COMBINATORICA
Bolyai Society – Springer-Verlag

COMBINATORICA **24** (3) (2004) 389–401

# ON GRAPHS WITH SMALL RAMSEY NUMBERS, II

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Received February 1, 2001

There exists a constant $C$ such that for every $d$-degenerate graphs $G_1$ and $G_2$ on $n$ vertices, Ramsey number $R(G_1,G_2)$ is at most $Cn\Delta$, where $\Delta$ is the minimum of the maximum degrees of $G_1$ and $G_2$.

## 1. Introduction

For arbitrary graphs $G_1$ and $G_2$, define the *Ramsey number* $R(G_1,G_2)$ to be the minimum positive integer $N$ such that in every bicoloring of edges of the complete graph $K_N$ with, say, red and blue colors, there is either a red copy of $G_1$ or a blue copy of $G_2$. The classical Ramsey number $r(k,l)$ is in our terminology $R(K_k,K_l)$.

Call a family $\mathcal{F}$ of graphs *linear Ramsey* if there exists a constant $C = C(\mathcal{F})$ such that for every $G \in \mathcal{F}$,

$$R(G,G) \le C|V(G)|.$$

Burr and Erdős [3] conjectured that for every $\Delta$ and $d$,
(a) *the family of graphs with maximum degree at most $\Delta$ is linear Ramsey;*
(b) *the family $\mathcal{D}_d$ of $d$-degenerate graphs is linear Ramsey.*

*Mathematics Subject Classification (2000):* 05C55, 05C35
* The work of this author was supported by the grants 99-01-00581 and 00-01-00916 of the Russian Foundation for Fundamental Research and the Dutch-Russian Grant NWO-047-008-006.
† The work of this author was supported by the NSF grant DMS-9704114.

Recall that a graph is *d-degenerate* if every its subgraph has a vertex of degree (in this subgraph) at most $d$. Equivalently, a graph $G$ is $d$-degenerate if for some linear ordering of the vertex set of $G$ every vertex of $G$ is adjacent to at most $d$ vertices of $G$ that precede it in the ordering.

The first conjecture was proved by Chvátal, Rödl, Szemerédi, and Trotter [5]. The $C(\Delta)$ in their proof grows with $\Delta$ very rapidly. Recently, Eaton [6] improved the upper bound for $C(\Delta)$ to a function of the form $2^{2^{c\Delta}}$ and Graham, Rödl and Ruciński [7] reduced it to $c^{\Delta \log^2 \Delta}$. Moreover, they proved in [8] that for every bipartite graph $G$ on $n$ vertices with maximum degree $\Delta \geq 1$,

$$R(G,G) \leq 8(8\Delta)^{\Delta} n. \tag{1}$$

On the other hand, they showed in [7] and [8] that $C(\Delta)$ grows exponentially.

The second conjecture (which is much stronger) is still wide open. In recent years, some subfamilies of the family $\mathcal{D}_d$ were shown to be linear Ramsey.

Let $\mathcal{W}_d$ denote the family of graphs in which the vertices of degree greater than $d$ form an independent set. Alon [1] proved that $\mathcal{W}_2$ is linear Ramsey.

A graph $G$ is called *p-arrangeable*, if there exists an ordering $v_1, \dots, v_n$ of its vertices with the following property: for every $i$, $1<i<n$, the number of $v_j$ with $j<i$ having a common neighbor $v_s$ for some $s>i$ with $v_i$ is less than $p$. Let $\mathcal{A}_d$ denote the family of $d$-arrangeable graphs. Observe that $\mathcal{A}_d \subset \mathcal{D}_d$ for $d \geq 2$. On the other hand, $\mathcal{A}_{10}$ contains all planar graphs and $\mathcal{A}_{p^8}$ contains all graphs with no $K_p$-subdivisions (see [10]). Chen and Schelp [4] proved that $\mathcal{A}_d$ is linear Ramsey for every $d$.

In [9] and this paper, we attack the second Burr–Erdős conjecture from another angle. In [9], it is proved that the family $\mathcal{W}_d$ is "almost" linear Ramsey: for every $\epsilon > 0$, there exists $C = C(d,\epsilon)$ such that for every graph $G \in \mathcal{W}_d$,

$$R(G,G) \leq C|V(G)|^{1+\epsilon}.$$

Our main result yields that even if $\mathcal{D}_d$ were not linear Ramsey, anyway, it is 'polynomially Ramsey'.

