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The Dimension of Suborders of the Boolean Lattice

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Abstract. We consider the order dimension of suborders of the Boolean lattice $\mathbf{B}_n$. In particular we show that the suborder consisting of the middle two levels of $\mathbf{B}_n$ has dimension at most $6 \log_3 n$. More generally, we show that the suborder consisting of levels s and $s+k$ of $\mathbf{B}_n$ has dimension $O(k^2 \log n)$.

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1. Introduction

For any positive integer n, let $[n] = \{1, 2, \ldots, n\}$, let $\mathcal{B}_n$ be the collection of subsets of $[n]$, and let $\mathbf{B}_n = (\mathcal{B}_n, \subseteq)$ denote the Boolean lattice, where the subsets of $[n]$ are ordered by inclusion. For a finite set A, let $C(A, k)$ denote the collection of k-element subsets of A. For integers n, s and t with $0 \leqslant s < t \leqslant n$, let $\mathbf{B}_n(s,t)$

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denote the restriction of $\mathcal{B}_n$ to $C([n], s) \cup C([n], t)$. Finally, let $\dim(s, t; n)$ denote the (order) dimension of $\mathcal{B}_n(s, t)$. We refer the reader to the monograph [7] for additional background material on dimension theory.

The function $\dim(s, t; n)$ was first studied by Dushnik [1] in 1950, but estimates for the function are surprisingly poor, except in the case $s = 1$. In this case, Dushnik noted the following useful reformulation of the problem.

PROPOSITION 1.1. *For all positive integers t and n, $1 < t < n$, $\dim(1, t; n)$ is the least positive integer d for which there exists a set Σ of d linear orderings of $[n]$ such that for all $X \in C([n], t)$ and all $y \in [n] - X$, there exists $L \in \Sigma$ such that in L, y is greater than every element of X.*

With the aid of Proposition 1.1, Dushnik [1] proved the following result, establishing the exact value for $\dim(1, t; n)$ when $t \geqslant 2\sqrt{n} - 2$.

THEOREM 1.2 [1]. *Let n and t be positive integers with $n \geqslant 4$ and $2\sqrt{n} - 2 \leqslant t \leqslant n - 1$. Then let j be the unique integer with $2 \leqslant j \leqslant \sqrt{n}$ for which*

$$\left\lfloor \frac{n - 2j + j^2}{j} \right\rfloor \leqslant t < \left\lfloor \frac{n - 2(j-1) + (j-1)^2}{j - 1} \right\rfloor.$$

Then

$$\dim(1, t; n) = n - j + 1.$$

In the remainder of this paper, we will discuss *estimates* for the dimension of ordered sets. For this reason, we will omit “floors” and “ceilings” from expressions which only have meaning for integers.

For fixed t, Spencer [6] established the asymptotic behavior of $\dim(1, t; n)$.

THEOREM 1.2 [6]. *For fixed t,*

$$\dim(1, t; n) = \Theta(\log \log n).$$

The following elementary result is an exercise in [7] and follows easily from Dushnik's proof of Theorem 1.2.

PROPOSITION 1.4. *For all positive integers t and n with $t^2 \leqslant n$,*

$$t^2/4 < \dim(1, t; n).$$

In view of Proposition 1.4, the following result of Füredi and Kahn [4] establishes the value of $\dim(1, t; n)$ within a multiplicative factor of order $\log t$, if $t = \Omega(n^{\varepsilon})$. The proof is simply a matter of taking d linear orderings of $[n]$, uniformly at random from the set of all possible linear orderings, and noting that the probability that these do not form a family Σ as in Proposition 1.1 tends to 0.

PROPOSITION 1.5 [4]. *For all positive integers* t, n, *with* $t < n$, *if* d *is a positive integer satisfying*

$$n\binom{n-1}{t}\left(\frac{t}{t+1}\right)^d < 1,$$

then $\dim(1, t; n) \leqslant d$. *In particular,*

$$\dim(1, t; n) \leqslant (t+1)^2 \log n.$$

Determining $\dim(1, t; n)$ for t a small growing function of n remains an intriguing open problem. Moreover, until recently, very little was known for the case $s > 1$. Here are two well known trivial bounds.

PROPOSITION 1.6. *For all positive integers* $s \leqslant s' < t' \leqslant t \leqslant n' \leqslant n$,

$$\dim(s', t'; n') \leqslant \dim(s, t; n).$$

PROPOSITION 1.7. *For all positive integers* $r < s < t < n$,

$$\dim(s - r, t - r; n - r) \leqslant \dim(s, t; n).$$

The next two results are given by Hurlbert, Kostochka and Talysheva in [5].

