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Equitable Colourings of d-degenerate Graphs

A. V. KOSTOCHKA[1]† and K. NAKPRASIT[2]

[1,2] University of Illinois, Urbana, IL 61801, USA

[1] Institute of Mathematics, Novosibirsk, Russia 630090

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A proper vertex coloring of a graph is called *equitable* if the sizes of colour classes differ by at most 1. In this paper, we find the minimum number $l = l(d, \Delta)$ such that every d-degenerate graph with maximum degree at most Δ admits an equitable t-colouring for every $t \geqslant l$ when $\Delta \geqslant 27d$.

1. Introduction

In many applications of graph colouring the sizes of colour classes should not be too large. For example, in scheduling jobs (some of which could be performed at the same time), it is not good if the resulting schedule requires many jobs to occur at some specific time. An application of this type is discussed in [8]. A possible formalization of this restriction is the notion of equitable colouring. A proper vertex colouring of a graph is called *equitable* if the sizes of colour classes differ by at most 1.

A graph may have an equitable k-colouring (*i.e.*, an equitable colouring with k colours) but have no equitable $(k+1)$-colouring. For example, the complete bipartite graph $K_{7,7}$ has an equitable k-colouring for $k = 2$, 4, 6 and 8, but has no equitable k-colouring for $k = 3$, 5 and 7. Thus, it is natural to look for the minimum number, $\mathrm{eq}(G)$, such that, for every $k \geqslant \mathrm{eq}(G)$, G has an equitable k-colouring. A good survey on equitable colourings of graphs is given in [5]. Hajnal and Szemerédi [3], answering a question of Erdős, proved that, for every graph G, $\mathrm{eq}(G) \leqslant \Delta(G) + 1$. Recently, Pemmaraju [7] used equitable colourings to give new bounds on the tail of the distribution of the sum of random variables. He applied different theorems on equitable colourings for different situations. If the dependence graph of variables had a bounded maximum degree, he applied the above-mentioned Hajnal–Szemerédi theorem [3]; for trees he used a bound of Bollobás

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and Guy [1]; and for outerplanar dependence graphs he derived his own bound using the Bollobás–Guy result.

Meyer [6] proved that every tree with maximum degree Δ has an equitable k-colouring for $k = 1 + \lfloor \frac{\Delta}{2} \rfloor$ and it was observed later that the result holds for every $k \geqslant 1 + \lfloor \frac{\Delta}{2} \rfloor$. The example of the star $K_{1,\Delta}$ shows that one cannot take $k < 1 + \lfloor \frac{\Delta}{2} \rfloor$.

Yap and Zhang [10] proved that every outerplanar graph with maximum degree $\Delta \geqslant 3$ has an equitable Δ-colouring, and conjectured that every outerplanar graph with maximum degree $\Delta \geqslant 3$ is equitably k-colourable for every $k \geqslant 1 + \Delta/2$. This conjecture was proved in [4].

The aim of this paper is to find upper bounds on $\mathrm{eq}(G)$ for d-degenerate graphs with given maximum degree. Recall that a graph G is *d-degenerate* if the minimum degree of every subgraph of G is at most d (see, *e.g.*, [9]). Thus we can destroy any d-degenerate graph by consecutive deleting of vertices of degree at most d. Clearly, forests are exactly 1-degenerate graphs. It is also well known that every outerplanar graph is 2-degenerate (see, *e.g.*, [9]), and every planar graph is 5-degenerate. Note that the above-mentioned conjecture of Yap and Zhang [10] on outerplanar graphs does not extend to all 2-degenerate graphs. To see this, consider the graph $G(d, \Delta) = K_d + \overline{K}_{\Delta-d+1}$, obtained from the complete graph K_d by adding $\Delta - d + 1$ vertices, so that each of them is adjacent to vertices of our K_d and only to them. This graph is d-degenerate and has maximum degree Δ. In every proper colouring of $G(d, \Delta)$, d colour classes must be singletons containing the d all-adjacent vertices. Therefore, every equitable colouring of $G(d, \Delta)$ has at least

$$v(d, \Delta) = d + \left\lceil \frac{\Delta - d + 1}{2} \right\rceil = \left\lceil \frac{\Delta + d + 1}{2} \right\rceil$$

colour classes. In particular, $G(2, \Delta)$ needs at least

$$v(2, \Delta) = \left\lceil \frac{\Delta + 3}{2} \right\rceil$$

colours for an equitable colouring, which is greater than $1 + \lfloor \frac{\Delta}{2} \rfloor$ for even Δ.

We will show that $v(d, \Delta)$ colours is enough for an equitable colouring of an arbitrary d-degenerate graph with maximum degree Δ provided that Δ/d is large.