**Theorem 1.** *Let $C = C(d) = (8d)^{4d^2+d}$. Then for every d-degenerate graphs $G_1$ and $G_2$ on $n$ vertices,*

$$R(G_1,G_2) \leq Cn\Delta(G_1),$$

*where $\Delta(G_1)$ is the maximum degree of $G_1$.*

**Corollary 1.** *Let $C=C(d)=(8d)^{4d^2+d}$. Then for every $d$-degenerate graph $G$,*

$$R(G,G) \le C|V(G)|\Delta(G) \le C|V(G)|^2.$$

We also improve the constant factor in the statement of Theorem 1 for $d$-degenerate graphs with chromatic number less than $d$.

**Theorem 2.** *Let $G_1$ be an arbitrary $d$-degenerate graph on $n$ vertices with maximum degree $\Delta$ and let $G_2$ be an arbitrary $d$-degenerate graph on $n$ vertices with chromatic number $\chi$. Let $m = 4(d+1)(\chi-1)$ and $C=m^{d+1}(4m^{d-1})^{\chi-2}$. Then*

$$R(G_1,G_2) \le Cn\Delta.$$

*In particular, if $G_2$ is bipartite, then*

$$R(G_1,G_2) \le (4(d+1))^{d+1}n\Delta. \tag{2}$$

For large $d$, (2) is a bit better than (1) even if $d=\Delta$.

For $n>d$, we say that a graph $H$ possesses $(d,n)$-property if

$$\forall v_1,\dots,v_d \in V(H), \quad |N_H(v_1)\cap\dots\cap N_H(v_d)| \ge n-d. \tag{3}$$

It is easy to observe (see Lemma 2 in the next section) that each graph with $(d,n)$-property contains every $d$-degenerate graph on $n$ vertices. In view of this, Frieze and Reed asked the following question:

*Is it true that for every positive integer $d$, there exists a constant $C=C(d)$ such that for every graph $H$ on $Cn$ vertices, either $H$ or $\overline{H}$ contains a subgraph with $(d,n)$-property?*

Answering the question in the positive would imply the Burr–Erdős conjecture. The following is a weaker ('polynomial') result in this spirit.

**Theorem 3.** *For every positive integer $d$, there exists a positive constant $C=C(d)$ such that for every graph $H$ on $(Cn)^d$ vertices, either $H$ or $\overline{H}$ contains a subgraph $H_1$ possessing $(d,n)$-property.*

To derive Theorems 1 and 3, we prove statements on graphs in which every 'big' subgraph has 'many' edges. To be exact, a graph $H$ will be called $(d,s)$-*thick*, if for every $s\le k\le |V(H)|$ and every induced subgraph $H'$ of $H$ on $k$ vertices,

$$|E(H')| \ge \frac{1}{2d}\binom{k}{2}.$$

Since for every graph $H$ on at least $4n$ vertices which is not $(d,4n)$-thick, the complement $\overline{H}$ of $H$ contains a subgraph with $(d,n)$-property (see Lemma 3 in the next section), the following two theorems imply Theorems 1 and 3, respectively.

**Theorem 1′.** *Let $M \geq (8d)^{4d^2+d}\Delta n$ and $G$ be a $d$-degenerate graph on $n$ vertices with maximum degree $\Delta$. Then every $(d,4dn)$-thick graph $H$ on $M$ vertices contains $G$.*

**Theorem 3′.** *Let $d \geq 2$, $n \geq (8d)^{d+1}$ and $M \geq \left(8(8d)^{5d}n\right)^d$. Then every $(d,4dn)$-thick graph on $M$ vertices contains a subgraph $H_1$ possessing $(d,n)$-property.*

In the next section, we prove simple statements used above to motivate results of the paper. In Section 3 we discuss a useful notion of reducing pairs. Sections 4, 5, and 6 are devoted to the proofs of Theorems 1′ (and 1), 3′ (and 3), and 2, respectively.

