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On nice graphs

P. Hell[a,1], A.V. Kostochka[b,2], A. Raspaud[c,3], E. Sopena[c,*,4]

[a] *Simon Fraser University, School of Computing Science, Burnaby, B.C., Canada V5A 1S6*

[b] *Institute of Mathematics, 630090, Novosibirsk, Russia and University of Illinois at Urbana-Champaign, IL, USA*

[c] *LaBRI, Université Bordeaux 1, 351, cours de la Liberation, 33405 Talence, Cedex, France*

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Abstract

A digraph G is k-nice for some positive integer k if for every two (not necessarily distinct) vertices x and y in G and every pattern of length k, given as a sequence of pluses and minuses, there exists a walk of length k linking x to y which respects this pattern (pluses corresponding to forward edges and minuses to backward edges). A digraph is then nice if it is k-nice for some k. Similarly, a multigraph H, whose edges are coloured by a set of p colours, is k-nice if for every two (not necessarily distinct) vertices x and y in H and every pattern of length k, given as a sequence of colours, there exists a path of length k linking x to y which respects this pattern. Such a multigraph is nice if it is k-nice for some k.

In this paper we study the structure of nice digraphs and multigraphs. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

If G is a graph or a digraph we denote by $V(G)$ its vertex set and by $E(G)$ its edge or arc set. The graphs we consider are simple, that is they have neither multiple edges nor loops, and connected. If multiple edges or loops are allowed we shall speak about multigraphs. Let G and G' be either two graphs or two digraphs. A homomorphism

* Corresponding author.

E-mail address: sopena@labri.u-bordeaux.fr (E. Sopena).

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of G to G' is a mapping $f: V(G) \to V(G')$ such that $f(x)f(y)$ is an edge or an arc in G' whenever xy is an edge or an arc in G. The existence of a homomorphism of G to G' is denoted by $G \to G'$. Homomorphisms of graphs and digraphs have been considered in the literature as a generalization of colourings. For instance, a graph G has chromatic number k if and only if G admits a homomorphism to K_k, the complete graph on k vertices, and no homomorphism to K_{k-1}.

An orientation of a graph G is any digraph obtained by assigning to each edge of G one of its two possible orientations. A digraph is an oriented graph if it is an orientation of some graph. Homomorphisms of oriented graphs have been first considered in [7,8]. In particular, the notion of the oriented chromatic number of an oriented graph G, denoted by $\boldsymbol{\chi}(G)$, has been defined as the minimum order of an oriented graph H such that $G \to H$. The oriented chromatic number of a graph is then defined as the maximum oriented chromatic number of its orientations.

The bounds on the oriented chromatic number of planar graphs in terms of girth have been considered in [6]. The connection with the maximum average degree parameter, defined as the maximum of the average degrees of all subgraphs, has been studied in [5]. These two papers substantially used the property that every planar graph with sufficiently large girth, or every graph with sufficiently small maximum average degree, contains either a vertex with degree one or a long path whose internal vertices have degree two. Let k be any given positive integer. Suppose that H is an oriented graph such that for every vertices u and v in $V(H)$, not necessarily distinct, and every pattern of length k, given as a sequence of pluses and minuses, there exists a walk of length k in H linking u to v which respects this pattern (pluses corresponding to forward edges and minuses to backward edges). Then it can be deduced from the property of planar graphs mentioned earlier that every oriented planar graph with sufficiently large girth, or every oriented graph with sufficiently small maximum average degree, can be homomorphically mapped to H. We will say that an oriented graph H with the above property is *k-nice*.

In [1] Alon and Marshall studied homomorphisms of edge-coloured graphs. (Homomorphisms of edge-coloured graphs have also been studied in [2,3].) Let p be any fixed non-negative integer; a *p-edge-coloured graph* is a graph whose edges are coloured by a set of p colours. A homomorphism between two edge-coloured graphs must also preserve the colours of edges. One can make similar conclusions about edge-coloured planar graphs of sufficiently large girth, respectively, edge-coloured graphs with sufficiently small maximum average degree, mapping homomorphically to an edge-coloured graph H, if for any vertices u and v in $V(H)$, not necessarily distinct, and any pattern of length k, given as a sequence of colours, there exists a path of length k in H linking u to v which respects this pattern. Thus, we will also consider the notion of nice edge-coloured graphs or, more generally, multigraphs.

A natural extension of the notion of niceness can be obtained by only requiring that for every two vertices u and v and every pattern there exists a walk (or a path) with this pattern starting at u and containing v. Graphs having this property will be called *half-nice*. Clearly, every nice graph is half-nice.

The definitions and the main results are introduced in the next section. In particular, we consider the problem of the characterization of nice or half-nice graphs and edge-coloured multigraphs and apply our results to the problem of determining the minimum number of edges in such graphs or multigraphs. Sections 3–6 are devoted to the proofs of our results.

