Discrete Mathematics 240 (2001) 123–143

DISCRETE MATHEMATICS

www.elsevier.com/locate/disc

Choosability conjectures and multicircuits

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Received 16 November 1998; revised 22 May 2000; accepted 19 June 2000

Abstract

This paper starts with a discussion of several old and new conjectures about choosability in graphs. In particular, the list-colouring conjecture, that $\mathrm{ch}' = \chi'$ for every multigraph, is shown to imply that if a line graph is $(a:b)$-choosable, then it is $(ta:tb)$-choosable for every positive integer t. It is proved that $\mathrm{ch}(H^2)=\chi(H^2)$ for many "small" graphs H, including inflations of all circuits (connected 2-regular graphs) with length at most 11 except possibly length 9; and that $\mathrm{ch}''(C)=\chi''(C)$ (the total chromatic number) for various multicircuits C, mainly of even order, where a *multicircuit* is a multigraph whose underlying simple graph is a circuit. In consequence, it is shown that if any of the corresponding graphs H^2 or $T(C)$ is $(a:b)$-choosable, then it is $(ta:tb)$-choosable for every positive integer t. © 2001 Elsevier Science B.V. All rights reserved.

Keywords: Graph colouring; List-colouring conjecture; List-edge-colouring conjecture; List-total- colouring conjecture; List-square-colouring conjecture; Choosability conjectures; Total choosability; List chromatic number; List total chromatic number; Inflation of a graph; Square of a graph; Multicircuit

1. Introduction

Let $G=(V,E)$ be a multigraph with vertex-set $V(G)=V$ and edge-set $E(G)=E$. Let $f: V\cup E \to \mathbb{N}$ be a function into the positive integers. We say that G is *totally-f-choosable* if, whenever we are given sets ('lists') A_x of 'colours' with $|A_x| = f(x)$ for each $x \in V \cup E$, we can choose a colour $c(x) \in A_x$ for each element x so that no two adjacent vertices or adjacent edges have the same colour, and no vertex

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[1] Part of this work was carried out while the first author was visiting Nottingham, funded by Visiting Fellowship Research Grant GR/L54585 from the Engineering and Physical Sciences Research Council. The work of this author was also partly supported by grants 96-01-01614 and 97-01-01075 of the Russian Foundation for Fundamental Research.

PII: S0012-365X(00)00371-X

has the same colour as an edge incident with it; in this case we say loosely that G can be *totally coloured from its lists*. The *list total chromatic number* $\mathrm{ch}''(G)$ of G is the smallest integer k such that G is totally-f-choosable when $f(x)=k$ for each x. The *list (vertex) chromatic number* $\mathrm{ch}(G)$, and the *list edge chromatic number* (or *list chromatic index*) $\mathrm{ch}'(G)$, are defined similarly in terms of colouring vertices alone, or edges alone, respectively; and so are the concepts of *(vertex-)f-choosability* and *edge-f-choosability*. The ordinary vertex, edge and total chromatic numbers of G are denoted by $\chi(G)$, $\chi'(G)$ and $\chi''(G)$. Of course, multiple edges are irrelevant to vertex-colourings. We shall denote the *simple* line graph and total graph of G by $L(G)$ and $T(G)$, respectively. Then $\mathrm{ch}'(G)=\mathrm{ch}(L(G))$, $\mathrm{ch}''(G)=\mathrm{ch}(T(G))$, $\chi'(G)=\chi(L(G))$ and $\chi''(G)=\chi(T(G))$.

Clearly $\mathrm{ch}(G)\geqslant\chi(G)$. It is easy to see (by considering complete-bipartite graphs, cf. [2,10]) that there is no upper bound for $\mathrm{ch}(G)$ in terms of $\chi(G)$ in general. In contrast, the first of the following conjectures was made independently by Vizing, by Gupta, by Albertson and Collins, and by Bollobás and Harris (see [4,5]), and the second was made in [1].

The List-Edge-Colouring Conjecture (LECC). For every multigraph G, $\mathrm{ch}'(G)=\chi'(G)$.

The List-Total-Colouring Conjecture (LTCC). For every multigraph G, $\mathrm{ch}''(G)=\chi''(G)$.