Theorem 1.1. *Let $d \geqslant 2$, $\Delta \geqslant 27d$, $k \geqslant (d + \Delta + 1)/2$. Then every d-degenerate graph with maximum degree at most Δ is equitably k-colourable.*

The above example of $G(d, \Delta)$ shows that the bound on k cannot be weakened. Since every planar graph is 5-degenerate, we obtain the following consequence.

Corollary 1.2. *Let $\Delta \geqslant 135$, $k \geqslant 3 + \Delta/2$. Then every planar graph with maximum degree at most Δ is equitably k-colourable.*

Note that, by the above-mentioned Hajnal and Szemerédi theorem, the statement of Theorem 1.1 also holds for $d = \Delta$. We conjecture that it holds for every $d \leqslant \Delta$.

Chen, Lih and Wu [2] conjectured that, apart from $K_{1+\Delta}$ and $K_{\Delta,\Delta}$, every connected graph with maximum degree $\Delta \geqslant 3$ has an equitable Δ-colouring. This conjecture is proved for graphs in some classes, such as interval graphs, trees, and so on (for a survey see [5]). If $k \geqslant 14d + 1$ and the maximum degree of a d-degenerate graph G is at most k, then G satisfies the conditions of Theorem 1.1 for $\Delta = 2k - 1 - d$. Thus we get the following.

Corollary 1.3. *Let $d \geqslant 2$ and $k \geqslant 14d + 1$. Then every d-degenerate graph with maximum degree at most k is equitably k-colourable.*

In fact, we can prove the corollary under restrictions somewhat weaker than $k \geqslant 14d + 1$, along the lines of the proof of Theorem 1.1. But we do not present this here.

In the next section, we prove some auxiliary statements, and Section 3 is devoted to the proof of the theorem.

2. Preliminaries

Claim 2.1. *Let $G = (V, E)$ be a d-degenerate graph ($d \geqslant 2$), $V = \{v_1, \dots, v_n\}$ and $\deg_G(v_1) \geqslant \deg_G(v_2) \geqslant \cdots \geqslant \deg_G(v_n)$. Then $\deg_G(v_i) < d(1 + \frac{n}{i})$ for every $i = 1, \dots, n$.*

Proof. Consider an arbitrary i. Let $V_i = \{v_1, \dots, v_i\}$. Then, since $G(V_i)$ is d-degenerate,

$$|E| \geqslant \sum_{j=1}^{i} \deg_G v_j - |E(G(V_i))| > i \deg_G v_i - d \cdot i = i(\deg_G v_i - d).$$

Since G itself is d-degenerate, we conclude that $dn > i(\deg_G v_i - d)$. This yields the claim. □

Claim 2.2. *Let $\Delta \geqslant d \geqslant 2$ and*

$$k > \frac{d + \Delta}{2}. \tag{2.1}$$

Let $G = (V, E)$ be a d-degenerate graph on $2k$ vertices with maximum degree at most Δ. Then G is equitably k-colourable.

Proof. Assume that G is not equitably k-colourable. This means that the complement $\overline{G}$ of G has no perfect matching. Then, by Tutte's criterion of existence of a perfect matching in a graph (taking into account that $2k$ is even), there exists some nonnegative integer p and $P \subset V(G)$ with $|P| = p$ such that $\bar{G} - P$ is the union of some $q \geqslant p + 2$ components G_i. Let $W_i = V(G_i)$ for $i = 1, \dots, p + 1$ and $W_{p+2} = \bigcup_{j=p+2}^{q} V(G_j)$. Then G contains the complete $(p + 2)$-partite graph with parts $W_1, \dots, W_{p+2}$. Let $W = \bigcup_{i=1}^{p+2} W_i$. Clearly, $|W| = 2k - p$. We may assume that $|W_i| = w_i$ and that $w_1 \leqslant \cdots \leqslant w_{p+2}$. Then $\Delta \geqslant |W| - w_1$ and $d \geqslant |W| - w_{p+2}$. Therefore,

$$d + \Delta \geqslant |W| + (|W| - w_1 - w_{p+2}) = 2k - p + \sum_{i=2}^{p+1} w_i \geqslant 2k,$$

a contradiction to (2.1). □

Claim 2.3. *Let $\Delta \geqslant 22d$, $d \geqslant 2$ and let k satisfy* (2.1). *Let $G = (V, E)$ be a d-degenerate graph on $3k$ vertices with maximum degree at most Δ. Then G is equitably k-colourable.*

Proof. Let $G_0 = G$. For $i = 1, 2, \ldots,$ consider the following procedure. If G_{i-1} contains three mutually non-adjacent vertices $v_{i,1}$, $v_{i,2}$, and $v_{i,3}$ of degree at least $6d + 1$, then let $M_i = \{v_{i,1}, v_{i,2}, v_{i,3}\}$, $G_i = G_{i-1} - M_i$, and go to step $i + 1$. Otherwise, stop.