2. Definitions and main results

A *pattern* Q is a (non-empty) word in $\{+,-\}^+$. Let $Q = q_0q_1 \ldots q_{k-1}$ be a pattern of length k. A Q*-walk* in a digraph G is a walk $P = x_0x_1 \ldots x_k$ such that for every i, $0 \leqslant i \leqslant k-1$, $x_ix_{i+1} \in E(G)$ if $q_i = +$ and $x_{i+1}x_i \in E(G)$ otherwise. For $X \subseteq V(G)$ we denote by $N_Q(X)$ the set of all vertices y such that there exists a Q-walk going from some vertex $x \in X$ to y. We then say that a digraph G is k*-nice* if for every pattern Q of length k and every vertex x in $V(G)$ we have $N_Q(\{x\}) = V(G)$. In other words, a digraph is k-nice if for all pairs of vertices x, y (allowing $x = y$) there is a k-walk from x to y for each of the 2^k possible oriented patterns. Observe that if a digraph G is k-nice for some k, then it is k'-nice for every $k' > k$. We say that a digraph is *nice* if it is k-nice for some k.

Recall that the *circulant* digraph $G = G(n; a_1, a_2, \ldots, a_\ell)$ is the digraph defined by $V(G) = \{0, 1, \ldots, n-1\}$ and $E(G) = \{xy : y = x + a_i \pmod n,\ 1 \leqslant i \leqslant \ell\}$. In fact, it has been proved in [6] that every circulant digraph of the form $G(n; 1, 2, \ldots, d)$ is $\lceil (n-1)/(d-1) \rceil$-nice.

Let G be a multigraph whose edges are p-coloured and let $\varphi : E(G) \to \{1, 2, \ldots, p\}$ denote the corresponding colouring function. For every pattern $Q = q_0q_1 \ldots q_{k-1}$ in $\{1, 2, \ldots, p\}^+$, a Q-path in G is a path $P = x_0x_1 \ldots x_k$ such that for every i, $0 \leqslant i \leqslant k-1$, the edge x_ix_{i+1} has colour q_i. As before, for $X \subseteq V(G)$ we denote by $N_Q(X)$ the set of all vertices y such that there exists a Q-path going from some vertex $x \in X$ to y. We then say that G is k*-nice* if for every pattern Q of length k and every vertex x in $V(G)$ we have $N_Q(\{x\}) = V(G)$. We say that G is *nice* if it is k-nice for some k.

First we shall consider characterizations of nice digraphs and nice edge-coloured multigraphs. A useful notion in that context is that of a black hole. Let G be a digraph or an edge-coloured multigraph. A *black hole* is a pair (A, Q) such that A is a proper subset of $V(G)$ and Q is a pattern such that $N_Q(A) \subseteq A$. If Q has length k, and if there exists no pattern Q' of length $k' < k$ such that (A, Q') is a black hole we say that the black hole (A, Q) has *depth* k.

Note that there exists a black hole (A, Q) with $N_Q(A) = \emptyset$ in a digraph G if and only if G contains a vertex with in- or out-degree zero. Moreover, if G is not strongly connected then G has at least one strong component C such that no arc leaves C. The component C is thus a black hole with pattern $+$ (similarly, there exists at least one strong component of G which is a black hole with pattern $-$).

Let $k \geqslant 1$. A *vicious circle* with pattern $Q = q_0q_1 \ldots q_{k-1}$ is a cyclic sequence of black holes $A_0, A_1, \ldots, A_{k-1}$ such that $N_{q_i}(A_i) = A_{i+1}$ for every i, $0 \leqslant i < k-1$, and $N_{q_{k-1}}(A_{k-1}) = A_0$. Then we have:

Observation 1. *Any not strongly connected digraph* $G=(V,E)$ *with* $E\neq\emptyset$ *contains a vicious circle.*

To see that, consider the digraph S whose vertices are the vertex sets of strong components of G and XY is an arc in S if and only if there is an arc xy in G such that $x\in X$ and $y\in Y$. The graph S is clearly acyclic. If some source or sink X in S is not a singleton then X is a vicious circle with the pattern $-$ or $+$. Otherwise, the pair (A,B), where

$$A=\{v\in V\mid \exists u\in V: vu\in E(G)\},\quad B=\{u\in V\mid \exists v\in V: vu\in E(G)\},$$

is a vicious circle with pattern $+-$. Similarly, if the subgraph H_q spanned by the edges of a colour q in a p-edge-coloured multigraph is not connected and has a component A with $|A|>1$, then A forms a vicious circle of length 1 with the pattern q.