## 2. Preliminaries

**Lemma 1.** *Let $|V(H)|=n$ and $|E(H)| \geq (c+\lambda)\binom{n}{2}$, where $c \geq 0$ and $\lambda \geq 0$. Then there exists $H' \subseteq H$ such that*

$$\forall v \in V(H'), \quad \deg_{H'}(v) \geq c(|V(H')|-1) + \frac{\lambda n}{2}. \tag{4}$$

**Proof.** If the lemma is false, then we can order the vertices of $H$: $v_1,\ldots,v_n$ in such a way that denoting $H_i = H \setminus \{v_1,\ldots,v_{i-1}\}$ $(i=1,\ldots,n-1)$, we have

$$\deg_{H_i}(v_i) < c(n-i) + \frac{\lambda n}{2}. \tag{5}$$

Since $H_n = K_1$, (5) yields

$$\begin{aligned} |E(H)| &< \sum_{i=1}^{n-1}\left(c(n-i) + \frac{\lambda n}{2}\right) = c\sum_{i=1}^{n-1}(n-i) + \frac{\lambda n(n-1)}{2} \\ &= (c+\lambda)\binom{n}{2} \leq |E(H)|. \end{aligned}$$

This contradiction proves the lemma. ∎

**Lemma 2.** *Suppose that a graph $H$ possesses the $(d,n)$-property. Then $H$ contains every $d$-degenerate graph on $n$ vertices.*

**Proof.** Let $G$ be a $d$-degenerate graph on $n$ vertices, and let $x_1,\dots,x_n$ be its vertices ordered so that for every $i=1,\dots,n$, at most $d$ neighbors of $x_i$ have indices less than $i$. We construct an embedding $\phi$ of $G$ into $H$ as follows. On Step $i$ we will find $\phi(x_i)$.

Step 1. Let $\phi(x_1)$ be an arbitrary vertex $v_1$ in $H$.

Step $i$ $(i>1)$. Suppose that $v_k=\phi(x_k)$ for $k=1,\dots,i-1$ and that $x_i$ is adjacent only to $x_{j_1},\dots,x_{j_h}$ among embedded vertices (where $h\le d$). If $h<d$, then take as $v_{j_{h+1}},\dots,v_{j_d}$ arbitrary vertices with indices less than $i$ (distinct, if possible). By (3), there are at least $n-d$ vertices in $N_H(v_{j_1})\cap\dots\cap N_H(v_{j_d})$. We choose as $\phi(x_i)$ any of them different from $v_1,\dots,v_{i-1}$. ∎

**Lemma 3.** *If $|V(H)|>4n$ and for some $s$, $4n\le s\le|V(H)|$, $H$ is not $(d,s)$-thick, then $\overline{H}$ contains a subgraph with $(d,n)$-property.*

**Proof.** Suppose that $H$ is not $(d,s)$-thick for some $s$, $4n\le s\le|V(H)|$. By the definition, this means that for some $k\ge s$ there exists an induced subgraph $H'$ of $H$ on $k$ vertices such that $|E(\overline{H'})|>(1-\frac{1}{2d})\binom{k}{2}$. Then by Lemma 1, (with $c=1-1/d$ and $\lambda=1/2d$), there exists a subgraph $H_1$ of $H'$ such that

$$\forall v\in V(H_1),\quad \deg_{\overline{H}_1}(v)\ge\frac{d-1}{d}(|V(H_1)|-1)+\frac{k}{4d}.$$

It follows that for all $v_1,\dots,v_d\in V(H_1)$,

$$|N_{\overline{H}_1}(v_1)\cap\dots\cap N_{\overline{H}_1}(v_d)|\ge(|V(H_1)|-d)-d\frac{1}{d}(|V(H_1)|-1)+d\frac{k}{4d}=\frac{k}{4}-d+1.$$

Since $n\le k/4$, we are done. ∎

## 3. Reducing pairs

Let $H_1$ be a graph with $|V(H_1)|=M_1$. Define $N_{H_1}(\emptyset)=V(H_1)$, and for $\emptyset\ne A\subseteq V(H_1)$, let

$$N_{H_1}(A)=\bigcap_{v\in A}N_{H_1}(v).$$

An $a$-tuple $A\subset V(H_1)$ is $(H_1,m)$-*good* if $|N_{H_1}(A)|\ge M_1m^{-a}$, and is $(H_1,m)$-*bad* otherwise.

In this section we prove two lemmas which later let us reduce the proofs of the theorems to the cases when in 'big' subgraphs of $H$ every 'good' $a$-tuple is contained in 'few' 'bad' $(a+1)$-tuples. We will need the notion of *reducing pairs*.

