

Monochromatic Matchings in the Shadow Graph of Almost Complete Hypergraphs

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Abstract. Edge colorings of r -uniform hypergraphs naturally define a multicoloring on the 2-shadow, i.e., on the pairs that are covered by hyperedges. We show that in any $(r - 1)$ -coloring of the edges of an r -uniform hypergraph with n vertices and at least $(1 - \varepsilon) \binom{n}{r}$ edges, the 2-shadow has a monochromatic matching covering all but at most $o(n)$ vertices. This result confirms an earlier conjecture and implies that for any fixed r and sufficiently large n , there is a monochromatic Berge-cycle of length $(1 - o(1))n$ in every $(r - 1)$ -coloring of the edges of $K_n^{(r)}$, the complete r -uniform hypergraph on n vertices.

Keywords: colored complete uniform hypergraphs, monochromatic matchings

1. Introduction

Let \mathcal{H} be an r -uniform hypergraph (a family of some r -element subsets of a set). The *shadow graph* of \mathcal{H} is defined as the graph $\Gamma(\mathcal{H})$ on the same vertex set, where two vertices are adjacent if they are covered by at least one edge of \mathcal{H} . A coloring of the edges of an r -uniform hypergraph \mathcal{H} , $r \geq 2$, induces a multicoloring on the edges of the shadow graph $\Gamma(\mathcal{H})$ in a natural way; every edge e of $\Gamma(\mathcal{H})$ receives the color of all hyperedges containing e . A subgraph of $\Gamma(\mathcal{H})$ is *monochromatic* if the color sets of its edges have a nonempty intersection.

A set of pairwise disjoint edges of the shadow graph covering $n - o(n)$ vertices is called an *almost perfect matching* of $\Gamma(\mathcal{H})$. Let $K_n^{(r)}$ denote the complete r -uniform

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hypergraph on n vertices. An r -uniform hypergraph is *almost complete*, if it has at least $(1 - o(1))\binom{n}{r}$ edges. We call an r -uniform hypergraph $(1 - \epsilon)$ -complete if it has at least $(1 - \epsilon)n^r/r!$ edges.

In this paper we prove the following conjecture from [3].

Theorem 1.1. *Assume that $r \geq 2$ is fixed, \mathcal{H} is an almost complete r -uniform hypergraph with n vertices, and its edges are colored with $r - 1$ colors. Then the induced multicoloring on $\Gamma(\mathcal{H})$ contains a monochromatic almost perfect matching.*

It is worth noting that Theorem 1.1 does not hold if we color with r colors instead of $r - 1$. An example in [3] gives an r -coloring of $K_n^{(r)}$ such that the largest number of vertices covered by any monochromatic matching is not larger than $\frac{(2r-2)n}{2r-1}$. In fact that is conjectured to be the best result ([3]) and proved for $r = 3$ ([5]).

It was proved in [3] that Theorem 1.1 implies a stronger result, namely, that the almost perfect monochromatic matching M guaranteed can be *connected* as well which means that the edges of M are in the same component of the hypergraph defined by the edges of the color of M . Moreover, it was shown in [3] how to combine this strengthening of Theorem 1.1 and a “weak version” of the hypergraph Regularity lemma to get a Ramsey-type result for Berge-cycles. An r -uniform Berge-cycle ([1]) of length ℓ is a sequence of distinct vertices v_1, v_2, \dots, v_ℓ together with a set of distinct edges e_1, \dots, e_ℓ such that e_i contains v_i, v_{i+1} ($v_{\ell+1} \equiv v_1$).

Corollary 1.2. *In every $(r - 1)$ -coloring of the edges of $K_n^{(r)}$ there is a monochromatic Berge-cycle of length at least $(1 - o(1))n$.*

Note that in [3] it was conjectured that for sufficiently large n this statement is true with a monochromatic Berge-cycle of length n , i.e., there is a monochromatic *Hamiltonian* Berge-cycle. However, at the moment we are unable to prove this stronger statement.

The way to obtain the above corollary from Theorem 1.1 illustrates a principle due to Łuczak (suggested in [8]): In many cases the task of finding a monochromatic path or cycle can be reduced to the easier task of finding a monochromatic matching via the Regularity lemma. This principle is applied in many recent Ramsey-type results such as [2–7].

2. Proof of Theorem 1.1

Suppose that \mathcal{H} is a hypergraph with vertex set V , $|V| = n$ and $0 < \delta < 1$ is fixed.

Informally, a sequence L of k distinct vertices of V is a δ -*bounded selection with respect to a family \mathcal{F} of forbidden sets*, if its elements are chosen in k consecutive steps so that in each step there is a set $F \in \mathcal{F}$, $|F| \leq \delta n$ such that vertices from F cannot be included as the next element of the sequence.