Our first proposition is simple:

Proposition 2. *The following statements are equivalent for a graph or a* p*-edge-coloured multigraph* G:

(1) G *is not nice*;
(2) G *has a black hole*;
(3) G *has a vicious circle.*

Proof. We just consider the case of a digraph G, the proof being similar for a p-edge-coloured multigraph. From the definitions, (3) implies (2) and (2) implies (1). Suppose now that G is not nice. That means that for every k there exists a vertex x and a pattern $Q=q_0q_1\dots q_{k-1}$ of length k such that $N_Q(\{x\})\neq V(G)$. We denote by Q_i the pattern $q_0q_1\dots q_{i-1}$ for every i, $0<i\leqslant k-1$. Let $k=2^{|V(G)|}$ and $X_0,X_1,\dots,X_{k-1}$ be the sequence defined by $X_0=\{x\}$ and $X_i=N_{Q_i}(\{x\})$ for every i, $0<i\leqslant k-1$. If X_i is empty for some i then G is not strongly connected and thus contains a vicious circle (Observation 1). Otherwise there exist two indices i and j, $i<j$, such that $X_i=X_j$. The sequence $(X_i,X_{i+1},\dots,X_j)$ is then a vicious circle with pattern $q_iq_{i+1}\dots q_j$. $\square$

The following theorem states that in order to check whether a digraph is nice or not it is enough to consider only short patterns

Theorem 3. *If a digraph G is not nice then it has a black hole with pattern either* $+$ *or* $+-$.

In case of p-edge-coloured multigraphs, we have the following characterization

Theorem 4. *If a* p*-edge-coloured multigraph G is not nice then it has a vicious circle either with pattern* qq *or with a non-periodic cyclic pattern* $q_0\dots q_{k-1}$ *such that* $q_i\neq q_{i+1}$ *for every* i, $0\leqslant i<k$ *(subscripts are taken modulo* k*).*

We now give a second characterization of nice digraphs by means of the existence of some special cycles. A walk of length $2\ell + 1$ is a *quasi-alternating walk* if it has pattern $(+-)^{\ell}+$. A vertex x is said to be *special* if there exists a quasi-alternating walk going from x to x. Then we have

Theorem 5. *A digraph is nice if and only if it is strongly connected and all its vertices are special.*

Theorems 3 and 5 will be proved in Section 3. Theorem 4 will be proved in Section 4. Moreover, we shall show in Section 4 that for every such non-periodic cyclic pattern one can construct an edge-coloured multigraph having only one vicious circle, precisely with this pattern.

We say that a digraph G is *k-half-nice*, if for every pattern Q of length k and every vertices x and y there is a Q-walk starting at x which contains y. Similarly, a p-edge-coloured multigraph G is *k-half-nice* if for every pattern Q of length k and every vertices x and y there is a Q-path starting at x which contains y. We say that a digraph or a p-edge-coloured multigraph is *half-nice* if it is k-half-nice for some k. Clearly, each nice digraph or p-edge-coloured multigraph is half-nice.

Our characterization of half-nice digraphs and half-nice multigraphs is the following:

Proposition 6. *The following statements are equivalent, for a digraph or a p-edge-coloured multigraph G:*

(1) *G is not half-nice;*
(2) *G has a vicious circle $\{A_0, A_1, \ldots, A_{k-1}\}$ with pattern $Q = q_0 q_1 \ldots q_{k-1}$ such that $V(G) \setminus \bigcup_{i=0}^{k-1} A_i \neq \emptyset$.*

Proof. We only consider the case of a digraph G, the proof being similar for a p-edge-coloured multigraph. If G satisfies condition (2) then for every $m \geqslant k$, the pattern $Q' = Q^{\lfloor m/k \rfloor} q_0 q_1 \ldots q_{\ell}$, with $\ell = m - 1 \pmod k$, is such that no Q'-walk starting at $a \in A_0$ goes through a vertex $b \in V(G) \setminus \bigcup_{i=0}^{k-1} A_i$.

Conversely, if G is not half-nice then for every k there exist two vertices x and y and a pattern $Q = q_0 q_1 \ldots q_{k-1}$ such that no Q-walk starting from x goes through y. Let $X_0 = \{x\}$ and $X_i = N_{q_0 q_1 \ldots q_{i-1}}(X_0)$ for every $i > 0$. If $k \geqslant 2^{|V(G)|}$ then there exist two indices i and j, $j > i$, such that $X_i = X_j$. If $X_i \neq \emptyset$ then the sequence $(X_i, X_{i+1}, \ldots, X_{j-1})$ is a vicious circle such that $y \notin \bigcup_{\ell=i}^{j-1} X_{\ell}$. If $X_i = \emptyset$ then G is not strongly connected and, as observed before, contains a vicious circle satisfying the claim. □

From Theorem 5, we deduce that a nice digraph cannot be bipartite. Moreover, we have:

Proposition 7. *A digraph is nice if and only if it is half-nice and non-bipartite.*

