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On Ramsey Numbers of Sparse Graphs

ALEXANDR KOSTOCHKA[1†] and BENNY SUDAKOV[2‡]

[1]University of Illinois, Urbana, IL 61801, USA
and
Institute of Mathematics, Novosibirsk, 630090, Russia
(e-mail: kostochk@math.uiuc.edu)

[2]Department of Mathematics, Princeton University, Princeton, NJ 08540, USA
and
Institute for Advanced Study, Princeton, NJ 08540, USA
(e-mail: bsudakov@math.princeton.edu)

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The *Ramsey number*, $r(G)$, of a graph G is the minimum integer N such that, in every 2-colouring of the edges of the complete graph K_N on N vertices, there is a monochromatic copy of G. In 1975, Burr and Erdős posed a problem on Ramsey numbers of *d-degenerate* graphs, *i.e.*, graphs in which every subgraph has a vertex of degree at most d. They conjectured that for every d there exists a constant $c(d)$ such that

$$r(G) \leqslant c(d)n$$

for any d-degenerate graph G of order n.

In this paper we prove that $r(G) \leqslant n^{1+o(1)}$ for each such G. In fact, we show that, for every $\epsilon > 0$, sufficiently large n, and any graph H of order $n^{1+\epsilon}$, either H or its complement contains a (d, n)-*common* graph, that is, a graph in which every set of d vertices has at least n common neighbours. It is easy to see that any (d, n)-common graph contains every d-degenerate graph G of order n. We further show that, for every constant C, there is an n and a graph H of order Cn such that neither H nor its complement contains a $(2, n)$-common graph.

1. Introduction

The Ramsey number of a graph G, denoted by $r(G)$, is the minimum integer N such that, in every 2-colouring of the edges of the complete graph K_N on N vertices, there is a monochromatic copy of G. The existence of $r(G)$ follows from a classical theorem of Ramsey and we refer to $r(G)$ as the *Ramsey number* of G. We say that a family of graphs

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$\mathcal{G}$ is a *Ramsey linear family* if there is a constant $c = c(\mathcal{G}) > 0$ such that $r(G) \leqslant cn$ for every $G \in \mathcal{G}$ of order n.

For dense graphs G, $r(G)$ is known to be exponential in the order of G. For example, in the extreme case when G is the complete graph of order n, we have $2^{n/2} \leqslant r(G) \leqslant 2^{2n}$. Therefore, to be Ramsey linear a family should contain relatively sparse graphs.

One obvious way to force a graph to be sparse is to bound its maximal degree. Another possibility which is less restrictive is to consider graphs in which every subgraph has a small average degree. A graph is *d-degenerate* if every one of its subgraphs contains a vertex of degree at most d. By definition, low degeneracy is equivalent to low average degrees of all subgraphs. Burr and Erdős [4] posed the problem of estimating the Ramsey numbers of sparse graphs. They put forward the following two conjectures.

Conjecture 1.1. *The family $\mathcal{B}_\Delta$ of graphs with maximum degree at most Δ is Ramsey linear.*

Conjecture 1.2. *The family $\mathcal{D}_d$ of d-degenerate graphs is Ramsey linear.*

The first conjecture was proved by Chvátal, Rödl, Szemerédi and Trotter [6]. They used the Regularity Lemma, and the constant $c(\mathcal{B}_\Delta)$ in their proof is very large. Better estimates for $c(\mathcal{B}_\Delta)$ were obtained in [8], [9], [10], and [13]. In addition, in the past two decades some other subfamilies of the family $\mathcal{D}_d$ were shown to be Ramsey linear. Alon [1] proved that the family $\mathcal{S}$ of graphs obtained by subdividing every edge of some other graph is Ramsey linear. Chen and Schelp [5] showed that for every k, the family $\mathcal{A}_k$ of the so-called *k-arrangeable graphs* is also Ramsey linear and that every planar graph is 10-arrangeable. Rödl and Thomas [14] used Chen and Schelp's result to deduce that for every k, the family of graphs with no subdivision of K_k is Ramsey linear. Conjecture 1.2 is still wide open. Recently Kostochka and Rödl [13] proved that the Ramsey number of any d-degenerate graph with n vertices and maximum degree Δ is bounded by $C(d)n\Delta$. If Δ is not restricted, this gives an $O(n^2)$ bound for every d-degenerate graph with n vertices and this is the first polynomial upper bound on the Ramsey numbers of graphs in $\mathcal{D}_d$.

For a pair of positive integers $n > d$, we say that a graph H is (d,n)-*common* if, for every d vertices $v_1, \ldots, v_d \in V(H)$, there are at least n vertices of H adjacent to all $v_i, 1 \leqslant i \leqslant d$. Let $F_d(n)$ denote the minimum positive integer N_0 such that, for every $N \geqslant N_0$ and every graph H on N vertices, either H or its complement $\overline{H}$ contains a (d,n)-common subgraph. It is easy to see from this definition (see Lemma 2.1, below) that every (d,n)-common graph contains every d-degenerate graph on n vertices. In view of this observation, the following question was considered in [13] (in slightly different terms).

Question 1.3. *Is it true that, for every positive integer d, there exists a constant $C = C(d)$ such that $F_d(n) \leqslant Cn$?*

By the above discussion, answering this question in the affirmative would imply Conjecture 1.2. In [13], the following polynomial bound on $F_d(n)$ was proved. *For every fixed d there exists a constant $C_1 = C_1(d)$ such that $F_d(n) \leqslant C_1 n^d$.*

In this paper we improve estimates on $F_d(n)$. Our first theorem gives an upper bound on $F_d(n)$ which is not far from linear.

Theorem 1.4. *For every $\epsilon > 0$ there exists $n_0 = n_0(\epsilon)$ such that, for every $n > n_0$ and every positive integer $d < 0.1\sqrt{\ln \ln n}$,*

$$F_d(n) < n^{1+\epsilon}.$$

As an immediate corollary we obtain the following new upper bound on the Ramsey number of d-degenerate graphs, which comes close to the one conjectured by Burr and Erdős.

Corollary 1.5. *For every $\epsilon > 0$ there exists $n_0 = n_0(\epsilon)$ such that, for every $n > n_0$ and every positive integer $d < 0.1\sqrt{\ln \ln n}$, the Ramsey number of every d-degenerate graph of order n is at most $n^{1+\epsilon}$.*

On the other hand, we will present a construction answering Question 1.3 in the negative: even for $d = 2$ the function $F_d(n)$ is superlinear. This is somewhat surprising and unfortunate, since this implies that another, more subtle, approach is needed to attack Conjecture 1.2.