Suppose that we stop after step q. Let S be the set of vertices in G_q of degree (in G_q) at least $6d + 1$ and $s = |S|$. Since G is d-degenerate,

$$|E(G_q)| \geqslant \sum_{v \in S} \deg_{G_q}(v) - |E(G(S))| \geqslant (6d + 1)s - ds = (5d + 1)s$$

and $|E(G)| < 3kd$. Observe that $|E(G_i)| \leqslant |E(G_{i-1})| - 3(6d + 1)$ for every $i \leqslant q$. Hence

$$3(6d + 1)q + (5d + 1)s < 3kd \quad \text{and hence} \quad q + \frac{5}{18}s < \frac{k}{6}. \tag{2.2}$$

Since the independence number of $G(S)$ is at most two, and G is d-degenerate,

$$s \leqslant 2(d + 1). \tag{2.3}$$

We will greedily construct s disjoint independent sets $M_{q+1}, \ldots, M_{q+s}$ of size 3 in G_q so that each of them contains exactly one vertex in S. At step j, $1 \leqslant j \leqslant s$, take a vertex $w_j \in S$ in G_{q+j-1} and choose in $V(G_{q+j-1})$ two vertices so that, together with w_j, they form an independent set, say, M_{q+j}; then let $G_{q+j} = G_{q+j-1} - M_{q+j}$.

To see that we will be able to make all s steps, let us count how many non-neighbours w_j has in G_{q+j-1}. This number is at least $3k - 3(q + j - 1) - \Delta - 1$. If this number is at least $d + 2$, then we can choose two mutually non-adjacent non-neighbours of w_j. Thus, the bad situation might occur only when $3k - 3(q + s - 1) - \Delta - 1 \leqslant d + 1$. Since $k \geqslant (\Delta + d + 1)/2$, this implies that $k \leqslant 3q + 3s - 2$. By (2.2), we then get that $k/2 < (3 - \frac{5}{6})s - 2$. Now, by (2.3),

$$\frac{\Delta + d + 1}{4} < \frac{13(d + 1)}{3} - 2 = \frac{13d}{3} + \frac{7}{3},$$

and hence

$$\Delta < \frac{49d}{3} + 9.$$

But this contradicts the conditions $22d \leqslant \Delta$ and $d \geqslant 2$.

Since the maximum degree of G_{q+s} is at most $6d$, by the Hajnal–Szemerédi theorem, it has an equitable l-colouring for every $l \geqslant 6d + 1$. Thus, if $k - q - s \geqslant 6d + 1$, then we are done. Assume that $k - q - s \leqslant 6d$. Then by (2.2) and (2.3), respectively,

$$\frac{5k}{6} - \frac{13s}{18} < 6d \quad \text{and} \quad \frac{5k}{6} < \frac{13(d + 1)}{9} + 6d.$$

This again contradicts the conditions $22d \leqslant \Delta$ and $d \geqslant 2$. $\square$

3. Proof of Theorem 1.1

For a given d, let G_0 be a counterexample to the theorem with minimal number of vertices. Let $|V(G_0)| = n_0$ and $n_0 = tk - r$, where $0 \leqslant r < k$.

Claim 3.1. $r < d$.

Proof. Assume $r \geqslant d$. Since G_0 is d-degenerate, there exists $v \in V(G_0)$ with $\deg_{G_0}(v) \leqslant d$. By the minimality of G_0, there exists an equitable k-colouring f of $G' = G_0 - v$. In this colouring, exactly $r + 1$ colour classes are of size $t - 1$. Since $r \geqslant d$, at least one of these classes does not contain neighbours of v. We can add v to any such class. □

Let $G = (V, E)$ be obtained from G_0 by adding a copy of K_r disjoint from G_0. Then $n = |V(G)| = tk$ and by Claim 3.1, G is d-degenerate. If $t = 1$ then the statement is trivial. If $t = 2$ or $t = 3$, it follows from Claims 2.2 and 2.3. So below we assume $t \geqslant 4$.