Proof. Let G be a non-bipartite half-nice digraph, and suppose that G is not nice. Since G is half-nice, it is strongly connected. By Theorem 3 and Proposition 6 it has a vicious circle (A,B) with pattern $+-$ such that $A\cup B=V(G)$. Moreover, since G is strongly connected, we have $A\backslash B\neq\emptyset$ and $B\backslash A\neq\emptyset$. We first claim that $A\cap B=\emptyset$. If not, let $x\in A\cap B$; since there is no arc from x to A or from B to x, we conclude that $(A\backslash B,B\backslash A)$ is a vicious circle with pattern $+-$ such that $A\cup B\neq V(G)$, in contradiction with Proposition 6. Finally, observe that since $N_+(A)=B$ and $N_-(B)=A$, both sets A and B are independent. We thus see that G is bipartite, a contradiction. □

Similarly, an edge-coloured multigraph cannot be nice if one of the subgraphs induced by monochromatic edges is bipartite. However, a half-nice edge-coloured multigraph such that none of its monochromatic subgraphs is bipartite is not necessarily nice. To see that, consider the 2-edge-coloured multigraph G_{2n}, $n\geqslant 3$, defined by $V(G_{2n})=\{0,1,\dots,2n-1\}$, $E(G_{2n})=E^1_{2n}\cup E^2_{2n}$ with $E^1_{2n}=\{ij:(i=2k+1,\ j=2k'+1)$ or $(i=2k,\ j=2k+1)$, $0\leqslant k<k'\leqslant n\}$ and $E^2_{2n}=\{ij:(i=2k,\ j=2k')$ or $(i=2k-1,\ j=2k)$, $0\leqslant k<k'\leqslant n\}$ (vertices are taken modulo $2n$). Assume that all the edges of E^1_{2n} are 1-coloured and that all the edges of E^2_{2n} are 2-coloured. Clearly, the only vicious circle is the sequence $(\{0\},\{1\},\dots,\{2n-1\})$ with pattern $(12)^n$ and thus G_{2n} is half-nice but not nice. Moreover, each monochromatic subgraph contains triangles and thus is not bipartite.

We now consider the problem of determining the minimum number of edges in a nice or half-nice digraph. We have

Theorem 8. *The minimum number of edges in a nice digraph with n vertices is* $2n-1$. *The minimum number of edges in a half-nice digraph with n vertices is* $2n-2$.

In case of p-edge-coloured multigraphs, we have

Theorem 9. *The minimum number of edges in a nice* p*-edge-coloured multigraph* G *with* n *vertices is* pn. *The minimum number of edges in a half-nice* p*-edge-coloured multigraph* G *with* n *vertices is* $p(n-1)$.

Theorems 8 and 9 will be proved in Section 5.

Our initial motivation for the study of nice graphs was the problem of determining the oriented chromatic number of some classes of graphs. More precisely, we say that a graph G is *universal* for some class of graphs $\mathscr{C}$, or shortly $\mathscr{C}$-universal, if every graph H in $\mathscr{C}$ has a homomorphism to G [4,6]. Denote by $\mathscr{P}_k$ (respectively, $\mathscr{O}\mathscr{P}_k$) the class of planar (respectively, outerplanar) oriented graphs with girth at least k. In particular, $\mathscr{P}_3$ (respectively, $\mathscr{O}\mathscr{P}_3$) is the class of all planar (respectively, outerplanar) oriented graphs. Evidently, $\mathscr{P}_3\supset\mathscr{P}_4\supset\mathscr{P}_5\dots$, which yields that any $\mathscr{P}_k$-universal graph is also $\mathscr{P}_\ell$-universal for every $\ell>k$ (the same is true for outerplanar graphs). The next fact was used in [6]

Proposition 10. *For each* $k\geqslant 3$ *any* k*-nice oriented graph is* $\mathscr{P}_{5k-4}$*-universal.*

For completeness, we shall include a proof of this proposition in Section 6 together with the proof of the following theorem

Theorem 11. *Let g be a positive integer. If H is $\mathcal{OP}_g$-universal and inclusion minimal with respect to this property, then H is nice.*

Using a similar proof technique, we can also prove

Theorem 12. *Let g be a positive integer. If H is $\mathcal{P}_g$-universal and inclusion minimal with respect to this property, then H is nice.*

In [4], it has been proved that every planar $\mathcal{P}_g$-universal graph must contain a cycle of length at most five. Theorems 8 and 9, together with Euler's formula, give the following improvement:

Corollary 13. *Every nice or half-nice oriented planar graph has a triangle.*

3. Characterizations of nice digraphs

We first make some easy observations concerning black holes. Let $\bar{q} = +$ if $q = -$ and $\bar{q} = -$ if $q = +$.