Theorem 1.6. *There exists a real $c > 0$ such that, for every integer n, there exists a graph H of order $c\frac{n \ln^{1/4} n}{\ln \ln n}$ with the property that neither H nor its complement contains a $(2, n)$-common subgraph, that is,*

$$F_2(n) \geqslant c\frac{n \ln^{1/4} n}{\ln \ln n}.$$

The rest of this paper is organized as follows. In the next section we illustrate our main ideas by obtaining bounds on Ramsey numbers of bipartite d-degenerate graphs and deduce Corollary 1.5 from Theorem 1.4. In Section 3 we prove Theorem 1.6, thus answering Question 1.3 in the negative. Our construction uses the isoperimetric properties of the Hamming space. Next, in Section 4 we treat (d, n)-common subgraphs of large graphs and present the proof of Theorem 1.4. The last section contains some concluding remarks.

We close this section by introducing some notation. Given a graph $G = (V, E)$, the *neighbourhood* $N_G(v)$ of a vertex $v \in V$ is the set of all vertices of G adjacent to it and $d_G(v) = |N_G(v)|$ is the *degree* of v. For a subset $W \subset V$, we let $N_G(W) = \bigcap_{v \in W} N_G(v)$ denote the set of vertices of G adjacent to all the vertices in W. We will frequently write simply $N(v)$ and $N(W)$, when it is clear from the context what graph is under consideration. Similarly, given a set $Y \subseteq V$, we let $N_Y(v)$ denote the set of all vertices in Y adjacent to v and let $N_Y(W)$ denote the set of vertices of Y adjacent to all the vertices in W. We let ln denote the natural logarithm. Throughout the paper we assume, whenever necessary, that n is sufficiently large. Finally, for the sake of clarity of presentation, we will omit some floor and ceiling signs in places where it does not affect the argument.

2. Main ideas: bipartite case

In this section we illustrate our main ideas by giving a nearly linear upper bound on Ramsey numbers of bipartite sparse graphs. We make no attempt to optimize our constants here and in the rest of the paper.

We say that a bipartite graph $H = (U_1, U_2; E)$ is (d, n)-*quasi-common* if, for each $i = 1, 2$, every set of d vertices $v_1, \dots, v_d$ in U_i has at least n common neighbours in U_{3-i}. The following folklore lemma shows why we are interested in (d, n)-common graphs and in Question 1.3. In particular, the second statement of the lemma shows that Theorem 1.4 implies Corollary 1.5.

Lemma 2.1. *Let n and d be two positive integers. Then any (d, n)-quasi-common graph contains every d-degenerate bipartite graph of order n. Furthermore, any (d, n)-common graph contains every d-degenerate graph of order n.*

Proof. Let $H = (U_1, U_2; E)$ be a (d, n)-quasi-common graph and let $G = (V_1, V_2; E')$ be a d-degenerate bipartite graph of order n. By the definition of d-degenerate graphs, there exists a labelling $v_1, \dots, v_n$ of vertices of G such that, for every i, the number of neighbours v_j of v_i with $j < i$ is at most d. Using this labelling we can construct embedding $f : G \to H$ greedily so that the vertices in V_l will be embedded into set U_l, $l = 1, 2$.

Without loss of generality we assume that $v_1 \in V_1$, and let $f(v_1)$ be an arbitrary vertex in U_1. Suppose that we have already embedded vertices $v_1, \dots, v_{i-1}$, and suppose that $v_i \in V_l$. Let $D = \{f(v_j) \mid (v_j, v_i) \in E(G), j < i\}$. Then D is a subset of U_{3-l} of size at most d, and hence the set $N_{U_l}(D)$ of common neighbours of D in U_l has size at least n, which is the order of G. Therefore it is always possible to choose $f(v_i)$ to be a vertex in $N_{U_l}(D)$ different from $f(v_1), \dots, f(v_{i-1})$. This process clearly embeds G into H.

The proof of the second statement of the lemma is very similar. It is even shorter, since we do not need to control the parts, and we omit it here. □

The main theorem of this section is the following Turán-type result. Its proof is based on the approach introduced in [12], [7] and [15]. The crucial new idea here and also in the proof of Theorem 1.4 is to find the way to apply these arguments in both directions.

Theorem 2.2. *Let $0 < c \leqslant 1$ be a constant and let d, N and n be positive integers satisfying*

$$d \leqslant \frac{1}{64} \ln n \quad \textit{and} \quad N = n \left(\frac{2e}{c} \right)^{2d^{1/3} \ln^{2/3} n}. \tag{2.1}$$

Then every bipartite graph $G = (V_1, V_2; E)$ with $|V_1| = |V_2| = N$ and $|E| = cN^2$ contains a (d, n)-quasi-common graph $H = (U_1, U_2; E')$.

Proof. Let $x_1, \dots, x_s$ be a sequence of $s = d^{1/3} \ln^{2/3} n$, not necessarily distinct vertices of V_2, which we choose uniformly and independently at random, and denote $S = \{x_1, \dots, x_s\}$. Let U_1' denote the set $N_{V_1}(S)$ of common neighbours of vertices in S. Note that the size of U_1' is a random variable and that $S \subseteq N(v)$ for every $v \in U_1'$. Then, using Jensen's inequality and (2.1), we can estimate the expected size of U_1' as follows:

$$\begin{aligned}\mathbf{E}\big(|U_1'|\big) &= \sum_{v \in V_1} \mathbf{Pr}\big(v \in U_1'\big) = \sum_{v \in V_1} \left(\frac{|N(v)|}{N} \right)^s = \frac{\sum_{v \in V} \big(d(v)\big)^s}{N^s} \geqslant \frac{N \left(\frac{\sum_{v \in V} d(v)}{N} \right)^s}{N^s} \\ &= \frac{N \big(|E(G)|/N\big)^s}{N^s} \geqslant c^s N = n \, (4e^2/c)^{d^{1/3} \ln^{2/3} n}.\end{aligned}$$

On the other hand, by definition, the probability that a given set of vertices $W \subset V_1$ is contained in U_1' equals $\big(|N(W)|/N\big)^s$. Let Z denote the number of subsets W of U_1' of size $d^{2/3}\ln^{1/3} n$ with $|N(W)| < n$. Then by (2.1) the expected value of Z is at most

$$\begin{aligned}
\mathbf{E}(Z) &= \sum_{|W|=d^{2/3}\ln^{1/3} n,\, |N(W)|<n} \mathbf{Pr}\big(W \subset U_1'\big) \leqslant \binom{N}{d^{2/3}\ln^{1/3} n}\left(\frac{n}{N}\right)^s \\
&\leqslant N^{d^{2/3}\ln^{1/3} n}\left(\frac{n}{N}\right)^{d^{1/3}\ln^{2/3} n} = n^{d^{2/3}\ln^{1/3} n}\left(\frac{n}{N}\right)^{-d^{2/3}\ln^{1/3} n + d^{1/3}\ln^{2/3} n} \\
&\leqslant e^{d^{2/3}\ln^{4/3} n}\left(\frac{n}{N}\right)^{(1/2)d^{1/3}\ln^{2/3} n} = \left(\frac{c}{2}\right)^{d^{2/3}\ln^{4/3} n} < 1.
\end{aligned}$$