**Definition.** *For a graph $H_1$ with $|V(H_1)| = M_1$, an $(H_1,r,m,d)$-*reducing pair *is a pair of disjoint subsets $R$ and $S$ of $V(H_1)$ such that*

$$|R| = r, \quad |S| \geq \frac{3M_1}{4m^{d-1}} \quad \text{and} \quad |N_{H_1}(v) \cap S| \leq \frac{4|S|}{3m} \quad \forall v \in R.$$

**Lemma 4.** *Let $m \geq 2$. Let $H_1$ be a graph with $|V(H_1)| = M_1 \geq 2rm^d$. If for some $0 \leq a \leq d-1$, an $(H_1,m)$-good $a$-tuple $A$ is contained in at least $r$ $(H_1,m)$-bad $(a+1)$-tuples, then $H_1$ contains an $(H_1,r,m,d)$-reducing pair.*

**Proof.** Suppose that $R$ is any set of vertices having fewer than $M_1 m^{-a-1}$ neighbors in $N_{H_1}(A)$ with $|R| = r$. Let $S = N_{H_1}(A) - R$. Since $A$ is $(H_1,m)$-good, we have

$$|S| \geq |N_{H_1}(A)| - r \geq \frac{M_1}{m^a} - \frac{M_1}{2m^d} \geq \frac{M_1}{m^a}\left(1 - \frac{1}{2m^{d-a}}\right) \geq \frac{3M_1}{4m^a}.$$

By the choice of $R$, every $x \in R$ has less than $\frac{M_1}{m^{a+1}} \leq \frac{4|S|}{3m}$ neighbors in $S$. ∎

**Lemma 5.** *Let $d \geq 2, r \geq 2$, and $m \geq 8d$. Let $|V(H)| = M \geq 2rm^{4d^2+d}$. If every subgraph $H_1$ of $H$ with $|V(H_1)| \geq M \cdot m^{-4d^2}$ contains an $(H_1,r,m,d)$-reducing pair, then $H$ contains a subgraph $H'$ on $4dr$ vertices with $|E(H')| < \frac{1}{2d}\binom{4dr}{2}$. In particular, then $H$ is not $(d,4dr)$-thick.*

**Proof.** Let $H_0 = H$. For $k = 1,\dots,4d-1$ we proceed as follows:
(a) Choose an $(H_{k-1},r,m,d)$-reducing pair $(R_k,S_k)$;
(b) Since $|E_H(R_k,S_k)| \leq \frac{4|R_k|\cdot|S_k|}{3m}$, there exists $S'_k \subseteq S_k$ such that $|S'_k| \geq |S_k|/3$ and

$$|N_H(v) \cap R_k| \leq \frac{2}{m}|R_k| \quad \forall v \in S'_k. \tag{6}$$

(c) Let $H_k$ be the subgraph of $H$ induced by $S'_k$ and note that by the definitions of $S'_k$ and reducing pairs,

$$|V(H_k)| \geq \frac{1}{3}|S_k| \geq \frac{1}{3}\frac{3|V(H_{k-1})|}{4m^{d-1}} > \frac{|V(H_{k-1})|}{m^d} > \dots > \frac{|V(H_0)|}{m^{kd}}.$$

Denote by $R_{4d}$ any subset of $S'_{4d-1}$ of cardinality $r$.

Consider $\widetilde{R} = \bigcup_{k=1}^{4d} R_k$ and $\widetilde{H} = H(\widetilde{R})$. We have $|\widetilde{R}| = 4dr$. By (6),

$$|E_H(R_i,R_j)| \leq \frac{2r^2}{m} \quad \forall i \neq j.$$

Thus,

$$|E(\widetilde{H})| \leq 4d\binom{r}{2} + \binom{4d}{2}\frac{2r^2}{m} < 2dr(4dr-1)\left(\frac{1}{4d} + \frac{2}{m}\right) \leq \binom{4dr}{2}\left(\frac{1}{4d} + \frac{2}{8d}\right).$$

This proves the lemma. ∎

## 4. Proof of Theorem 1′

**Lemma 6.** *Let $n > \Delta \geq d \geq 2$, $m \geq d$, $\alpha \geq 1$, and $M_0 = m^d \Delta\alpha n$. If a graph $H_1$ on $M_1 > M_0$ vertices has no $(H_1, \alpha n, m, d)$-reducing pairs, then every $d$-degenerate graph $G$ on $n$ vertices with maximum degree $\Delta$ can be embedded into $H_1$.*

**Proof.** Let $x_1, \dots, x_n$ be the vertices of $G$ ordered so that for every $i = 1, \dots, n$, at most $d$ neighbors of $x_i$ have indices less than $i$. Let $X(i)$ denote the set of neighbors of $x_i$ having indices less than $i$. We will construct an embedding $f$ of $V(G)$ into $V(H_1)$. On Step $k$ we will map $x_k$ and we will maintain property

$$\forall j = k+1, \dots, n, \quad f\left(X(j) \cap \{x_1, \dots, x_k\}\right) \text{ is } (H_1, m)\text{-good.} \tag{7}$$

STEP 1. Since we assume that $H_1$ has no $(H_1, \alpha n, m, d)$-reducing pairs, Lemma 4 (applied with $a = 0$ and $r = n\alpha$) yields that there are fewer than $n$ $(H_1, m)$-bad vertices. Thus, we can choose a vertex $v_1$ which is not $(H_1, m)$-bad and let $v_1 = f(x_1)$.