Observation 14. *If $(A, Q = q_0 q_1 \dots q_{k-1})$ is a black hole then so is $(\bar{A}, \bar{Q})$, where $\bar{A} = V(G) \backslash A$ and $\bar{Q} = \bar{q}_{k-1} \dots \bar{q}_0$.*

Observation 15. *If $(A, Q = q_0 q_1 \dots q_{k-1})$ is a black hole of minimal depth k then so is $(N_{q_0 q_1 \dots q_i}(A), q_{i+1} \dots q_{k-1} q_0 \dots q_i)$, for every i, $0 \leqslant i < k-1$.*

Moreover, since for every subset X of vertices we have $X \subseteq N_{+-}(X)$ and $X \subseteq N_{-+}(X)$, we obtain

Observation 16. *If $(A, u+-v)$ or $(A, u-+v)$ is a black hole such that uv is nonempty then (A, uv) is a black hole.*

We now turn to the proof of Theorem 3. Recall that the condition 'G has no black hole with pattern $+$' means that G is strongly connected.

Proof of Theorem 3. Suppose that G is not nice. By Proposition 2, G has a black hole. Let (A, Q) be a black hole of *minimal* depth k. If $k = 1$ then G is not strongly connected and thus contains a strong component which is a black hole with pattern $+$. We have two cases more to consider:

Case 1: $k = 2$.

By Observations 14 and 15 it suffices to consider the cases $Q=++$ and $Q=+-$. If $Q=+-$, there is nothing to prove. Suppose that $Q=++$, and let $A_+=N_+(A)\backslash A$. Note first that for every $x\in V(G)\backslash A$ there is no edge from A_+ to x. Moreover, A_+ is an independent set of vertices. Let $X=V(G)\backslash(A\cup A_+)$. By the definition of A_+ there is no edge from A to X and by the above discussion no edge from A_+ to X. If X is not empty then G is not strongly connected. Suppose that X is empty. Let aa' be an edge from A to $N_+(A)$. Since A is a black hole, we have $d_A^-(a)=0$. Since G is strongly connected, A is an independent set of vertices. The graph G is thus bipartite and $(A,+-)$ is a black hole.

Case 2: $k>2$.

By Observations 14 and 16 we have to consider onlythe case $Q=(+)^k$. We first prove that $(A,(-)^k)$ is also a black hole. Let $a\in A$ and $b\in N_{(-)^k}(a)$. If G is strongly connected then there exists a directed path from a to b and thus a circuit C (not necessarily simple) passing through a and b. Let ℓ be the length of that circuit. The vertex a is the ℓkth successor of itself in C and b is the $(\ell-1)k$th successor of a in C. Hence $b\in A$ and we are done. We thus see that $(A,(+)^k(-)^k)$ is a black hole. By Observation 16 we then conclude that $(A,+-)$ is a black hole. This completes the proof. □

Proof of Theorem 5. The ‘if’ part follows directly from the definitions. Suppose now that G is strongly connected and that every vertex x in G is special. We shall prove that G cannot have a black hole with pattern $+-$. To the contrary, assume that A is such a black hole, and let $B=N_+(A)$. We first claim that for every vertex $a\in A$, there exists some vertex $a'\in A$ such that $a'a$ is an arc in G. Since a is special, there exists a closed walk $a=a_0a_1a_2\ldots a_{2\ell}a$ in G such that for every i, $1\leqslant i\leqslant\ell$, $a_{2i}a_{2i-1}$ and $a_{2i}a_{2i+1}$ belong to $E(G)$ (subscripts are taken modulo $2\ell+1$). We then get that for every i, $1\leqslant i\leqslant\ell$, a_{2i} also belongs to A and $a'=a_{2\ell}$ is the required vertex. Hence we have $A\subseteq B$. Since B is a black hole with pattern $-+$, we conclude symmetrically that $B\subseteq A$. Therefore $A=B$, a contradiction since G is strongly connected. This completes the proof. □

From Theorem 3 and Observation 14, we get that if a strongly connected digraph G is not nice then, starting with any 1-element subset $X_0=\{x\}$ of $V(G)$ and defining $X_i=N_{+-}(X_{i-1})$ for every $i>0$, we necessarily find some j such that X_j is a black hole with pattern $+-$ (j is the minimal index such that $X_{j+1}=X_j$). Moreover, since in strongly connected graphs $N_{+-}(X)\subseteq X$ for every set X, we get that $X_{i+1}=X_i\cup N_{+-}(X_i\backslash X_{i-1})$ for every $i>0$. This proves that the algorithm given in Fig. 1 decides whether a given digraph is nice or not in time $O(|E(G)|\times|V(G)|)$.

4. A characterization of nice p -edge-coloured multigraphs

We first prove Theorem 4.

```
ALGORITHM NICE

Input: A digraph G
Question: Is G nice ?

    if G is not strongly connected then answer NO end-if
    Otherwise, pick any vertex x in G
    Set X := {x} and Y := {x}
    repeat
        Set Y := N+-(Y) \ X
        if Y = ∅ then answer NO end-if
        Set X := X ∪ Y
        if X = V(G) then answer YES end-if
    end-repeat
```

Fig. 1. An algorithm for deciding whether a digraph is nice or not.