THEOREM 1.8 [5]. *For each positive integer* n *with* $n \geqslant 5$,

$$\dim(2, n-2; n) = n - 1.$$

THEOREM 1.9 [5]. *For each positive integer* n *with* $n \geqslant 6$,

$$\dim(2, n-3; n) = n - 2.$$

In fact, it is shown in [5] that if $2\sqrt{n} < t < n-2$ and t is not an integer of the form $j - 2 + (n-1)/j$, for some positive integer j, then $\dim(2, t; n) = \dim(1, t-1; n-1)$.

While preparing this manuscript, we have just learned that Füredi [2] has proven the following result.

THEOREM 1.10 [2]. *For each integer* $k \geqslant 3$, *there exists* n_0 *so that if* $n > n_0$, *then*

$$\dim(k, n-k; n) = n - 2.$$

In this note, we provide the following upper bound on $\dim(s, t; n)$ in terms of the parameters $\dim(1, 2(t-s); n)$ and $t - s$.

THEOREM 1.11. *For all positive integers k, n with $2k \leqslant n$, there exists a collection Σ of at most* $\dim(1, 2k; n) + 18k \log n$ *linear extensions of $\mathcal{B}_n$ such that for any incomparable pair $(S, T) \in \mathcal{B}_n \times \mathcal{B}_n$ with $|S| < |T| \leqslant k + |S|$, there exists $L \in \Sigma$ such that $T < S$ in L. In particular,*

$$\dim(s, s+k; n) \leqslant \dim(1, 2k; n) + 18k \log n,$$

for every positive integer s, with $s + k \leqslant n$.

Using Theorem 1.5, we have the following corollary.

COROLLARY 1.12. *For all positive integers s, k and n, with $s + k \leqslant n$,*

$$\dim(s, s+k; n) = O(k^2 \log n).$$

When $k = 1$, we can do a little better.

THEOREM 1.13. *For every positive integer n, there exists a collection Σ of $6 \log_3 n$ linear extensions of $\mathcal{B}_n$ such that for any incomparable pair $(S, T) \in \mathcal{B}_n \times \mathcal{B}_n$ with $|T| = 1 + |S|$, there exists $L \in \Sigma$ such that $T < S$ in L. In particular,*

$$\dim(s, s+1; n) \leqslant 6 \log_3 n,$$

for every positive integer s with $s + 1 \leqslant n$.

For some values of s and k, we know that the inequalities in Theorems 1.11 and 1.13 are far from tight. For example, the following asymptotic formula is proved in [7], based on work [3], and following earlier results of Spencer [6].

THEOREM 1.14.

$$\dim(1, 2; n) = \lg \lg n + \big(1/2 + o(1)\big) \lg \lg \lg n.$$

For the middle two levels of the Boolean lattice, our upper and lower bounds are

$$\lg \lg n + \big(1/2 + o(1)\big) \lg \lg \lg n < \dim(s, s+1; 2s+1) \leqslant 6 \log_3 n.$$

However, we should comment that when $k \geqslant \log n$, but k and s are both $o(n)$, the inequality in Theorem 1.11 is relatively tight. This follows from the observation that

$$\dim(s, s+k; n) \geqslant \dim(1, k+1; n-s+1).$$

Our upper bound is not too far this lower bound whenever $\dim(1, k; n)$ and $\dim(1, 2k; n)$ are relatively close (see Problem 4.2).

2. Some Combinatorial Lemmas

To prove Theorem 1.11, we need to provide an appropriate family Σ of linear extensions of $\mathcal{B}_n$. This family will be made up of two sets of extensions; the first set is designed to deal with those pairs (S, T) where $T - S$ is small, and the second set is designed to handle the remaining pairs. Our first lemma concerns the first of these sets; in the next section, we shall apply it with $c = 2k$.

LEMMA 2.1. *For all positive integers c and n with $1 < c \leqslant n$, there exist $d = \dim(1, c; n)$ linear extensions $M_1, M_2, \ldots, M_d$ of $\mathcal{B}_n$ with the property that for all incomparable pairs $(S, T) \in \mathcal{B}_n \times \mathcal{B}_n$ with $|T - S| \leqslant c$, there exists $i \in [d]$ such that $T < S$ in M_i.*

Proof. For any linear ordering σ of $[n]$, define the *lexicographical ordering* $L(\sigma)$ on $\mathcal{B}_n$ with respect to σ as follows. For two sets $S, T \in \mathcal{B}_n$, $T < S$ in $L(\sigma)$ if and only if the σ-largest element of $S \Delta T = (S - T) \cup (T - S)$ is in S. Clearly, any such $L(\sigma)$ is a linear extension of $\mathcal{B}_n$.