Here we used that, by (2.1), $d^{1/3} \leqslant (1/4)\ln^{1/3} n < (1/2)\ln^{1/3} n$ and that $c/2 \leqslant 1/2 < 1$. Therefore, by linearity of expectation, there exists a particular choice of $x_1, \ldots, x_s$ for which $|U_1'| - Z \geqslant n(4e^2/c)^{d^{1/3}\ln^{2/3} n} - 1$. Fix these $x_1, \ldots, x_s$ and delete a vertex from every subset W of U_1' of size $d^{2/3}\ln^{1/3} n$ with $|N(W)| < n$. This produces a set $U_1 \subseteq V_1$ of size at least $n\,(4e^2/c)^{d^{1/3}\ln^{2/3} n} - 1 \geqslant n\,(2e^2/c)^{d^{1/3}\ln^{2/3} n}$ such that every subset of $d^{2/3}\ln^{1/3} n$ vertices has at least n common neighbours in V_2.

Next, let $q = d^{2/3}\ln^{1/3} n - d$. By (2.1), $q \geqslant (3/4)d^{2/3}\ln^{1/3} n$. Take a sequence $y_1, \ldots, y_q$ of not necessarily distinct vertices of U_1, which we choose uniformly and independently at random, and denote $Q = \{y_1, \ldots, y_q\}$. Let U_2 denote the set $N_{V_2}(Q)$. Note that a set of vertices $W' \subset V_2$ is contained in U_2 if and only if $Q \subseteq N_{U_1}(W')$, and the probability that this happens equals $(|N_{U_1}(W')|/|U_1|)^q$. Let Z' denote the number of subsets W' of U_2 of size d with $|N_{U_1}(W')| < n$. Then, using (2.1) and the fact that $d^{1/3} \leqslant (1/4)\ln^{1/3} n$, we obtain

$$\begin{aligned}
\mathbf{E}(Z') &\leqslant \binom{N}{d}\left(\frac{n}{|U_1|}\right)^q \leqslant N^d\left(\left(\frac{2e^2}{c}\right)^{d^{1/3}\ln^{2/3} n}\right)^{-q} \\
&\leqslant n^d\left(\frac{2e}{c}\right)^{2d^{4/3}\ln^{2/3} n}\left(\frac{2e^2}{c}\right)^{-(3/4)d\ln n} \leqslant e^{d\;\ln n}\left(\frac{2e}{c}\right)^{(1/2)d\ln n}\left(\frac{2e^2}{c}\right)^{-(3/4)d\ln n} \\
&= \left(\frac{c}{2}\right)^{(1/4)d\ln n} < 1.
\end{aligned}$$

Since Z' is an integer, by the definition of expectation, there exists a particular choice of $y_1, \ldots, y_q$ for which $Z' = 0$. Fix such $y_1, \ldots, y_q$ and the corresponding set U_2. By construction, every set of d vertices in U_2 has at least n common neighbours in U_1. Observe that, *vice versa*, every set of d vertices in U_1 has at least n common neighbours in U_2. Indeed, let D be a subset of U_1 of size d. Then the set $Y = D \cup Q$ is a subset of U_1 of size at most $d + q = d + (d^{2/3}\ln^{1/3} n - d) = d^{2/3}\ln^{1/3} n$. By the choice of U_1 there are at least n vertices in V_2 adjacent to all vertices in Y. Hence, to complete the proof of the theorem one should only notice that all these vertices belong to U_2. This indeed, follows easily from the facts that U_2 contains all common neighbours of $Q = \{y_1, \ldots, y_q\}$ and $Q \subset Y$. $\square$

This theorem immediately yields the following corollary, which provides a nearly linear upper bound on Ramsey numbers of bipartite d-degenerate graphs.

Corollary 2.3. *Let d be a fixed integer and let G be a bipartite d-degenerate graph of order n. Then*

$$r(G) \leqslant ne^{O(\ln^{2/3} n)} = n^{1+o(1)}.$$

Proof. Let $N = n(4e)^{2d^{1/3}\ln^{2/3} n} = ne^{O(\ln^{2/3} n)}$ and suppose that edges of the complete graph K_{2N} are 2-coloured. Consider any partition of vertices of K_{2N} into two equal parts of size N each. Then clearly at least $N^2/2$ edges between these parts have the same colour. These edges form a monochromatic bipartite graph which satisfies the conditions of Theorem 2.2 with $c = 1/2$. Therefore this graph contains a (d,n)-quasi-common subgraph H, and we can now finish the proof by applying Lemma 2.1. □

3. A lower bound on $F_2(n)$

In this section we show that the results of previous section, and more generally of Theorem 1.4, are in some sense tight. More precisely, we present a construction that proves Theorem 1.6 and gives a negative answer to Question 1.3, even for $d = 2$.

Our construction is based on the isoperimetric properties of the binary cube. Let $\{0,1\}^m$ be the set of all binary vectors of length m. For any two vectors $x, y \in \{0,1\}^m$, let $\rho(x,y)$ denote their *Hamming distance*, that is, the number of coordinates in which they differ. We use the well-known fact that any sufficiently large subset of $\{0,1\}^m$ contains two almost antipodal vectors. More precisely, we apply the following classical result of Kleitman [11].

Lemma 3.1. *Let $t < m/2$, $A \subseteq \{0,1\}^m$, and*

$$|A| > \sum_{i=0}^{t} \binom{m}{i}.$$

Then there is a pair of vectors a_1 and a_2 in A such that $\rho(a_1,a_2) \geqslant 2t+1$.

We will also use the following standard Chernoff estimates (see, *e.g.*, [2, Appendix A, Theorem A.4]) for binomial distributions.

Lemma 3.2. *Let λ and μ be positive integers, $\lambda < \mu/2$. Then*

$$\sum_{0 \leqslant a \leqslant \mu/2-\lambda} \binom{\mu}{a} \leqslant 2^{\mu}\, e^{-2\lambda^2/\mu}.$$

Having finished all the necessary preparations, we are now ready to complete the proof of Theorem 1.6. Our approach here was influenced by the well-known construction of Bollobás and Erdős [3] of dense K_4-free graphs without large independent sets.