Formally, to capture that the forbidden set F for the next vertex of the sequence depends on the choices of previous vertices, we assume that the sets of \mathcal{F} are indexed with certain sequences of vertices, called valid sequences. This is achieved by the following assumptions. The empty sequence \emptyset is valid and there is a set $F(\emptyset) \in \mathcal{F}$. A one-element sequence $x_1 \in V$ is valid if and only if $x_1 \notin F(\emptyset)$. For $1 \leq i < k$

and for each valid sequence (x_1, x_2, \dots, x_i) of distinct vertices of V there is a set $F(x_1, x_2, \dots, x_i) \in \mathcal{F}$ and furthermore a sequence $(x_1, x_2, \dots, x_i, x_{i+1})$ is valid if and only if $x_{i+1} \notin F(x_1, x_2, \dots, x_i)$.

Now, with the assumptions of the previous paragraph, a sequence $L = (x_1, x_2, \dots, x_k)$ of k distinct vertices is called a δ -bounded selection with respect to \mathcal{F} if L is a valid sequence and each set in \mathcal{F} has at most δn vertices. As a trivial example, every sequence is δ -bounded with respect to \mathcal{F} that contains only empty sets. For the convenience of the reader a slightly less trivial example is described in the next paragraph.

Let \mathcal{H} be a balanced complete tripartite graph on n vertices and let $A(x)$ denote the partite class containing $x \in V$. Then define \mathcal{F} by $F(\emptyset) = \emptyset$ and $F(x_1, x_2, \dots, x_i) = \cup_{j=1}^i A(x_j)$. With respect to these forbidden sets every $\frac{1}{3}$ -bounded selection $L = (x_1, x_2)$ is an edge in \mathcal{H} ; every $\frac{2}{3}$ -bounded selection $L = (x_1, x_2, x_3)$ is a triangle in \mathcal{H} . In fact, no matter how forbidden sets are defined, the numbers $\frac{1}{3}, \frac{2}{3}$ can not be lowered without violating the statements above, that is, for a $\delta < \frac{1}{3}$ not every δ -bounded selection $L = (x_1, x_2)$ will be an edge in \mathcal{H} , and similarly, for a $\delta' < \frac{2}{3}$ not every δ' -bounded selection $L = (x_1, x_2, x_3)$ will be a triangle in \mathcal{H} .

The following lemma from [3] ensures that in almost complete hypergraphs there exists \mathcal{F} , such that with respect to \mathcal{F} every δ -bounded selection of at most r vertices is contained in many edges of the hypergraph.

Lemma 2.1. *Assume that $\varepsilon > 0, \delta = \varepsilon^{2-r}$ and \mathcal{H} is a $(1 - \varepsilon)$ -complete r -uniform hypergraph on n vertices. Then there exists \mathcal{F} , such that with respect to \mathcal{F} there exist δ -bounded selections and every δ -bounded selection $L \subset V(\mathcal{H})$ of at most r vertices has the following property: L is contained in at least $(1 - \delta) \frac{n^{r-|L|}}{(r-|L|)!}$ edges of \mathcal{H} . In particular, for $|L| = r, L$ is an edge of \mathcal{H} .*

Now we are ready to prove Theorem 1.1 by induction on r . Let $\varepsilon > 0$ be arbitrary, \mathcal{H} is a $(1 - \varepsilon)$ -complete r -uniform hypergraph with n vertices whose edges are colored with $r - 1$ colors. We shall prove that there is a monochromatic matching M in $\Gamma(\mathcal{H})$ covering all but at most αn vertices, where α tends to 0 if ε tends to 0.

Set $p = \sqrt{\delta n} + 1$. For $r = 2$ we have a $(1 - \varepsilon)$ -complete graph (colored with one color). Select a maximum matching M in the graph. Observe that the set of vertices uncovered by M forms an independent set. Since $\binom{p}{2} > \frac{\delta n^2}{2} > \frac{\varepsilon n^2}{2}$, less than $p = o(n)$ vertices are uncovered by M .

Assume that Theorem 1.1 is true for every $q < r$. Consider an $(r - 1)$ -coloring of a $(1 - \varepsilon)$ -complete r -uniform hypergraph \mathcal{H} with $r \geq 3$. Set $\delta = \varepsilon^{2-r}$ and apply Lemma 2.1 to define \mathcal{F} . Consider all δ -bounded selections of r vertices with respect to \mathcal{F} . Let G be the graph defined on the set of vertices that appear as the first vertices of these δ -bounded selections where the edges are those edges of the shadow graph $\Gamma(\mathcal{H})$ that appear as the first two vertices of these δ -bounded selections. Note that G has minimum degree at least $(1 - 2\delta)n$. Indeed, if $x_1 \in V(G)$ then at most δn vertices are forbidden for x_2 and another at most δn vertices are not in $V(G)$. On the other hand, for the remaining $(1 - 2\delta)n$ vertices Lemma 2.1 (with $|L| = 2$) ensures that $\{x_1, x_2\}$ is covered by at least $(1 - \delta) \frac{n^{r-2}}{(r-2)!} > 0$ edges of \mathcal{H} , thus $x_1 x_2 \in E(G)$.