If H is a graph, define its *square* H^2 to be the graph with the same vertex-set as H in which two vertices are adjacent if their distance apart in H is at most 2. Note that if H is obtained by placing a vertex in the middle of every edge of a multigraph G, then $H^2=T(G)$. Thus the following conjecture (LSCC) implies the LTCC; indeed, the LTCC is equivalent to the special case of the LSCC for bipartite graphs in which every vertex in one partite set has degree 2.

The List-Square-Colouring Conjecture (LSCC). For every graph G, $\mathrm{ch}(G^2)=\chi(G^2)$.

As a result of the work of Galvin [3], Peterson and Woodall [9] and Woodall [11], the LECC is now known to hold for every graph in which every block is bipartite or a multicircuit or has at most four vertices or has underlying simple graph of the form $K_{1,1,p}$, where a *multicircuit* is a multigraph whose underlying simple graph is a circuit. In contrast with the LECC, there is little hard evidence in support of the LTCC or LSCC. An easy inductive argument proves the LTCC and the LSCC for any multigraph whose underlying simple graph G_0 is a forest; indeed, if G is such a multigraph then $\mathrm{ch}''(G)=\chi''(G)=\Delta(G)+1$ and $\mathrm{ch}(G^2)=\chi(G^2)=\Delta(G_0)+1$. Because $T(K_3)=L(K_4)$ (the octahedron), if G is a multigraph with underlying simple graph K_3 then there is a multigraph H with at most four vertices such that $T(G)=L(H)$, and so the truth of the LECC for H (proved in [9]) implies the truth of the LTCC for G; it follows that the LTCC holds for multigraphs with at most three vertices. If G is bipartite then Galvin's result that $\mathrm{ch}'(G)=\Delta(G)$ implies that $\mathrm{ch}''(G)\leqslant\Delta(G)+2$, and this proves the LTCC for any bipartite graph G for which $\chi''(G)>\Delta(G)+1$.

In Section 3 we define the *inflation* of a graph, and prove the LSCC for every inflation of a graph with at most seven vertices. In Section 4 we prove the LSCC for inflations of all circuits with length at most 11 except length 9, and hence prove the LTCC for multicircuits of orders 3, 4 and 5. In Section 5 we prove the LTCC for a reasonably wide class of multicircuits of even order. The results that we have obtained all support the following conjecture, which is discussed further in Corollary 2.4 below.

Conjecture. For a multicircuit C with n vertices, m edges and maximum degree Δ,

$$\mathrm{ch}''(C) = \chi''(C) = \max\left\{\Delta + 1, \left\lceil \frac{m}{\lfloor \frac{1}{2}n \rfloor} \right\rceil, \left\lceil \frac{m+n}{\lfloor \frac{2}{3}n \rfloor} \right\rceil \right\}. \tag{1.1}$$

We believe that we have recently proved this conjecture [6,7]. In the present paper we are interested in a wider range of conjectures, which we now explore.

2. The $(a:b)$-choosability conjectures

If G is a (simple) graph, let $G_{(t)}$ be the graph obtained from G by replacing each vertex v of G by a copy H_v of the complete graph K_t, with $x \in H_v$ being adjacent to $y \in H_w$ if and only if v and w are adjacent in G. In the terminology of Section 3 below, $G_{(t)}$ is a *uniform inflation* of G. If G is a multigraph, let $G'_{(t)}$ be the multigraph obtained from G by replacing each edge of G by t parallel edges; $G'_{(t)}$ is a *uniform edge inflation* of G. Clearly $L(G'_{(t)}) = L(G)_{(t)}$.

We say that a graph $G = (V, E)$ is $(a:b)$-*choosable* if, whenever each vertex is assigned a list of a colours, we can give each vertex a set of b colours from its list in such a way that adjacent vertices get disjoint sets of colours; so $(a:1)$-*choosable* means the same as a-*choosable*. It is easy to see that G is $(a:b)$-choosable if $G_{(b)}$ is a-choosable, but it is not clear whether the converse holds; we conjecture that it does. Erdös et al. [2] asked whether, for $a, b, t \in \mathbb{N}$, every graph that is $(a:b)$-choosable is necessarily $(ta:tb)$-choosable. Since it appears that no counterexample to this has been found in over 20 years, perhaps the time has come to state it as a conjecture:

The Weak $(a:b)$-Choosability Conjecture (Weak $(a:b)$-CC). For all $a, b, t \in \mathbb{N}$, if a graph G is $(a:b)$-choosable, then G is $(ta:tb)$-choosable.