Proof of Theorem 1.6. Let $m = \log_2 n + (\log_2\log_2 n)/4 - \log_2\log_2\log_2 n$ and let $V = \{0,1\}^m$ be the set of binary vectors of length m. Let H be the graph on the vertex set V in which two vertices $x, y \in V$ are adjacent if and only if their Hamming distance

$\rho(x,y) \leqslant m/2$. We claim that neither graph H nor its complement $\overline{H}$ contains a $(2,n)$-common subgraph. We assume that H or $\overline{H}$ contains a $(2,n)$-common subgraph G, and obtain a contradiction.

Let U denote the set of vertices of G. Since G is $(2,n)$-common, by definition, the order of U is at least $n = (1+o(1))2^m \frac{\log_2 m}{m^{1/4}} > 2^m/\sqrt{m}$. Therefore, by Lemma 3.2 (with $\mu = m$ and $\lambda = 0.5\sqrt{m \log_2 m}$) and Lemma 3.1 (with $t = 0.5m - 0.5\sqrt{m\log_2 m}$)), U contains a pair of vertices u_1, u_2 such that $\rho(u_1,u_2) \geqslant m - \sqrt{m\log_2 m}$. We obtain a contradiction by proving that the total number of vertices in H and also in $\overline{H}$ adjacent to both u_1 and u_2 is less than n. Without loss of generality we can assume that u_1 is the all-zero vector, and u_2 has 0 in its first $k \leqslant \sqrt{m\log_2 m}$ coordinates and 1 in the remaining $m-k$ coordinates.

Case 1: G is contained in H.
Let x be a vertex of V which has precisely a ones in first k coordinates and b ones in the remaining $m-k$ coordinates. The number of such vertices in H is $\binom{k}{a}\binom{m-k}{b}$. If x is adjacent to both u_1 and u_2 then, by the definition of H, $\rho(x,u_1) = a+b \leqslant m/2$ and $\rho(x,u_2) = a + (m-k-b) \leqslant m/2$. This implies that $m/2 - k + a \leqslant b \leqslant m/2 - a$ and $a \leqslant k/2$.

First we consider the case when $k \leqslant m^{1/4}$. By the above discussion the total number of vertices of H adjacent to both u_1, u_2 is at most

$$\begin{aligned}|N_H(u_1,u_2)| &= \sum_{a\leqslant k/2}\ \sum_{m/2-k+a\leqslant b\leqslant m/2-a} \binom{k}{a}\binom{m-k}{b} \leqslant \sum_{a\leqslant k/2}\binom{k}{a}\binom{m-k}{\frac{m-k}{2}}(k-2a)\\ &\leqslant O\left(k\frac{2^{m-k}}{\sqrt{m-k}}\,2^k\right) = O\left(\frac{2^m}{m^{1/4}}\right) = o(n).\end{aligned}$$

In these inequalities we use the fact that the largest binomial coefficient is the central one, Stirling's formula, and the estimate $n = (1+o(1))2^m \frac{\log_2 m}{m^{1/4}}$.

Next suppose that $k \geqslant m^{1/4}$. In this case, first note that by Lemma 3.2

$$\sum_{|a-k/2|>\lambda}\binom{k}{a} \leqslant 2e^{-2\lambda^2/k}2^k.$$

By choosing $\lambda = \sqrt{k\ln k}$ and using the facts that $\sqrt[4]{m} \leqslant k = O(\sqrt{m\log_2 m})$, $n = (1+o(1))2^m \frac{\log_2 m}{m^{1/4}}$ together with Stirling's formula, we obtain that the total number of vertices of H adjacent to both u_1, u_2 is at most

$$\begin{aligned}|N_H(u_1,u_2)| &= \sum_{a\leqslant k/2}\ \sum_{m/2-k+a\leqslant b\leqslant m/2-a}\binom{k}{a}\binom{m-k}{b}\\ &\leqslant \sum_{|a-k/2|\geqslant\sqrt{k\ln k}}\ \sum_{b}\binom{k}{a}\binom{m-k}{b} + \sum_{|2a-k|\leqslant\sqrt{k\ln k}}\ \sum_{m/2-k+a\leqslant b\leqslant m/2-a}\binom{k}{a}\binom{m-k}{b}\\ &\leqslant 2\frac{2^k}{k^2}\sum_b\binom{m-k}{b} + \sum_{|a-k/2|\leqslant 2\sqrt{k\ln k}}\binom{k}{a}\binom{m-k}{\frac{m-k}{2}}(k-2a)\\ &\leqslant 2\frac{2^k2^{m-k}}{k^2} + O\left(\sqrt{k\ln k}\frac{2^{m-k}}{\sqrt{m-k}}\,2^k\right) \leqslant \frac{2^{m+1}}{\sqrt{m}} + O\left(\frac{2^m\ln^{3/4}m}{m^{1/4}}\right) = o(n).\end{aligned}$$

This completes the proof of Case 1.

Case 2: G is contained in $\overline{H}$.
Let v_i, $i = 1, 2$ be the vertex in V antipodal to u_i (*e.g.*, v_1 is the all-one vector). Then $\rho(v_1, v_2) = \rho(u_1, u_2)$ and a vertex x is adjacent to both u_1 and u_2 in $\overline{H}$ if and only if x is adjacent to both v_1 and v_2 in H. According to Case 1, there are only $o(n)$ vertices with this property. This proves the theorem. □

4. Embedding (d, n)-common graphs

In this section we prove the following statement, which implies Theorem 1.4.

Theorem 4.1. *Let d, n and N be positive integers such that $d < 0.1\sqrt{\ln\ln n}$,*

$$n \leqslant N \exp\bigl(-9^{4d}(\ln N)^{2d/(2d+1)}\bigr), \tag{4.1}$$

and let H be a graph of order N. Then either H or its complement $\overline{H}$ contains a (d, n)-common subgraph.

First we show how Theorem 1.4 follows from Theorem 4.1 (which will be proved in the next three subsections). Suppose some $0 < \epsilon < 1$ is given and $n > n_0(\epsilon)$. Let $d < 0.1\sqrt{\ln\ln n}$ and let H be a graph on $N = n^{1+\epsilon}$ vertices. Checking that our d, n and N satisfy (4.1) is equivalent to checking that

$$1 \leqslant n^{\epsilon} \exp\bigl(-9^{0.4\sqrt{\ln\ln n}}((1+\epsilon)\ln n)^{2d/(2d+1)}\bigr).$$

The last inequality would follow from

$$\epsilon \ln n > 9^{0.4\sqrt{\ln\ln n}} 2(\ln n)^{1-2/\sqrt{\ln\ln n}}. \tag{4.2}$$

Since

$$\frac{9^{0.4\sqrt{\ln\ln n}}}{(\ln n)^{2/\sqrt{\ln\ln n}}} = \left(\frac{9^{0.4}}{e^2}\right)^{\sqrt{\ln\ln n}} = o(1),$$

inequality (4.2) holds for sufficiently large n. Therefore, by Theorem 4.1, either H or $\overline{H}$ contains a (d, n)-common subgraph.