For any $v \in V(G)$ and $1 \leq i \leq r - 1$, let A_i be the set of vertices w such that color i is not on the edge $vw \in E(G)$, without loss of generality, $|A_1| \leq |A_2| \leq \dots \leq |A_{r-1}|$. Assume that $y_i \in A_i$ for $1 \leq i \leq r - 1$ are distinct vertices. Then $e = \{v, y_1, \dots, y_{r-1}\} \notin E(\mathcal{H})$ because no color can be assigned to the r -tuple e . Therefore, with

$$D = \frac{|A_1|(|A_2| - 1)(|A_3| - 2) \cdots (|A_{r-1}| - r + 2)}{(r - 1)!},$$

v is contained in at least D r -tuples of $V(\mathcal{H})$ that are not edges of \mathcal{H} . However, Lemma 2.1 (with $|L| = 1$) guarantees that the number of these r -tuples is less than

$$\binom{n - 1}{r - 1} - (1 - \delta) \frac{n^{r-1}}{(r - 1)!} < \frac{\delta n^{r-1}}{(r - 1)!}.$$

Now it follows that

$$\frac{(|A_1| - r + 2)^{r-1}}{(r - 1)!} < D < \frac{\delta n^{r-1}}{(r - 1)!},$$

implying that $|A_1| < n\delta^{\frac{1}{r-1}} + r - 2$. Using this and the minimum degree condition in G , v is adjacent in color 1 to at least $(1 - 2\delta - \rho)n$ vertices of G , where $\rho = \delta^{\frac{1}{r-1}} + \frac{r-2}{n}$.

The argument above can be repeated for any $v \in V(G)$, showing that $V(G) = \cup_{i=1}^{r-1} X_i$ where $v \in X_i$ has the property that at least $(1 - 2\delta - \rho)n$ edges of G of color i are incident to v .

Let M_i be a maximum matching in color i in the subgraph of G induced by $V(G) \setminus X_i$ and set $Y_i = V(G) \setminus (V(M_i) \cup X_i)$. Observe that, from the choice of M_i , no edge of G within Y_i is colored with color i .

If

$$p(i - 1) + \frac{|X_i|}{1 - \delta} \geq |Y_i|$$

holds for some i , $1 \leq i \leq r - 1$, then we have the required large matching in color i . Indeed, almost every edge of G incident to X_i has color i thus M_i can be extended to a matching that misses at most $p(i - 1) + (3\delta + \rho)n = o(n)$ vertices of G .

Assume that $p(i - 1) + \frac{|X_i|}{1 - \delta} < |Y_i|$ for every i , $1 \leq i \leq r - 1$. This implies that

$$\begin{aligned} \sum_{i=1}^{r-1} |Y_i| &> p \sum_{i=1}^{r-1} (i - 1) + \frac{1}{1 - \delta} \sum_{i=1}^{r-1} |X_i| \\ &\geq p \binom{r - 1}{2} + \frac{|V(G)|}{1 - \delta} \\ &\geq p \binom{r - 1}{2} + \frac{(1 - \delta)n}{1 - \delta} \\ &= p \binom{r - 1}{2} + n. \end{aligned}$$

We claim that this inequality implies $|Y_i \cap Y_j| \geq p$ for some $1 \leq i < j \leq r-1$. Indeed, otherwise

$$n \geq |\cup_{i=1}^{r-1} Y_i| > \sum_{i=1}^{r-1} |Y_i| - \binom{r-1}{2} p,$$

contradicting to the inequality above. This proves the claim.

Select Y_i, Y_j from the claim, without loss of generality, $|Y_{r-2} \cap Y_{r-1}| \geq p$. From the definition of p , $\binom{p}{2} > \frac{\delta n^2}{2}$ follows, implying from the minimum degree condition in G that there are $x, y \in Y_{r-2} \cap Y_{r-1}$ such that $xy \in E(G)$ — this implies that the colors $r-2, r-1$ are not among the colors of xy . Notice that for $r=3$ we have a contradiction on this branch of the proof, $xy \in E(G)$ thus $\{x, y\}$ is covered by an edge of \mathcal{H} (a triple) but it cannot have a color. Thus we may assume that $r \geq 4$.

Consider the $(r-2)$ -uniform colored hypergraph \mathcal{H}^* with edge set $\{e \setminus \{x, y\} : e \in \mathcal{H} \text{ and } \{x, y\} \subset e\}$, where the color of the edge $e \setminus \{x, y\}$ is by definition the color of the edge e in \mathcal{H} . Note that x, y are the first two vertices in a δ -bounded selection of Lemma 2.1 thus \mathcal{H}^* is a $(1-\delta)$ -complete hypergraph. Moreover, since $x, y \in Y_{r-2} \cap Y_{r-1}$, \mathcal{H}^* is colored with $r-3$ colors (colors $r-1$ and $r-2$ cannot be used). Thus, since $r-2 \geq 2$ - induction applies, \mathcal{H}^* has an almost perfect monochromatic matching M in its shadow graph. Observing that M is a monochromatic matching in the shadow graph of \mathcal{H} as well, the proof is finished. ■

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