Proof of Theorem 4. Let G be a non-nice multigraph, and let $(A_0, A_1, \ldots, A_{k-1})$ be a vicious circle with pattern $Q = q_0 q_1 \ldots q_{k-1}$ of minimum length and such that $\sum_{i=0}^{k-1} |A_i|$ is minimum among those with minimum length.

We first claim that either $k = 2$ and $Q = qq$ or $q_i \neq q_{i+1}$ for every i, $0 \leqslant i < k-1$. To see that, assume to the contrary that there exists some i with $q_i = q_{i+1} = q$. We then have $A_i \subseteq A_{i+2}$ and there exists a vicious circle with pattern $q_0 \ldots q_{i-1} q_{i+2} \ldots q_{k-1}$, contradicting our minimality assumption unless the initial pattern was qq.

Assume now that the pattern is periodic and has the form $(q_0 q_1 \ldots q_{r-1})^m$, with $m > 1$. We denote by A_i^j the set $N_{(q_0 \ldots q_{r-1})^j q_0 \ldots q_{i-1}}(A_0)$, for every i, j such that $0 \leqslant i \leqslant r-1$, $0 \leqslant j < m$. We claim that $A_i^j \cap A_i^\ell = \emptyset$ for every $j \neq \ell$. Assume to the contrary that there exists $x \in A_i^j \cap A_i^\ell$. From x we can reach all vertices belonging to $N_{q_i}(A_i^j)$ by using the periodic pattern, since otherwise we could find a vicious circle contradicting our minimality assumption. It follows that $N_{q_i}(A_i^j) = N_{q_i}(A_i^\ell)$. Thus, $A_{i+1}^j = A_{i+1}^\ell$, and we again have a vicious circle which contradicts our minimality assumption.

Let $B_i = \bigcup_{j=0}^{j=m-1} A_i^j$. If $B_i \neq V(G)$ for every i, $0 \leqslant i \leqslant k-1$, then the sets B_i form a shorter vicious circle, a contradiction. If we have $B_i = V(G)$ for some i, then we claim that there is a vicious circle with a pattern of length two having two identical colours. W.l.o.g. we assume that $B_0 = V$; then $N_{q_0}(A_0^1) = A_1^1$. Observe that $N_{q_0}(A_1^1) \subset A_0^1$: if not, there exists a vertex $x \notin A_0^1$ belonging to $N_{q_0}(A_1^1)$. Hence $x \in A_0^\ell$ for some ℓ, $0 < \ell \leqslant m-1$. Then $x \in N_{q_0}(A_0^l)$ but $A_1^1 \cap A_1^\ell = \emptyset$, a contradiction. It follows that A_0^1 and A_1^1 form a vicious circle for the pattern $q_0 q_0$. This completes the proof. □

For every non-periodic cyclic pattern $Q = q_0 q_1 \ldots q_{k-1}$ in $\{1, 2, \ldots, p\}^k$ with $q_i \neq q_{i+1}$ for every i, $1 \leqslant i < k$, one can construct a p-edge-coloured multigraph G_Q having vicious circles only with this pattern. The construction is as follows. The graph G_Q has $k+1$ vertices, denoted by $\{v_0, v_1, \ldots, v_{k-1}, w\}$. The vertex w has loops of every colour c, $1 \leqslant c \leqslant p$, and for every i, j, $0 \leqslant i, j < k$, with $|i - j| > 1 \pmod{k}$, we join v_i and v_j by edges of every colour distinct from q_i and q_j. Finally, for every i, $0 \leqslant i < k$, we add an edge from v_i to v_{i+1} with colour q_i and edges from v_i to w of every colour except q_i. We will say that the vertex v_i is *of type* q_i. Clearly, the sequence $\{v_0\}, \{v_1\}, \ldots, \{v_{k-1}\}$

is a vicious circle with pattern Q. Assume that there exists another vicious circle $B_0, B_1, \ldots, B_{\ell-1}$ with pattern $b_0 b_1 \ldots b_{\ell-1}$. We have two cases to consider

Case 1: $w \in B_0 \cup \cdots \cup B_{\ell-1}$.

Since w has all possible coloured loops, w belongs to every B_i. We may assume that B_0 is not smaller than every other B_i. By construction, B_1 contains all vertices of type different from b_0. Moreover, if a vertex v_i is of type b_1 and $v_{i-1} \in B_0$ then $v_i \in B_1$. It follows that $|B_1| \geqslant |B_0|$ and that all vertices in $V(G_Q) \backslash B_1$ are of type b_0.