Let

$$s = s(t) = \begin{cases} 1, & \text{if } 4 \leqslant t \leqslant 9; \\ \lceil \frac{t}{8} \rceil, & \text{if } t \geqslant 10, \end{cases}$$

and let $\lambda = \lambda(t) = 1 + \frac{t}{s(t)}$. Observe that $\lambda(4) = 5$, $\lambda(9) = 10$, and $6 \leqslant \lambda(t) \leqslant 9$ for every other $t > 3$. Order the vertices of G so that $\deg_G(v_1) \geqslant \deg_G(v_2) \geqslant \cdots \geqslant \deg_G(v_n)$. Let $V' = \{v_1, \ldots, v_\mu\}$, where $\deg_G(v_\mu) \geqslant \lambda d$ and $\deg_G(v_{\mu+1}) < \lambda d$.

By Claim 2.1, $\mu \leqslant \frac{n}{\lambda - 1}$. Since $|V'| = \mu \leqslant n/4 < n_0$, the minimality of G_0 implies that there exists an equitable k-colouring f' of $G(V')$. (If $t \leqslant 9$ then all vertices in V' get different colours.)

Let G_i be the subgraph of G induced by vertices $v_1, \ldots, v_i$. We will now complete f' to obtain a colouring of G by colouring consecutively vertices v_i for $i = 1 + \mu, 2 + \mu, \ldots, n$, in such a way that:

(i) after step i, vertices of G_i will be coloured;
(ii) at every step, every colour class is of size at most t;
(iii) no vertex in V' will be recoloured at any step.

Case 1: $1 + \mu \leqslant i < 3n/4$.
In the current colouring f_{i-1}, there are at least $k - \deg_G(v_i)$ colour classes not containing neighbours of v_i. If the size of at least one of them is less than t, we can move v_i into that class. Otherwise, $i > t(k - \deg_G(v_i))$. Since $\deg_G(v_{\mu+1}) < \lambda d \leqslant 10d$, we have $i > t(k - 10d) \geqslant 2n/15$.

By Claim 2.1, $\deg_G(v_i) < d(1 + \frac{n}{i})$. It follows that

$$n - i < td\frac{n + i}{i} = \frac{nd(n + i)}{ik}.$$

Thus,

$$\frac{\Delta}{d} < \frac{2k}{d} - 1 \leqslant \frac{2n(n + i)}{i(n - i)} - 1. \tag{3.1}$$

Denoting $\alpha = \frac{i}{n}$, we have $\frac{2}{15} \leqslant \alpha < \frac{3}{4}$ and (3.1) yields

$$\frac{\Delta}{d} < g(\alpha) = 2\frac{1+\alpha}{\alpha(1-\alpha)} - 1.$$

We will show that $g(\alpha) < 27$ when $\frac{2}{15} \leqslant \alpha < \frac{3}{4}$.

Since

$$g'(\alpha) = 2\frac{(1+\alpha)^2 - 2}{\alpha^2(1-\alpha)^2},$$

$g(\alpha)$ decreases when $\alpha < \sqrt{2} - 1$ and increases when $\alpha > \sqrt{2} - 1$. Thus, it is enough to check the inequality $g(\alpha) < 27$ only for $\alpha = \frac{2}{15}$ and $\alpha = \frac{3}{4}$. Clearly,

$$g\left(\frac{2}{15}\right) = 2\frac{17/15}{(2/15)(13/15)} - 1 = \frac{255}{13} - 1 < 19$$

and

$$g\left(\frac{3}{4}\right) = 2\frac{7/4}{(3/4)(1/4)} - 1 = \frac{56}{3} - 1 < 18.$$

This proves the case.

Case 2: $3n/4 \leqslant i \leqslant n$.
In this case, by Claim 2.1,

$$\deg_G(v_i) < d\left(1 + \frac{n}{3n/4}\right) = 7d/3. \tag{3.2}$$

Let $M_1, \ldots, M_k$ be the current colour classes. Let Y_0 denote the set of colour classes of cardinality less than t. If some $M_j \in Y_0$ contains no neighbours of v_i, then we colour v_i with M_j and go to the next step. Otherwise, let Y_0-*candidate* be a vertex $w \in V - V'$ such that there exists a colour class $M(w) \in Y_0$, with $w \notin M(w)$ and $N_G(w) \cap M(w) = \emptyset$. Let Y_1 be the set of colour classes containing a Y_0-candidate. If a member M_j of Y_1 does not contain a neighbour of v_i, then we colour v_i with M_j and recolour some Y_0-candidate $w \in M_j$ with $M(w)$. For $h \geqslant 1$, let a Y_h-*candidate* be a vertex $w \in V - V' - \cup_{M \in Y_0 \cup \cdots \cup Y_h} M$ such that there exists $M(w) \in Y_h$ with $N_G(w) \cap M(w) = \emptyset$. Let Y_{h+1} be the set of colour classes containing a Y_h-candidate. If a member M_j of Y_{h+1} does not contain a neighbour of v_i, then we colour v_i with M_j, and, as above, recolour a sequence of candidates. Finally, let $Y = \bigcup_{j=0}^{\infty} Y_j$ and $y = |Y|$. Then, by the above, Y possesses the following properties:

(a) every member of Y contains a neighbour of v_i and thus $y \leqslant \deg_G(v_i)$,
(b) every vertex $u \in V - V' - \cup_{M \in Y} M$ has a neighbour in every $M \in Y$ (otherwise the colour class of u would be in Y).