Our important tool will be the Tripartite Lemma proved in Section 4.1. It is an elaboration of similar lemmas proved in [7], [12] and [15]. The difference from previous applications is that we have managed to keep some useful properties on all steps of a procedure. After proving the Tripartite Lemma and a technical lemma, we conclude the proof of Theorem 4.1 in Section 4.3 by presenting a procedure that, for every graph H satisfying the conditions of the theorem either in H or in $\overline{H}$, finds $d+1$ disjoint vertex subsets $X_{j_1}, \ldots, X_{j_{d+1}}$ with the property that each d-tuple of vertices in $X' = \bigcup_{i=1}^{d+1} X_{j_i}$ has at least n common neighbours in X'. The above-mentioned technical lemma helps to control the sizes of current sets and their neighbourhoods during the procedure.

4.1. Tripartite lemma

Let G be a graph and let X and Y be two disjoint subsets of G. Then we let $e(X, Y)$ denote the number of edges of G incident with exactly one vertex from X and one from Y.

Lemma 4.2. *Let $G = (V, E)$ be a tripartite graph with parts X, Y and Z such that $|X| = m$, $|Y| + |Z| \leqslant m^2$, and suppose that*

$$e(X, Y) \geqslant \frac{m|Y|}{a}, \tag{4.3}$$

for some $a > 0$. Let s and r be positive integers and let α be a positive real number such that

$$s \ln \frac{m}{\alpha} \geqslant 2r \ln m. \tag{4.4}$$

Suppose also that $|Y| \geqslant 2a^s$. Then there exist $S \subset X$ and $T \subset (Y \cup Z) \cap N(S)$ such that

(a) $|S| \leqslant s$,
(b) $|T \cap Y| \geqslant 0.5|Y|a^{-s}$,
(c) $|N(R) \cap X| \geqslant \alpha$ *for every subset* $R \subset T$ *of size* r,
(d) $|((Y \cup Z) \cap N(S)) \setminus T| \leqslant 2a^s$.

Proof. Let $x_1, \ldots, x_s$ be a sequence of not necessarily distinct vertices of X which we choose uniformly and independently at random, and denote $S = \{x_1, \ldots, x_s\}$. The probability that a given vertex $y \in Y$ is in $N(S)$ is $(|N(y) \cap X|/m)^s$. Thus, using (4.3) and Jensen's inequality, we obtain that the expected value of $|N(S) \cap Y|$ is

$$\sum_{y \in Y} \left(\frac{|N(y) \cap X|}{m} \right)^s \geqslant \frac{|Y|}{m^s} \left(\frac{\sum_{y \in Y} |N(y) \cap X|}{|Y|} \right)^s = \frac{|Y|}{m^s} \left(\frac{e(X, Y)}{|Y|} \right)^s \geqslant \frac{|Y|}{a^s}.$$

Let $\mu(S)$ denote the number of r-tuples of vertices in $(Y \cup Z) \cap N(S)$ having at most α common neighbours in X. If some r-tuple $R \subset Y \cup Z$ has at most α common neighbours in X, then the probability that $R \subset N(S)$ is at most $(\alpha/m)^s$. Therefore, by (4.4), the expectation of $\mu(S)$ is at most

$$\mathbf{E}\big(\mu(S)\big) \leqslant \binom{|Y| + |Z|}{r} \left(\frac{\alpha}{m} \right)^s \leqslant (m^2)^r (\alpha/m)^s = \exp\left(2r \ln m - s \left(\ln \frac{m}{\alpha} \right) \right) \leqslant 1.$$

Hence, by linearity, the expectation of $|N(S) \cap Y| - 0.5|Y|a^{-s}\mu(S)$ is at least

$$\frac{|Y|}{a^s} - \frac{|Y|}{2a^s} \mathbf{E}\big(\mu(S)\big) \geqslant \frac{|Y|}{a^s} - \frac{|Y|}{2a^s} = 0.5 \frac{|Y|}{a^s}.$$

Thus there exists a particular choice of S such that $|S| \leqslant s$, and

$$|N(S) \cap Y| - 0.5|Y|a^{-s}\mu(S) \geqslant 0.5|Y|a^{-s}. \tag{4.5}$$

Fix such a set S and delete a vertex from every r-tuple $R \subset (Y \cup Z) \cap N(S)$ having fewer than α common neighbours in X. This produces a set T that together with S satisfies statements (a) and (c) of the lemma. Next we use (4.5) together with the fact that $|Y| \geqslant 2a^s$ to conclude that

$$|T \cap Y| \geqslant |N(S) \cap Y| - \mu(S) \geqslant |N(S) \cap Y| - 0.5|Y|a^{-s}\mu(S) \geqslant 0.5|Y|a^{-s}.$$

This implies that T also satisfies statement (b).

Finally, note that we have deleted at most $\mu(S)$ vertices, and (4.5) yields

$$\mu(S) < \frac{|N(S) \cap Y|}{0.5|Y|a^{-s}} \leqslant \frac{|Y|}{0.5|Y|a^{-s}} = 2a^s.$$

This proves (d) and completes the proof of the lemma. □

Remark. To prove Theorem 4.1 we will use the assertion of this lemma only for $a=2$. Nevertheless we include here the proof of a slightly more general result, since it can be applied to obtain a multicoloured version of Theorem 4.1.

4.2. A technical lemma

To dispose of boring calculations in the proof of Theorem 4.1, we deal with them in the present subsection, which can be omitted at first reading. The relations we prove are routine, but we fix them to be on the safe side. Let d, n and N be the positive integers which satisfy the conditions of Theorem 4.1, and let $t_0 = N$. For $i = 1, 2, \ldots, 2d$, let us define inductively integers s_i, r_i, t_i, m_i and reals α_i as follows.