STEP $k$. Suppose that $X(k) = \{x_{i_1}, \dots, x_{i_a}\}$ (where $a \leq d$). Let $A = f(X(k))$. Due to (7),

$$|N_{H_1}(A)| \geq \frac{N_1}{m^a}. \tag{8}$$

Let $h_1, \dots, h_s$ (where $s \leq \Delta$) be the indices greater than $k$ of the neighbors of $x_k$, and $\widetilde{X}_k(h_i) = f(X(h_i) \cap \{x_1, \dots, x_{k-1}\})$. Assume that taking $f(x_k) = v \in N_{H_1}(A) \setminus \{f(x_1), \dots, f(x_{k-1})\}$ creates an $(H_1, m)$-bad $l$-tuple $L = \{f(x_{j_1}), \dots, f(x_{j_{l-1}}), v\}$ for some $l \in \{1, \dots, d\}$ and the $(l-1)$-tuple $L' = L - v$ is some $\widetilde{X}_{k-1}(h_i)$. By (7), $L'$ is $(H_1, m)$-good. Then in view of our assumptions on $H_1$, by Lemma 4, $L'$ participates in at most $\alpha n - 1$ such $(H_1, m)$-bad $l$-tuples. Therefore, at most $\alpha n - 1$ vertices $v$ can create an $(H_1, m)$-bad $l$-tuple with this $L'$. The total number of such $L'$ is at most $\Delta$. Moreover, if it equals $\Delta$, then $a = 0$.

If $a > 0$, then

$$|N_{H_1}(A)| - (k-1) - (\alpha n - 1)(\Delta - 1) \geq \frac{M_1}{m^a} - \alpha n\Delta > \frac{M_0}{m^d} - \alpha n\Delta \geq 0.$$

If $a = 0$, then

$$|N_{H_1}(A)| - (k-1) - (\alpha n - 1)\Delta \geq M_1 - \alpha n(\Delta + 1) > M_0 - \alpha n(\Delta + 1) > 0.$$

In both cases we can choose $f(x_k)$ so that (7) still holds. ∎

**Proof of Theorem 1′.** Let $M \geq (8d)^{4d^2+d}\Delta n$ and $H$ be a $(d,4dn)$-thick graph on $M$ vertices. Assume that $H$ does not contain $G$. Then by Lemma 6 (with $\alpha = 1$ and $m = 8d$), every subgraph $H_1$ of $H$ on at least $M(8d)^{-4d^2}$ vertices has an $(H_1,n,8d,d)$-reducing pair. But in this case, by Lemma 5 (with $r=n$), $H$ is not $(d,4dn)$-thick. This contradiction proves the theorem. ∎

**Proof of Theorem 1.** Let $H$ be an arbitrary graph on $M \geq (8d)^{4d^2+d}\Delta n$ vertices. If $H$ is $(d,4dn)$-thick, then by Theorem 1′, it contains $G_1$. If $H$ is not $(d,4dn)$-thick, then by Lemmas 3 and 2, $\overline{H}$ contains $G_2$. This proves the theorem. ∎

## 5. Proof of Theorem 3′

We shall use the following form of Chernoff-Hoeffding type inequality (cf. [2], Appendix A).

**Lemma 7.** *Let $Y$ be the sum of mutually independent indicator random variables, $\mu = \mathbf{E}(Y)$. For each $0<\epsilon<1$,*

$$\mathbf{P}\{Y < \mu(1-\epsilon)\} < \exp\{-\epsilon^2\mu/2\}. \tag{9}$$

**Lemma 8.** *Let $M \geq C^d n^d$ where $C = 4(8d)^{5d}$. Let $H_1$ be a graph on $M_1 \geq M(8d)^{-4d^2}$ vertices. Let $r \leq M_1/2m^d$. If $H_1$ has no $(H_1,r,8d,d)$-reducing pairs, then for every $1 \leq a \leq d$, the number of $(H_1,8d)$-bad $a$-tuples is at most*

$$M_1^{a-1} r \sum_{i=1}^{a} \frac{1}{i!}.$$

*In particular, the number of $(H_1,8d)$-bad $d$-tuples is at most $2M_1^{d-1}r$.*

**Proof.** We prove the lemma by induction on $a$. By Lemma 4, there are at most $r-1$ $(H_1,8d)$-bad 1-tuples (i.e., vertices). Thus, the lemma holds for $a=1$.