Let $d = \dim(1, c; n)$; choose d linear orderings $\sigma_1, \sigma_2, \ldots, \sigma_d$ on $[n]$ such that for all $X \in \mathcal{B}_n$ with $1 \leqslant |X| \leqslant c$ and all $y \in [n] - X$, there exists $i \in [d]$ such that y is greater than every element of X in σ_i. Let $M_i = L(\sigma_i)$, for all $i \in [d]$.

Consider an incomparable pair $(S, T) \in \mathcal{B}_n \times \mathcal{B}_n$ such that $|T - S| \leqslant c$. Choose $y \in S - T$ and let $X = T - S$. Then there exists $i \in [d]$ such that y is greater than every element of X in σ_i. Thus $T < S$ in M_i. □

For positive integers a, b, k, t and n with $k < b \leqslant n$ and $k < a$, we define a sequence $\{f_i\colon\ i \in [n]\}$ of functions from $[t]$ to $[a]$ to be (a, b, k, t, n)-*good* if, for each $X \in C([n], b)$, there exists $\tau \in [t]$ with $|\{f_i(\tau)\colon\ i \in X\}| > k$.

LEMMA 2.2. *For positive integers a, b, k, t, n with $k < b \leqslant n$ and $k < a$, if*

$$\binom{n}{b} e^{kt} (k/a)^{(b-k)t} < 1,$$

then there exists an (a, b, k, t, n)-good sequence.

Proof. Let S be the set of all functions from $[t]$ to $[a]$, and choose functions $f_1, \ldots, f_n$ independently uniformly at random from S. We estimate the probability that this sequence is not (a, b, k, t, n)-good. For each $\tau \in [t]$ and each $X \in C([n], b)$,

$$\operatorname{Prob}\Big[|\{f_i(\tau)\colon\ i \in X\}| \leqslant k \Big] < \binom{a}{k} (k/a)^b \leqslant e^k (k/a)^{b-k}$$

and so

$$\begin{aligned}&\text{Prob}\left[\exists X \in C([n], b)\ \forall \tau \in [t]\left|\{f_i(\tau)\colon\ i \in X\}\right| \leqslant k\right]\\&\leqslant \binom{n}{b} e^{kt}(k/a)^{(b-k)t} < 1.\end{aligned}$$

The lemma follows. □

LEMMA 2.3. *Let a, b, k, t and n be positive integers with $k < b \leqslant n$ and $k < a$. If there exists an (a, b, k, t, n)-good sequence, then there exists a set*

$$\Sigma = \{L(\alpha, \tau, j)\colon\ \alpha \in [a],\ \tau \in [t] \text{ and } j \in [2]\}$$

of $2at$ linear extensions of $\mathcal{B}_n$ such that for all incomparable pairs $(S, T) \in \mathcal{B}_n \times \mathcal{B}_n$ with both $|S| < |T| \leqslant k + |S|$ and $|T \Delta S| \geqslant b$, there exists $L \in \Sigma$ such that $T < S$ in L.

Proof. Let $\{f_i\colon\ i \in [n]\}$ be an (a, b, k, t, n)-good sequence. Let M_1 and M_2 be two linear extensions of $\mathcal{B}_n$ such that if $S, T \in \mathcal{B}_n$ satisfy $|S| = |T|$, then $T < S$ in M_1 if and only if $S < T$ in M_2. For $S \in \mathcal{B}_n$, $\alpha \in [a]$, and $\tau \in [t]$, let $S(\alpha, \tau) = \{i \in S\colon\ f_i(\tau) = \alpha\}$. For all $\alpha \in [a]$, $\tau \in [t]$, and $j \in [2]$, define partial extensions $M(\alpha, \tau, j)$ on $\mathcal{B}_n$ by $T < S$ in $M(\alpha, \tau, j)$ if and only if either $|T(\alpha, \tau)| < |S(\alpha, \tau)|$ or both $|T(\alpha, \tau)| = |S(\alpha, \tau)|$ and $T(\alpha, \tau) < S(\alpha, \tau)$ in M_j. It is easy to check that each $M(\alpha, \tau, j)$, is a partial order which extends $\mathcal{B}_n$. Finally, let $L(\alpha, \tau, j)$ be any linear extension of $M(\alpha, \tau, j)$ for all $\alpha \in [a]$, $\tau \in [t]$, and $j \in [2]$.