Let

$$m_1 = \lfloor t_0/3 \rfloor, \qquad s_1 = \left\lfloor 9^{2d} d (\ln N)^{2d/(2d+1)} \right\rfloor + 1, \qquad r_1 = \left\lfloor 0.5 s_1 (\ln N)^{-1/(2d+1)} \right\rfloor,$$
$$t_1 = \left\lceil \frac{N}{3} 2^{-s_1} \right\rceil \qquad \text{and} \qquad \alpha_1 = m_1 \exp\left(-(\ln N)^{2d/(2d+1)}\right).$$

For $i = 2, 3, \ldots, 2d$, let

$$s_i = \lfloor 0.5 r_{i-1} \rfloor, \qquad r_i = \left\lfloor 0.5 s_i (\ln N)^{-1/(2d+1)} \right\rfloor, \qquad t_i = \lceil N 2^{-s_1 - s_2 - \cdots - s_i} 3^{-i} \rceil,$$
$$m_i = \left\lfloor \frac{t_{i-1}}{3} \right\rfloor \qquad \text{and} \qquad \alpha_i = m_i \exp\left(-(\ln N)^{2d/(2d+1)}\right).$$

Lemma 4.3. *Let the numbers s_i, r_i, t_i, m_i and α_i be defined as above. Then*

(p1) $r_i \geqslant 3 \cdot 9^{2d-i} d (\ln N)^{(2d-i)/(2d+1)}$ *and* $s_i \geqslant 9^{2d+1-i} d (\ln N)^{(2d+1-i)/(2d+1)}$ *for every* i, $1 \leqslant i \leqslant 2d$,

(p2) $\sum_{j=i}^{2d} s_j < \frac{4 s_i}{3}$ *and* $\sum_{j=i}^{2d} r_j < \frac{4 r_i}{3}$ *for every* i, $1 \leqslant i \leqslant 2d$,

(p3) $r_i - \sum_{j=i+1}^{2d} s_j \geqslant \frac{r_i}{3}$ *for every* i, $1 \leqslant i \leqslant 2d$,

(p4) $t_i > 7n \exp\left((\ln N)^{2d/(2d+1)}\right)$ *for every* $i \leqslant 2d$,

(p5) $t_i > 3 \cdot 2^{s_{i+1}}$ *for every* $i \leqslant 2d-1$,

(p6) $\frac{s_i}{2r_i} \ln \frac{m_i}{\alpha_i} \geqslant \ln m_i$ *for every* $i \leqslant 2d$,

(p7) $\alpha_i - \sum_{j=i+1}^{2d} 2 \cdot 2^{s_j} \geqslant n$ *for every* i, $1 \leqslant i \leqslant 2d$.

Proof. Since, for every real $q \geqslant 6$, $\lfloor \frac{q}{2} \rfloor \geqslant \frac{q}{3}$, it is easy to see that r_1 and s_1 satisfy (p1). If (p1) holds for r_i and s_i, it also holds for s_{i+1}, since

$$\begin{aligned} s_{i+1} = \lfloor 0.5 r_i \rfloor \geqslant \frac{r_i}{3} \geqslant \frac{3 \cdot 9^{2d-i} d (\ln N)^{(2d-i)/(2d+1)}}{3} \\ = 9^{2d+1-(i+1)} d (\ln N)^{(2d+1-(i+1))/(2d+1)}, \end{aligned}$$

and then also for r_{i+1}, since

$$\begin{aligned} r_{i+1} &= \lfloor 0.5 s_{i+1} (\ln N)^{-1/(2d+1)} \rfloor \\ &\geqslant \lfloor 0.5 \cdot 9^{2d+1-(i+1)} d (\ln N)^{(2d+1-(i+1))/(2d+1)} (\ln N)^{-1/(2d+1)} \rfloor \\ &\geqslant \frac{9^{2d+1-(i+1)} d (\ln N)^{(2d-(i+1))/(2d+1)}}{3} \\ &= 3 \cdot 9^{2d-(i+1)} d (\ln N)^{(2d-(i+1))/(2d+1)}. \end{aligned}$$

To prove (p2), it is enough to observe that $r_{i+1} \leqslant 0.5 s_{i+1} \leqslant 0.5(0.5 r_i) \leqslant 0.5(0.25 s_i)$ and thus $r_{i+1} \leqslant r_i/4$ and $s_{i+1} \leqslant s_i/4$. The same observation together with (p2) proves (p3).

Next, from (4.1), definitions of t_i and (p2), it follows that the inequality

$$t_i \geqslant t_{2d} \geqslant N 2^{-4s_1/3} 3^{-2d} > 7n \exp\big((\ln N)^{2d/(2d+1)}\big),$$

holds if

$$\exp\big(9^{4d} (\ln N)^{2d/(2d+1)}\big) > 7 \cdot 3^{2d} 2^2 2^{(4/3) 9^{2d} d (\ln N)^{2d/(2d+1)}} \exp\big((\ln N)^{2d/(2d+1)}\big).$$

This in turn is true if

$$9^{4d} (\ln N)^{2d/(2d+1)} > \ln 28 + 2d \ln 3 + \big((4/3) 9^{2d} d \ln 2 + 1\big) (\ln N)^{2d/(2d+1)}.$$

Since the last inequality holds for every $d \geqslant 1$, we have (p4). The relation (p5) follows from (p4) and the facts that $s_{i+1} \leqslant s_1$ and that $d < 0.1\sqrt{\ln \ln n}$.

By the definitions, $s_i \geqslant 2 r_i (\ln N)^{1/(2d+1)}$ and $\frac{m_i}{\alpha_i} = \exp\big((\ln N)^{2d/(2d+1)}\big)$. Therefore, to obtain (p6) note that

$$\frac{s_i}{2r_i} \ln \frac{m_i}{\alpha_i} \geqslant (\ln N)^{1/(2d+1)} (\ln N)^{2d/(2d+1)} = \ln N > \ln m_i.$$

Finally, to prove (p7), observe first that the inequalities $s_{i+1} \leqslant s_i/4$ and $d < 0.1\sqrt{\ln \ln n}$ yield

$$\sum_{j=i+1}^{2d} 2 \cdot 2^{s_j} \leqslant 4 \cdot 2^{s_1} = o(n).$$

So, it suffices to prove that $\alpha_i \geqslant 2n$, which would follow from $m_i \geqslant 2n \exp\big((\ln N)^{2d/(2d+1)}\big)$. This, in turn, follows from (p4) and the fact that $m_i = \lfloor \frac{t_{i-1}}{3} \rfloor$. The lemma is proved. □

4.3. Proof of Theorem 4.1

Let the numbers s_i, r_i, t_i, m_i and α_i be as defined in the previous subsection and let H be a graph which satisfies conditions of Theorem 4.1. We will now construct an auxiliary graph G using the following procedure.

Step 1. Let $T_0 = V(H)$ and let X_1 be a subset of $V(H)$ of size m_1 and $Y_1 = T_0 - X_1$. Define $H_1 = H$ if $e_H(X_1, Y_1) \geqslant \frac{m_1(N-m_1)}{2}$ and $H_1 = \overline{H}$ otherwise. If $H_1 = H$, then we will say that 1 is an H-*number*, and otherwise we will say that 1 is an $\overline{H}$-*number*. Let G_1 be the graph with $V(G_1) = V(H)$ and $E(G_1) = E_{H_1}(X_1, Y_1)$. Then G_1 is a bipartite graph with at least $\frac{m_1(N-m_1)}{2}$ edges between X_1 and Y_1. Observe that the graph G_1, together with the sets X_1, Y_1, and $Z_1 = \emptyset$, satisfies the conditions of Lemma 4.2 with $a = 2, s = s_1, m = m_1$ and

$r = r_1$. This implies that there exist $S_1 \subset X_1$ and $T_1 \subset Y_1$ such that:

(a) $|S_1| \leqslant s_1$,
(b) $|T_1| = t_1$,
(c) $|X_1 \cap N_{G_1}(R)| \geqslant \alpha_1$ for every $R \subset T_1$ of size r_1.