Suppose that the lemma is proved for every $a < a_0$. We say that an $(H_1,8d)$-bad $a_0$-tuple *is of type 1* if it contains an $(H_1,8d)$-bad $(a_0-1)$-tuple and that it *is of type* 2 otherwise. By the induction assumption, the number of $(H_1,8d)$-bad $a_0$-tuples of type 1 is at most

$$M_1 \cdot M_1^{a_0-2} r \sum_{i=1}^{a_0-1} \frac{1}{i!}.$$

If $A$ is an $(H_1,8d)$-bad $a_0$-tuple of type 2, then it contains $a_0$ $(H_1,8d)$-good $(a_0-1)$-tuples, and by Lemma 4, every $(H_1,8d)$-good $(a_0-1)$-tuple is contained in less than $r$ $(H_1,8d)$-bad $a_0$-tuples. Therefore by the induction assumption, the number of $(H_1,8d)$-bad $a_0$-tuples of type 2 is less than

$$\binom{M_1}{a_0-1} r \cdot \frac{1}{a_0} \le \frac{M_1^{a_0-1} r}{a_0!},$$

and the total number of $(H_1,8d)$-bad $a_0$-tuples is less than

$$M_1^{a_0-1} r \sum_{i=1}^{a_0-1} \frac{1}{i!} + \frac{M_1^{a_0-1} r}{a_0!}.$$

This proves the lemma. ∎

**Lemma 9.** *Let $d \ge 2$, $n \ge (8d)^{d+1}$ and $M \ge (Cn)^d$ where $C = 8(8d)^{5d}$. If a graph $H_1$ on $M_1 \ge M(8d)^{-4d^2}$ vertices has no $(H_1,n,8d,d)$-reducing pairs, then it contains a subgraph $G$ possessing $(d,n)$-property.*

**Proof.** Let $p = \frac{cn}{M_1}$ (where $c = 4(8d)^d$) and $\mathcal{G} = \mathcal{G}_p(H_1)$ be the random variable whose values are induced subgraphs of $H_1$, and every vertex of $H_1$ belongs to $\mathcal{G}_p(H_1)$ with probability $p$ independently of all other vertices.

Call a $d$-tuple $D$ of vertices of $\mathcal{G}$ *spoiled* if it is $(H_1,8d)$-good but the number of common neighbors of $D$ in $\mathcal{G}$ is less than $0.5cn(8d)^{-d}$.

The probability that a $d$-tuple $D$ is contained in $V(\mathcal{G})$ is $p^d$. Since by Lemma 8, the total number of $(H_1,8d)$-bad $d$-tuples is at most $2M_1^{d-1}r$, we conclude that for the expected number $f_1(\mathcal{G})$ of $(H_1,8d)$-bad $d$-tuples contained in $\mathcal{G}$ the following holds:

$$f_1(\mathcal{G}) \le \left(\frac{cn}{M_1}\right)^d 2M_1^{d-1} n = \frac{n^{d+1}}{M_1} \cdot 2c^d. \tag{10}$$

The fact that a $d$-tuple $D$ is $(H_1,8d)$-good means that $N_{H_1}(D) \ge M_1(8d)^{-d}$. So, the expected number $\mu$ of vertices in $N_{H_1}(D)$ belonging to $V(\mathcal{G})$ is at least $pM_1(8d)^{-d} = cn(8d)^{-d}$. By Lemma 7 (with $\epsilon = 0.5$), the probability that a fixed $(H_1,8d)$-good $d$-tuple $D$ is contained in $V(\mathcal{G})$ and the number of common neighbors of $D$ in $V(\mathcal{G})$ is less than $0.5\mu$ is at most $p^d \cdot \exp\{-\mu/8\}$. Thus, (remembering that $c = 4(8d)^d$ and $n \ge (8d)^{d+1}$) for the expected number $f_2(\mathcal{G})$ of spoiled $d$-tuples contained in $\mathcal{G}$ we have

$$\begin{aligned} f_2(\mathcal{G}) &\le \binom{M_1}{d}\left(\frac{cn}{M_1}\right)^d \exp\Big\{-\frac{cn}{8(8d)^d}\Big\} \\ &< \frac{(cn)^d}{d!}\exp\Big\{-\frac{n}{2}\Big\} \le \left(\frac{en^2}{2d^2}\right)^d \exp\Big\{-\frac{n}{2}\Big\} \le 0.2\ . \end{aligned} \tag{11}$$