Let $V'' = V' - \cup_{M \in Y} M$ and $V^+ = V - V' - \cup_{M \in Y} M$. By (b), at least $y|V^+|$ edges connect V^+ with $\cup_{M \in Y} M$. By the choice of V', $\sum_{v \in V''} \deg(v) \geqslant \lambda d|V''|$. Since G is d-degenerate, we conclude that at least $\lambda d|V''| - d|V''| = (\lambda - 1)d|V''|$ edges are incident with V''. Hence, the total number of edges in G is at least $y|V^+| + (\lambda - 1)d|V''|$. Recall that $y \leqslant \deg(v_i) < (\lambda - 1)d$. Hence G has at least $y(|V^+| + |V''|) = ty(k - y)$ but less than dkt edges. It follows that $y(k - y) < dk$, that is,

$$\varphi(y) = y^2 - ky + kd > 0. \tag{3.3}$$

Observe that

$$\varphi(4d) = 16d^2 - 4kd + kdd(16d - 3k) < 0$$

$$\text{and} \qquad \varphi(1.1d) = 1.21d^2 - 1.1kd + kd = d(1.21d - 0.1k) < 0.$$

In view of (a) and (3.2), this implies that $y < 1.1d$.

By construction, any M_i contains at most s vertices in V'. Let $M_1 \in Y_0$. Since at most s vertices in M_1 might be in V', the number of neighbours of M_1 is at most $s\Delta + (t - 1 - s)\lambda d$.

Subcase 2.1: $4 \leqslant t \leqslant 9$.
Here $s = 1$ and $\lambda = 1 + t$. By (b), the number of neighbours of M_1 is at least $(k - y)(t - 1) > (t - 1)(k - 1.1d)$. Hence,

$$(t - 1)(0.5(\Delta + d + 1) - 1.1d) < \Delta + (t - 2)(t + 1)d$$

and thus

$$(0.5(t - 1) - 1)\Delta < ((t + 1)(t - 2) + 0.6(t - 1))d.$$

Since $\Delta \geqslant 27d$, we should have

$$27 < 2\frac{t^2 - 0.4t - 2.6}{t - 3} = 2t + 5.2 + \frac{10.4}{t - 3}.$$

But $2t + 5.2 + \frac{10.4}{t-3} < 27$ for $4 \leqslant t \leqslant 9$.

Subcase 2.2: $t \geqslant 10$.
In this case, $\lambda = 1 + \frac{t}{s} = \frac{s+t}{s}$ and hence $6 \leqslant \lambda \leqslant 9$. By (b), the number of neighbours of M_1 is at least

$$(t - s)(k - y) \geqslant (t - s)\left(\frac{\Delta + d + 1}{2} - 1.1d\right).$$

Therefore,

$$s\Delta + (t - s)\lambda d > (t - s)\left(0.5\Delta - \frac{3d}{5}\right).$$

Dividing both parts by s, expressing $\frac{t-s}{s}$ as $\lambda - 2$, and rearranging, we get

$$\left(\frac{\lambda - 2}{2} - 1\right)\Delta < \left((\lambda - 2)\lambda + \frac{3}{5}(\lambda - 2)\right)d.$$

Since $\Delta \geqslant 27d$, we should have

$$27 < \frac{\lambda + \frac{3}{5}}{\frac{1}{2} - \frac{1}{\lambda - 2}} = 2\lambda + 5.2 + \frac{18.4}{\lambda - 4}. \tag{3.4}$$

We want to prove that (3.4) is false for $6 \leqslant \lambda \leqslant 9$. It is enough to check this only for $\lambda = 6$ and $\lambda = 9$, since the second derivative of

$$\psi(\lambda) = 2\lambda + 5.2 + \frac{18.4}{\lambda - 4}$$

is positive for $\lambda > 4$. And indeed

$$\psi(6) = 12 + 5.2 + \frac{18.4}{6-4} < 27 \quad \text{and} \quad \psi(9) = 18 + 5.2 + \frac{18.4}{9-4} < 27.$$

Thus (3.4) is false for all possible λ. This contradiction proves the theorem. □

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