We claim that

$$\Sigma = \{L(\alpha, \tau, j)\colon\ \alpha \in [a], \tau \in [t] \text{ and } j \in [2]\}$$

satisfies our requirement. Consider an incomparable pair $(S, T) \in \mathcal{B}_n \times \mathcal{B}_n$ with both $|S| < |T| \leqslant k + |S|$ and $|T \Delta S| \geqslant b$. Then there exists $X \subseteq T \Delta S$ with $|X| = b$. Since $\{f_i\colon\ i \in [n]\}$ is (a, b, k, t, n)-good, there exists $\tau \in [t]$ such that $|\{f_i(\tau)\colon\ i \in X\}| > k$. Since $|T| \leqslant k + |S|$, there exists $\alpha \in [a]$ such that either $|T(\alpha, \tau)| < |S(\alpha, \tau)|$ or both $\alpha \in \{f_i(\tau)\colon\ i \in X\}$ and $|T(\alpha, \tau)| = |S(\alpha, \tau)|$. In the first case, $T < S$ in $L(\alpha, \tau, j)$ for any $j \in [2]$. In the second case, there exists $i \in X \subseteq T \Delta S$ such that $f_i(\tau) = \alpha$. Thus $i \in T(\alpha, \tau) \Delta S(\alpha, \tau)$, so that $T(\alpha, \tau) \neq S(\alpha, \tau)$. It follows that there exists $j \in [2]$ such that $T < S$ in $L(\alpha, \tau, j)$. □

3. Proofs of Theorems 1.11 and 1.13

We first prove Theorem 1.11. The result is trivial if $18k \log n \geqslant n$, so we may assume that $18k \log n < n$. We now set $a = 3k$, $b = 3k$ and $t = 3 \log n$, and use the lemmas of the previous section. By Lemma 2.1, there is a collection Σ_1 of

$\dim(1, 2k; n)$ linear extensions of $\mathcal{B}_n$ such that, whenever S and T are incomparable elements of $\mathcal{B}_n$ with $|T - S| \leqslant 2k$, we have $T < S$ in some extension in Σ_1.

Next we note that

$$\binom{n}{3k} e^{3k \log n} (k/3k)^{(3k-k)3 \log n} \leqslant n^{3k} e^{3k \log n} 3^{-6k \log n} \leqslant (e/3)^{6k \log n} < 1,$$

so by Lemma 2.2, there is a $(3k, 3k, k, 3 \log n, n)$-good sequence. Now Lemma 2.3 tells us that there is a set Σ_2 of $18k \log n$ linear extensions of $\mathcal{B}_n$ such that, whenever S and T are incomparable sets with $|S| < |T| \leqslant k + |S|$ and $|T - S| \geqslant 2k$, we have $|T \Delta S| \geqslant 3k = b$, and so $T < S$ in some extension in Σ_2. The combined family $\Sigma = \Sigma_1 \cup \Sigma_2$ then has the desired property. This completes the proof of Theorem 1.11.

For the proof of Theorem 1.13, we need only apply Lemma 2.3 with $a = 3$, $b = 2$, $k = 1$, and $t = \lceil \lg n \rceil$. Observe first that any sequence $\{f_i\colon i \in [n]\}$ of distinct functions from $[t]$ to $[3]$ is $(3, 2, 1, t, n)$-good: the condition states exactly that any pair of functions differ for some argument. The collection Σ of $6 \log_3 n$ linear extensions of $\mathcal{B}_n$ provided by Lemma 2.3 now satisfies the requirements of the theorem, since if S and T are incomparable sets with $|T| = |S| + 1$, then $|T \Delta S| \geqslant 2$.

4. Concluding Remarks

In stating the principal results (Theorems 1.11 and 1.13) of this paper, we have chosen to express our upper bounds in a form which makes the analysis straightforward. This approach seems justified by the fact that for most of our inequalities, our upper and lower bounds differ by a multiplicative factor which is at least as large as $\log \log n$.

Our results suggest several new problems and reinforce the importance of some older ones, beginning of course with improvements to the various inequalities cited or derived in this paper. Here are two new problems which we consider to be particularly appealing.

PROBLEM 4.1. *For a fixed positive integer t, find (or estimate) the least number c_t so that* $\dim(1, t; n) \leqslant c_t \log \log n$.

PROBLEM 4.2. *For a fixed positive integer k, investigate the behavior of the ratio*

$$\dim(1, ks; n) / \dim(1, s; n).$$

For fixed values of k and n, what value of s makes this ratio maximum?

Note that Problem 4.2 is already interesting for small values of k, as the value $k = 2$ is featured in Theorem 1.11.

Note added in proof

After this manuscript was submitted, Kostochka improved the upper bound on $\dim(s, s+1; n)$ by showing that $\dim(s, s+1; n) = O(\log n / \log\log n)$. Kierstead showed that $\dim(1, k; n) \geqslant (1 - o(1))2^{k-2} \lg\lg n$, when $k < \lg\lg n - \lg\lg\lg n$. Kierstead also showed that $k^2 \lg n / 33 \lg k < \dim(1, k; n)$, when $2^{\lg^{1/2} n} \leqslant k \leqslant 2\sqrt{n} - 2$. Proofs will appear elsewhere.

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