Define $X_{1,1} = X_1$.

Step k ($2 \leqslant k \leqslant 2d$). Assume that at step $k-1$ we have an auxiliary graph G_{k-1}, a decreasing sequence of k sets $T_0 \supset T_1 \supset \cdots \supset T_{k-1}$, and a family of $k-1$ disjoint vertex sets $X_{1,k-1}, X_{2,k-1}, \ldots, X_{k-1,k-1}$ with the following properties.

(i) $|T_i| = t_i$ for every i, $1 \leqslant i \leqslant k-1$.
(ii) $X_{i,k-1} \subset T_{i-1} \setminus T_i$ for every i, $1 \leqslant i \leqslant k-1$.
(iii) For every i, $1 \leqslant i \leqslant k-1$, there is an $S_i \subset X_{i,k-1}$ such that

$$N_{G_{k-1}}(S_i) \supset T_i \supset X_{i+1,k-1} \cup \cdots \cup X_{k-1,k-1} \cup T_{k-1} \quad \text{and} \quad |S_i| \leqslant s_i. \tag{4.6}$$

(iv) For every i, $1 \leqslant i \leqslant k-1$, and every subset R of the set $U_{k-1} - X_{i,k-1}$, where $U_{k-1} = T_{k-1} \bigcup \cup_{j=1}^{k-1} X_{j,k-1}$ with $|R| \leqslant r_i - \sum_{j=i+1}^{k-1} s_j$,

$$|N_{G_{k-1}}(R) \cap X_{i,k-1}| \geqslant \alpha_i - \sum_{j=i+1}^{k-1} 2 \cdot 2^{s_j}. \tag{4.7}$$

(v) For every $1 \leqslant i \leqslant k-1$, the edges of G_{k-1} connecting $X_{i,k-1}$ with T_i either all belong to H or all belong to $\overline{H}$.

Note that we have properties (i)–(v) after step 1. Next we will describe step k, and prove that after step k we will still keep all these properties (with k in place of $k-1$).

Let X_k be a subset of T_{k-1} of size m_k and $Y_k = T_{k-1} - X_k$. Define $H_k = H$ if $e_H(X_k, Y_k) \geqslant \frac{m_k(t_{k-1}-m_k)}{2}$ and $H_k = \overline{H}$ otherwise. If $H_k = H$, then we will say that k is an H-*number*, and otherwise we will say that k is an $\overline{H}$-*number*. Let G_k be the graph with $V(G_k) = V(H)$ and $E(G_k) = E(G_{k-1}) \cup E_{H_k}(X_k, Y_k)$. Then G_k is a $(k+1)$-partite graph with at least $\frac{m_k(t_{k-1}-m_k)}{2}$ edges between X_k and Y_k. It follows that the sets $X = X_k$ and $Y = T_{k-1} - X_k$ satisfy condition (4.3) of Lemma 4.2 for $a = 2$. Now we check that the remaining conditions of this lemma are satisfied for $Z = \bigcup_{i=1}^{k-1} X_{i,k-1}$, $s = s_k$, $m = m_k$, $r = r_k$, $\alpha = \alpha_k$. Indeed, (p4) together with the fact that $m_k = \lfloor \frac{t_{k-1}}{3} \rfloor$ yields $|Y| + |Z| < N < n^2 < m_k^2$, (p5) yields that $|Y| \geqslant 2t_{k-1}/3 > 2 \cdot 2^{s_k}$, and (p6) yields (4.4).

Thus, by Lemma 4.2, there exist $S_k \subset X_k$ and $T \subset (Y \cup Z) \cap N_{G_k}(S_k)$ such that:

(a) $|S_k| \leqslant s_k$,
(b) $|T \cap Y| \geqslant 0.5|Y|2^{-s_k} = 0.5(t_{k-1} - m_k)2^{-s_k} \geqslant \frac{t_{k-1}}{3}2^{-s_k} \geqslant t_k$,
(c) $|X_k \cap N_{G_k}(R)| \geqslant \alpha_k$ for every $R \subset T$ of size r_k,
(d) $|((Y \cup Z) \cap N_{G_k}(S_k)) \setminus T| \leqslant 2 \cdot 2^{s_k}$.

Let T_k denote any subset of $T \cap Y$ of size t_k (the existence of such a subset follows from (b)). Let

$$X_{k,k} = X_k \quad \text{and} \quad X_{i,k} = X_{i,k-1} \cap T. \tag{4.8}$$

Next we check that properties (i)–(v) still hold after this step (with k in place of $k-1$).

The set T_k was chosen to satisfy (i). By construction, $X_{k,k} \subset T_{k-1}$, $X_{k,k} \cap T_k = \emptyset$ and $X_{i,k} \subseteq X_{i,k-1}$ for every $i = 1, \ldots, k-1$. This implies (ii). Now (iii) follows from the definition of S_k, (4.8), and the induction hypothesis.

To check (iv), consider an arbitrary i, $1 \leqslant i \leqslant k$, and $R \subset U_k - X_{i,k}$ with $|R| \leqslant r_i - \sum_{j=i+1}^{k} s_j$. If $i = k$, then $|R| \leqslant r_k$, $U_k - X_k \subset T$ and (4.7) reads $|X_{k,k} \cap N_{G_k}(R)| = |X_k \cap N_{G_k}(R)| \geqslant \alpha_k$, which is true, owing to (c). Let $1 \leqslant i \leqslant k-1$. Since the sets $X_{1,k-1}, \ldots, X_{k-1,k-1}$ and T_{k-1} are pairwise disjoint, by definition, we have $U_k - X_{i,k} \subseteq U_{k-1} - X_{i,k-1}$ for every $i \leqslant k-1$. Let $R' = R \cup S_k$. Then $|R'| \leqslant r_i - \sum_{j=i+1}^{k-1} s_j$. By property (iv) of the induction hypothesis, we have

$$\left|N_{G_{k-1}}(R') \cap X_{i,k-1}\right| \geqslant \alpha_i - \sum_{j=i+1}^{k-1} 2 \cdot 2^{s_j}. \tag{4.9}$$