Also by Lemma 7 (with $\epsilon = 0.5$), with probability greater than $1 - \exp\{-pM_1/8\} = 1 - \exp\{-cn/8\} > 0.8$, we have $|V(\mathcal{G})| > 0.5pM_1$. Together with (10) and (11), this implies that there exists a subgraph $H'$ of $H_1$ such that
(i) $|V(H')| > 0.5pM_1$,
(ii) the number of $(H_1, 8d)$-bad $d$-tuples contained in $H'$ is at most $\frac{n^{d+1}}{M_1} \cdot 4c^d$,
(iii) there are no spoiled $d$-tuples in $H'$.

Let $H_0$ be obtained from $H'$ by deleting a vertex from each $(H_1, 8d)$-bad $d$-tuple contained in $V(H')$. By (ii), we deleted at most $\frac{n^{d+1}}{M_1} \cdot 4c^d$ vertices. Since $H'$ has no spoiled $d$-tuples, every $d$-tuple of vertices in $H_0$ has at least

$$\frac{cn}{2(8d)^d} - \frac{n^{d+1}}{M_1} 4c^d = \frac{cn}{2(8d)^d} \left(1 - \frac{n^d}{M_1} 8c^{d-1}(8d)^d\right) \geq \frac{cn}{2(8d)^d} \left(1 - \frac{(8d)^{4d^2+d}}{C^d} 8c^{d-1}\right) \tag{12}$$

common neighbors. Since $c = 4(8d)^d$ and $C = 8(8d)^{5d} = 2(8d)^{4d}c$, the last expression in (12) is at least

$$\frac{cn}{2(8d)^d} \left(1 - \frac{8(8d)^d}{2^d c}\right) = 2n\left(1 - \frac{2}{2^d}\right) \geq 2n\left(1 - \frac{1}{2}\right) = n.$$

This proves the lemma. ∎

**Proof of Theorem 3′.** Let $d \geq 2$, $n \geq (8d)^{d+1}$, $M \geq \left(8(8d)^{5d}n\right)^d$ and let $H$ be a $(d, 4dn)$-thick graph on $M$ vertices. Assume that $H$ does not contain a subgraph $G$ possessing $(d, n)$-property. Then by Lemma 9, every subgraph $H_1$ of $H$ on at least $M(8d)^{-4d^2}$ vertices has an $(H_1, n, 8d, d)$-reducing pair. But in this case, by Lemma 5 (with $r = n$), $H$ is not $(d, 4dn)$-thick. This contradiction proves the theorem. ∎

**Proof of Theorem 3.** Let $n \geq (8d)^{d+1}$ and $H$ be an arbitrary graph on $M \geq (8(8d)^{5d}n)^d$ vertices. The statement of Theorem 3 for $d=1$ means that either $H$ or $\overline{H}$ contains a subgraph with minimum degree at least $n-1$, which is true, since $M > 4n$.

Let $d \geq 2$. If $H$ is $(d, 4dn)$-thick, then by Theorem 3′, it contains a subgraph $H_1$ possessing $(d, n)$-property. If $H$ is not $(d, 4dn)$-thick, then by Lemma 3, $\overline{H}$ contains contains a subgraph $H_2$ possessing $(d, n)$-property. This proves the theorem. ∎

## 6. Proof of Theorem 2

Say that a graph $H$ *possesses* $(k,d,n)$*-property* if the vertex set of $H$ can be partitioned into $k$ parts $W_1,\dots,W_k$ such that

$$\forall i\in\{1,\dots,k\},\ \forall v_1,\dots,v_d\in V(H)-W_i,\\ |N_H(v_1)\cap\dots\cap N_H(v_d)\cap W_i|\ge n-1. \tag{13}$$

**Lemma 10.** *Suppose that a graph $H$ possesses the $(k,d,n)$-property. Then $H$ contains every $k$-colorable $d$-degenerate graph on $n$ vertices.*