By the maximality of B_0 we conclude that $|B_1| = |B_0|$. We can continue in the same way and we see that all the sets B_i have the same cardinality and that all vertices in $V(G_Q) \backslash B_i$ are of type b_{i-1}. Moreover, we have $v_i \notin B_j$ if and only if $v_{i+1} \notin B_{j+1}$.

We may thus assume that $C_0 = V(G_Q) \backslash B_0$ contains only vertices of type q_0 (maybe not all of them). Then, $C_1 = V(G_Q) \backslash B_1$ contains only vertices of type q_1 and is obtained from C_0 by rotating one step around the circle $(v_0, v_1, \ldots, v_{k-1})$. In particular, we have $b_0 = q_0$. Continuing, we see that the pattern $q_0 q_1 \ldots q_{k-1}$ contains $b_0 b_1 \ldots b_{\ell-1}$ and, since $q_0 q_1 \ldots q_{k-1}$ is aperiodic, we have $q_0 q_1 \ldots q_{k-1} = b_0 b_1 \ldots b_{\ell-1}$.

Case 2: $w \notin B_0 \cup \cdots \cup B_{\ell-1}$.

Assume that $v_0 \in B_0$. If $b_0 \neq q_0$ then $w \in B_1$, a contradiction. Thus $b_0 = q_0$ and hence $v_1 \in B_1$. Repeating this, we get $b_1 = q_1$ and hence $v_2 \in B_2$, and so on. Finally, we conclude that $q_0 q_1 \ldots q_{k-1} = b_0 b_1 \ldots b_{\ell-1}$.

The reader can observe that if we delete the vertex w in the above constructed graph G_Q we get a graph G'_Q which is half-nice, but not nice, and that the sequence $\{v_0\}, \{v_1\}, \ldots, \{v_{k-1}\}$ and its complement $V(G'_Q) \backslash \{v_0\}, V(G'_Q) \backslash \{v_1\}, \ldots, V(G'_Q) \backslash \{v_{k-1}\}$ are again the only vicious circles.

5. Minimum number of edges in nice graphs

We first prove Theorem 8.

Proof of Theorem 8. We first consider the case of a nice digraph G and assume that $|V(G)| = n$. For every vertex $x \in V(G)$ we denote by $N_G^-(x)$ the set of predecessors of x. Let G' be the undirected graph constructed as follows: we set $V(G') = V(G)$ and, for every vertex x, we link in G' the vertices of $N_G^-(x)$ by a path. Since the digraph G is k-nice for some k, there exists a walk with pattern $(+-)^k$ between every pair of vertices. Therefore, the graph G' is connected and $|E(G')| \geqslant n-1$. On the other hand,

$$|E(G')| = \sum_{x \in V(G)} (|N_G^-(x)| - 1) = |E(G)| - n$$

and the result follows.

Suppose now that G is a half-nice digraph. By Proposition 7, if G is not nice then it is bipartite. Let $V(G) = A \cup B$ be the corresponding bipartition. Consider the auxiliary graph G' defined as above. Each class of the bipartition gives a connected component of G', denoted by G'_A and G'_B. Hence, $|E(G'_A)| = \sum_{x \in B} (|N_G^-(x)| - 1) \geqslant |A| - 1$ and, similarly, $|E(G'_B)| = \sum_{x\in A}(|N_G^-(x)| - 1) \geqslant |B| - 1$. From that, we obtain $E(G) = \sum_{x\in A}|N_G^-(x)| + \sum_{x\in B}|N_G^-(x)| \geqslant 2n-2$.

We now give examples of graphs achieving these bounds. Denote by G the circulant digraph $G(n;1,2)$ minus one edge. This graph has exactly $2n-1$ edges. Moreover, it is strongly connected and has no black hole with pattern $+-$ since the corresponding graph G' above defined is clearly connected. The graph G is thus nice. For the second claim, consider the bipartite oriented graph G given by $V(G) = X \cup Y$, with $X = \{x_0, x_2, \ldots, x_k\}$ and $Y = \{y_0, y_2, \ldots, x_k\}$ ($k \geqslant 4$), and $E(G) = \{x_0y_0\} \cup \{x_iy_{i-1}, x_iy_i : 1 \leqslant i \leqslant k\} \cup \{y_0x_{k-1}\} \cup \{y_ix_{i-1},\ y_ix_{i-2} : 1 \leqslant i \leqslant k\}$ (subscripts are taken modulo k). In fact, G is made of two alternating Hamiltonian walks. For every subset of X or Y, made of consecutive vertices (modulo k), both $N_+(X)$ and $N_-(X)$ consist of consecutive vertices. Moreover, the size of $N_+(X)$ or of $N_-(X)$ is strictly greater than the size of X except if X is reduced to one vertex from the set $\{x_0, y_0, x_k, y_k\}$. From that we deduce that for every sufficiently long walk we reach all the vertices. □

We now turn to the proof of Theorem 9.