According to (d), $X_{i,k}$ is obtained from $X_{i,k-1} \cap N_{G_k}(S_k)$ by deleting at most $2 \cdot 2^{s_k}$ vertices. Hence $X_{i,k} \cap N_{G_k}(R)$ is obtained from $X_{i,k-1} \cap N_{G_k}(S_k) \cap N_{G_k}(R)$ by deleting at most $2 \cdot 2^{s_k}$ vertices. But $X_{i,k-1} \cap N_{G_k}(S_k) \cap N_{G_k}(R) = X_{i,k-1} \cap N_{G_k}(R')$ and therefore, by (4.9),

$$|X_{i,k} \cap N_{G_k}(R)| \geqslant \left(\alpha_i - \sum_{j=i+1}^{k-1} 2 \cdot 2^{s_j}\right) - 2 \cdot 2^{s_k} = \alpha_i - \sum_{j=i+1}^{k} 2 \cdot 2^{s_j}.$$

This implies that (iv) still holds after step k. To finish the proof of this induction step, note that $X_{i,k} \subset X_{i,k-1}$ and that for $1 \leqslant i \leqslant k-1$ the edges of G_k connecting $X_{i,k-1}$ and T_i either all belong to H or all belong to $\overline{H}$. This implies that $X_{i,k}, 1 \leqslant i \leqslant k-1$ satisfies (v). Also, from the definition of H_k we conclude that (v) holds for $X_{k,k}$ as well.

Let $X_{1,2d}, \ldots, X_{2d,2d}$ be the disjoint sets obtained after step $2d$. Define $V_0 = \bigcup_{i=1}^{2d} X_{i,2d}$ and $G = G_{2d}$. By (iv) and (p7), for every i, $1 \leqslant i \leqslant 2d$ and every subset $R \subset V_0 - X_{i,2d}$ with $|R| \leqslant r_i - \sum_{j=i+1}^{2d} s_j$,

$$|N_G(R) \cap X_{i,2d}| \geqslant \alpha_i - \sum_{j=i+1}^{2d} 2 \cdot 2^{s_j} \geqslant n. \tag{4.10}$$

In addition, observe that, for every $i \leqslant 2d$, by (p3) and (p1) we have

$$r_i - \sum_{j=i+1}^{2d} s_j \geqslant r_i/3 \geqslant d.$$

Therefore, in our auxiliary graph G, for every i, $1 \leqslant i \leqslant 2d$, every set of d vertices in $V_0 - X_{i,2d}$ has at least n common neighbours in $X_{i,2d}$.

To finish the proof, observe that the set $\{1, 2, \ldots, 2d-1\}$, either contains d H-numbers, or contains d $\overline{H}$-numbers. Without loss of generality suppose that the former holds and assume that $1 \leqslant j_1 < \cdots < j_d \leqslant 2d-1$ are some H-numbers. Define $j_{d+1} = 2d$. Then the subgraph G' of G induced by the set $V' = \bigcup_{i=1}^{d+1} X_{j_i,2d}$ is also a subgraph of H.

Let $v_1, \ldots, v_d$ be arbitrary vertices of G'. Since the sets $X_{j_i,2d}$ are disjoint and there are $d+1$ of them, there is an index j_k such that $v_1, \ldots, v_d \notin X_{j_k,2d}$. Therefore, by (4.10), the number of common neighbours of $v_1, \ldots, v_d$ in $X_{j_k,2d}$ and thus also in graph G' is at least n.

This implies that G' is a (d,n)-common subgraph of H, and completes the proof of the theorem. □

5. Concluding remarks

(1) Note that, for sufficiently large n, the proof of Theorem 2.2 gives nearly linear bounds on N not only for fixed d and c. Indeed, if d and c are functions of n satisfying $d\ln^3(1/c) = o(\ln n)$, then the bound on N in (2.1) is still $n^{1+o(1)}$. This implies the following two conclusions of independent interest.

First, repeating the proof of Corollary 2.3, we observe that in any colouring of edges of the complete bipartite graph $K_{N,N}$ with a colours, some colour is used on at least N^2/a edges. Applying Theorem 2.2 and the previous remark to the graph spanned by the edges of the majority colour, we obtain the following extension of Corollary 2.3.

Corollary 5.1. *Let d and a be integer-valued functions of n such that $d\ln^3 a = o(\ln n)$. Then, for every family of bipartite d-degenerate graphs $G_1,\ldots,G_a$ of order n, the Ramsey number $r(G_1,\ldots,G_a)$ is $n^{1+o(1)}$.*

Second, note that, for fixed c, the condition $d\ln^3(1/c) = o(\ln n)$ is equivalent to $d = o(\ln n)$. In particular, we obtain that the Ramsey number $r(G,G)$ of each d-degenerate bipartite graph G of order n is still $n^{1+o(1)}$ even when d is as large as $\ln n/w(n)$, where $w(n)$ tends to infinity arbitrarily slowly together with n. This bound is nearly tight. For example, if $d = 3\log_2 n$ then the random colouring of $K_{n^{3/2}}$, where the colour of every edge is chosen independently with probability $1/2$, does not contain monochromatic $K_{d,d}$. Therefore the Ramsey number of $K_{d,d}$ is at least $n^{3/2}$.

(2) The proof of Theorem 1.4 shows that d can also grow, with n still keeping the bound on N as $n^{1+o(1)}$, but in the general case the restrictions on d are much stronger than in the bipartite one. Here d must grow no faster than $\sqrt{\ln\ln n}$. It would be interesting to determine how large d can be, as a function of n, such that the assertion of Theorem 1.4 still holds.

(3) Using the Tripartite Lemma for $a > 2$, we can generalize Theorem 4.1 to edge colourings of K_N with a colours. In this more general case the structure of the proof will remain the same, only the procedure in Section 4.3 will have ad steps, and the formulas will become somewhat uglier. We obtain that, for a and d which grow sufficiently slowly with n, in every a-colouring of edges of $K_{n^{1+o(1)}}$ there will be a monochromatic (d,n)-common graph. This immediately implies the corresponding bound on Ramsey number $r(G_1,\ldots,G_a)$ for every family of d-degenerate graphs of order n.

(4) Finally, note that Theorem 2.2 is a Turán-type theorem, but Theorem 1.4 is a Ramsey-type theorem. It is easy to see that it is not enough to have restrictions only on average degree of the whole graph to guarantee the conclusion of Theorem 1.4. Indeed, the complete bipartite graph $K_{N,N}$ has average degree N but does not contain triangles. On the other hand we can prove the following quasi-Turán-type analogue of Theorem 1.4.

Proposition 5.2. *For every $\epsilon > 0$ and positive integers d and a, there exists $n_0 = n_0(\epsilon, d, a)$ such that, for every $n > n_0$, the following holds. Let H be a graph of order $n^{1+\epsilon}$ in which every induced subgraph on $k \geqslant 8n$ vertices has at least $\frac{1}{a}\binom{k}{2}$ edges. Then H contains a (d, n)-common subgraph.* $\square$

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