**Proof.** Let $(W_1,\dots,W_k)$ be a partition of $V(H)$ satisfying (13). Let $G$ be an arbitrary $k$-colorable $d$-degenerate graph on $n$ vertices. Fix a coloring $f$ of $G$ with $k$ colors $1,\dots,k$. Then we simply repeat the proof of Lemma 2 with the only change that the image $\phi(x_i)$ of $x_i$ must belong to $W_{f(x_i)}$. ∎

**Proof of Theorem 2.** Let $G_1$ be an arbitrary $d$-degenerate graph on $n$ vertices with maximum degree $\Delta$ and let $G_2$ be an arbitrary $d$-degenerate graph on $n$ vertices with chromatic number $\chi$. Let $m=4(d+1)(\chi-1)$, $C=m^{d+1}(4m^{d-1})^{\chi-2}$, $M=Cn\Delta$ and $H$ be an arbitrary graph on $M$ vertices.

If some $H_1\subseteq H$ with at least $m^d\Delta 2(d+1)n$ vertices has no $(H_1,2(d+1)n,m,d)$-reducing pair, then, by Lemma 6, $H_1$ contains $G_1$. Thus, we assume below that every $H_1\subseteq H$ with at least $m^d\Delta 2(d+1)n$ vertices has an $(H_1,2(d+1)n,m,d)$-reducing pair.

Let $H_0=H$ and for $k=1,\dots,\chi-1$ we do the following:
(a) Choose an $(H_{k-1},2(d+1)n,m,d)$-reducing pair $(R_k,S_k)$;
(b) Since $|N_H(v)\cap S_k)|\le\frac{4|S_k|}{3m}\quad\forall v\in R_k$, there exists $S'_k\subseteq S_k$ such that $|S'_k|\ge\frac{|S_k|}{3}$ and

$$|N_H(v)\cap R_k|\le\frac{2|R_k|}{m}\quad\forall v\in S'_k. \tag{14}$$

(c) Take $H_k=H(S'_k)$ and note that by the definitions of $S'_k$ and reducing pairs,

$$|V(H_k)|\ge\frac{1}{3}|S_k|\ge\frac{1}{3}\frac{3|V(H_{k-1})|}{4m^{d-1}}=\frac{|V(H_{k-1})|}{4m^{d-1}}\ge\dots\ge\frac{|V(H_0)|}{(4m^{d-1})^k}. \tag{15}$$

Observe that since $M\ge 4^{\chi-1}m^{(\chi-2)(d-1)+d}\Delta(d+1)n$, by (15), for $k\le\chi-2$ we have $|V(H_k)|\ge m^d\Delta 2(d+1)n$ and we can make Step $k+1$.

Denote by $R_\chi$ any subset of $S'_{\chi-1}$ of cardinality $2(d+1)n$.

Observe that
(i) $|R_1|=\ldots=|R_\chi|=2(d+1)n$;
(ii) by (14), for every $i>k$ and every $v\in R_i$, $|N_H(v)\cap R_k|\le \frac{2\cdot 2(d+1)n}{m}=\frac{n}{\chi-1}$.

Now, we construct $T_1,\ldots,T_\chi$ as follows. Let $T_\chi$ be any subset of $R_\chi$ of size $(d+1)n$. Suppose that sets $T_{\chi-1}\subset R_{\chi-1}, T_{\chi-1}\subset R_\chi,\ldots,T_{k+1}\subset R_{k+1}$ of size $(d+1)n$ are chosen. By (ii), $|E_H(T_i,R_k)|\le (d+1)n\frac{n}{\chi-1}$ for every $i>k$. Hence the number of vertices in $R_k$ having more than $n$ neighbors in $T_i$ is at most $\frac{(d+1)n}{\chi-1}$. It follows that there are at least

$$|R_k|-(\chi-k)\frac{(d+1)n}{\chi-1}\ge |R_k|-(d+1)n=(d+1)n$$

vertices in $R_k$ with at most $n$ neighbors in each of $T_\chi,T_{\chi-1},\ldots,T_{k+1}$. Take as $T_k$ any set of $(d+1)n$ such vertices.

Now we have
(i') $|T_1|=\ldots=|T_\chi|=(d+1)n$;
(ii') for every $i\ne k$ and every $v\in R_i$, $|N_H(v)\cap R_k|\le n$.

Denote by $F$ the complement of the subgraph of $H$ induced by $\bigcup_{k=1}^{\chi} T_k$. By (i') and (ii'), $F$ possesses the $(\chi,d,n)$-property. Hence by Lemma 10, $G_2$ is embeddable in $F$. This proves the theorem. ∎

**Acknowledgement.** We thank the referees for helpful remarks.

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