Proof of Theorem 9. Let first G be a half-nice p-edge-coloured multigraph. Then for every colour c, the subgraph G_c of G spanned by the edges of colour c must cover all vertices and be connected. It follows that G_c has at least $n-1$ edges and thus G has at least $p(n-1)$ edges.

If G is nice, then each G_c must, furthermore, be non-bipartite. It follows that G_c has at least n edges and thus G has at least pn edges.

We now give examples of multigraphs achieving these bounds. We take the set of vertices $\{y, z, x_1, x_2, \ldots, x_{n-2}\}$ and add p edges, one for each colour, between vertices y and z, z and x_1, y and x_1, and x_i and x_{i+1} for every i, $1 \leqslant i < n-2$. Since all the edge colours play the same role, we get from Theorem 4 that this multigraph is not nice if and only if it has a vicious circle with pattern qq for some edge colour q. Due to the triangle yzx_1, such a vicious circle clearly cannot exist. For half-nice multigraphs, it suffices to consider a path such that every two adjacent vertices are linked by p edges, one for each colour. Note that it is also possible to construct nice or half-nice p-edge-coloured graphs (with no multiple edges) having this number of edges by arranging together edge disjoint Hamiltonian cycles or Hamiltonian paths in a more complicate way. □

6. Nice graphs and universal graphs

The aim of this section is to prove Proposition 10 and Theorem 11.

Proof of Proposition 10. Let H be k-nice, and let G be a graph with the smallest number of vertices in $\mathscr{P}_{5k-4}$ which has no homomorphism to H. Then G is connected.

Case 1: The graph G has a vertex v adjacent only to one vertex, w. By the minimality of G, there exists an oriented homomorphism f of $G-v$ to H. Since H is nice, the vertex $f(w)$ has both in- and out-neighbours. One of them is suitable for $f(v)$. This is a contradiction.

Case 2: The degree of every vertex in G is at least 2. Replacing every path with internal vertices of degree two and end-vertices of a larger degree by an edge, we obtain a planar graph G' with minimum degree at least three. By Euler's formula, it must contain a cycle C' of length at most five. Since the length of the corresponding cycle C in G is at least $5k-4$, C contains a subpath $v_1 \dots v_k$ whose internal vertices have degree two in G. Let $G_1 = G \setminus \{v_2, \dots, v_{k-1}\}$. By the minimality of G, there exists an oriented homomorphism f of G_1 to H. Since H is k-nice, f can be extended to $v_2, \dots, v_{k-1}$. □

Proof of Theorem 11. We deliver the proof in a series of claims. The first two of them are immediately implied by the minimality of H.

(i) *There is no homomorphism of H to any of its proper subgraphs.*
(ii) *For every arc e in H, there exists a graph $G_e \in \mathcal{OP}_g$ such that every homomorphism $f: G_e \to H$ maps some arc of G_e to the arc e.*
(iii) *Every component of H is strongly connected.*

Proof. Assume that there exists a partition of $V(H)$ into two parts W and U such that some arc e leads from W to U but no arc leads from U to W. We construct an auxiliary graph G' as follows: take a copy of the graph G_e from (ii). To every arc e' of G_e *lying in the outerface*, we 'glue' a copy of the circuit C_g by identifying the arc e' with an arc of C_g. The graph G' thus obtained is outerplanar and has girth g. Hence, there exists a homomorphism $f: G' \to H$. Since every homomorphism of G_e to H uses e, there is an arc $e'' = wu$ from G_e which is mapped to e. Since both w and u lie on the outer face and in the same component of G_e, there exists an arc xy on the outer face of G_e such that the arc $f(x)f(y)$ crosses the cut (W, U). Let G'_g be the corresponding copy of C_g which is glued to xy. Since C_g is a directed cycle, some arc of its image must cross the cut (W, U) in the direction opposite to $f(x)f(y)$. This contradicts the definition of W and U.

(iv) *For every $v \in V(H)$, for every outerplanar graph G with girth at least g and every $x \in V(G)$, there exists a homomorphism of G to H which maps x to v.*

Proof. Let e be an arc incident to x in H. We construct an auxiliary graph G' as follows: take a copy of the graph G_e from (ii). To every vertex w in G_e we 'glue' a copy of the graph G by identifying the vertex w with the vertex x of the corresponding copy of G. The graph G' thus obtained is outerplanar and has girth at least g. Thus, there exists a homomorphism $f: G' \to H$. Since every homomorphism of G_e to H uses e, some vertex of G_e is mapped to v. Thus the corresponding copy of G is mapped as requested.

From (iv) and the minimality of H we obtain

(v) *H is connected.*
(vi) *All vertices of H are special.*

Proof. This directly follows from (iv) by considering G to be a closed quasi-alternating walk of length g.

Now, the theorem follows directly from (iii), (v), (vi) and Theorem 5. □

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