s and t. Let $\dim(s, t; n)$ denote the (order) dimension of $\mathcal{B}_n(s, t)$. First results on $\dim(s, t; n)$ were proved by Dushnik [2] long ago. Clearly, it is enough to consider $s < n/2$. The best known upper bound for $\dim(s, s+1; n)$ independent of s was established in [1]:

$$\dim(s, s+1; n) \leqslant 6\log_3 n, \tag{1}$$

while $\dim(1, 2; n)$ is asymptotically $\log_2 \log_2 n$ (see Spencer [3]) and hence $\dim(s, s+1; n) \geqslant \dim(1, 2; (n+1)/2) \geqslant \log_2\log_2(n+1)/2$. We refer the reader to [1] for more details. The aim of the present note is to improve the upper bound (1) as follows.

THEOREM 1. *If* $k! \geqslant n$, *then* $\dim(s, s+1; n) \leqslant 2k$ *for any* $0 < s < n$. *In particular, for large* n, $\dim(s, s+1; n) \leqslant 3(\ln n)/\ln\ln n$.

Proof. Let $k! \geqslant n$. Then the numbers $1, \ldots, n$ we can cypher by distinct vectors $(a_1, \ldots, a_k)$ such that every a_i belongs to $\{1, \ldots, k\}$ $(i = 1, \ldots, k)$ and all the a_i-s are distinct.

For $D \in \mathcal{B}_n$, let $D(i, j) = \{d \in D \mid j\text{-th coordinate of } d \text{ is } i\}$. Let M be a linear extension of $\mathcal{B}_n$. Define linear extensions $L_1, \ldots, L_k, L^1, \ldots, L^k$ in rounds as follows. To define the relation between elements C and D of $\mathcal{B}_n$ in L_i and L^i, on round j, $j = 1, \ldots, k$, if decision was not made before, one compares $D(i, j)$ and $C(i, j)$:

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(C(i,j) and D(i,j))

(1) if $|D(i,j)| < |C(i,j)|$, then $D < C$ both in L_i and L^i;
(2) if $|D(i,j)| = |C(i,j)|$ and $D(i,j) \neq C(i,j)$, then in L_i, C and D are in the same order as in M, and in L^i – in the opposite order;
(3) if $D(i,j) = C(i,j)$, then we don't make the decision and go to the next round.

On round $k+1$ we do:

(1) if $|D| < |C|$, then $D < C$ both in L_i and L^i;
(2) if $|D| = |C|$ and $D \neq C$, then in L_i, C and D are in the same order as in M, and in L^i – in the opposite order.

It is easy to see that all L_i and L^i are linear extensions of $\mathcal{B}_n$. Let $|C| + 1 = |D|$, and $|C \setminus D| > 0$. We will show now that in some L_i or L^i we have $C > D$. How can it be that after the first round in no L_i and L^i we have $C > D$? If not all the vectors in $(C \setminus D) \cup (D \setminus C)$ have the same first coordinate, then for $j = 1$ and at least one i we have situation 1) or 2) when we make the decision favorable for C. Thus the only possibility is that for some $m(1)$, all elements of both $C \setminus D$ and $D \setminus C$ have the first coordinate equal to $m(1)$. That means that the decision $C < D$ was made only in $L_{m(1)}$ and $L^{m(1)}$. If after the second round we are still not satisfied, there should be $m(2)$ (note that $m(2) \neq m(1)$) such that all elements of both $C \setminus D$ and $D \setminus C$ have the second coordinate equal to $m(2)$, and so on. Thus, if after the k-th round we are still not satisfied, there should be distinct $m(1), m(2), \ldots, m(k)$ such that all elements of both $C \setminus D$ and $D \setminus C$ for any $i \in \{1, \ldots, k\}$ have the i-th coordinate equal to $m(i)$. But there is only one element with this code and

$$|C \setminus D| + |D \setminus C| > 2. \tag{2}$$

This contradiction completes the proof.

Remark. It is clear from the proof, in particular from (2), that it works already if $2(k!) \geqslant n$.

Note also that the same $2k$ linear orders serve for all s simultaneously.

References

1. Brightwell, G. R., Kierstead, H. A., Kostochka, A. V. and Trotter, W. T. (1994) The dimension of suborders of the Boolean lattice, *Order* **11**, 127–134.
2. Dushnik, B. (1950) Concerning a certain set of arrangements, *Proc. Amer. Math. Soc.* **1**, 788–796.
3. Spencer, J. (1972) Minimal scrambling sets of simple orders, *Acta Math. Hungar.* **22**, 349–353.
4. Trotter, W. T. (1991) *Combinatorics and Partially Ordered Sets: Dimension Theory*, The John Hopkins University Press, Baltimore, MD, U.S.A.