The Strong $(a:b)$-Choosability Conjecture (Strong $(a:b)$-CC). For all $a, b, t \in \mathbb{N}$, if a graph G is $(a:b)$-choosable, then $G_{(t)}$ is $(ta:b)$-choosable.

The $(a:b)$-Choosability Equivalence Conjecture ($(a:b)$-CEC). For all $a, b \in \mathbb{N}$, a graph G is $(a:b)$-choosable if and only if $G_{(b)}$ is a-choosable.

It is easy to see that the strong $(a:b)$-CC implies the weak $(a:b)$-CC, and if the $(a:b)$-CEC is true then the other two conjectures are equivalent. For certain families of graphs satisfying $\mathrm{ch} = \chi$, all three conjectures are true:

Theorem 2.1. *Let G be a graph such that $\mathrm{ch}(G_{(t)}) = \chi(G_{(t)})$ for all $t \in \mathbb{N}$. Then all three $(a:b)$-choosability conjectures hold for G.*

Proof. We claim that the following three statements are equivalent:

(i) G is $(a:b)$-choosable;
(ii) $\mathrm{ch}(G_{(b)}) \leqslant a$;
(iii) $\chi(G_{(b)}) \leqslant a$.

For, it is easy to see that (ii)⇒(i) (as already remarked) and (i) ⇒(iii) (by giving every vertex the same list of a colours), and (ii) and (iii) are equivalent by the hypothesis of the Theorem. Thus the $(a:b)$-CEC holds for G. But it is obvious that $\chi(G_{(tb)}) \leqslant t\chi(G_{(b)})$ (by splitting $G_{(tb)}$ into t copies of $G_{(b)}$, each coloured with a different set of $\chi(G_{(b)})$ colours), and so $\mathrm{ch}(G_{(tb)}) \leqslant t\,\mathrm{ch}(G_{(b)})$ by the hypothesis of the Theorem. Hence if G is $(a:b)$-choosable then $\mathrm{ch}((G_{(t)})_{(b)}) = \mathrm{ch}(G_{(tb)}) \leqslant t\,\mathrm{ch}(G_{(b)}) \leqslant ta$, and so $G_{(t)}$ is $(ta:b)$-choosable since (ii)⇒(i) for $G_{(t)}$. The strong and weak $(a:b)$-CCs immediately follow for G. □

For line graphs, Theorem 2.1 implies that all three conjectures would follow from the LECC:

Corollary 2.2. *Let G be a multigraph such that $\mathrm{ch}'(G'_{(t)}) = \chi'(G'_{(t)})$ for all $t \in \mathbb{N}$. Then all three $(a:b)$-choosability conjectures hold for $L(G)$.*

This holds because $\mathrm{ch}'(G'_{(t)}) = \chi'(G'_{(t)})$ is just another way of saying that $\mathrm{ch}(L(G)_{(t)}) = \chi(L(G)_{(t)})$; and together with results from [3,9,11] it implies that the three $(a:b)$-choosability conjectures hold for $L(G)$ whenever G is a multigraph in which every block is bipartite or a multicircuit or has at most four vertices or has underlying simple graph of the form $K_{1,1,p}$. There seems to be no similar way of rewriting the statement that $\mathrm{ch}(T(G)_{(t)}) = \chi(T(G)_{(t)})$, and so the truth of the LTCC would not apparently imply the truth of the $(a:b)$-choosability conjectures for total graphs. (Uniform inflations of line graphs are line graphs, but uniform inflations of total graphs are not necessarily total graphs.) However, $(G^2)_{(t)} = (G_{(t)})^2$, and so the truth of the LSCC would imply the truth of the conjectures for squares of graphs:

Corollary 2.3. *Let G be a graph such that $\mathrm{ch}(G^2_{(t)}) = \chi(G^2_{(t)})$ for all $t \in \mathbb{N}$. Then all three $(a:b)$-choosability conjectures hold for G^2.*

For total graphs of multicircuits, we have the following more specialized result, which will be of use in Sections 4 and 5 below.

Corollary 2.4. *Let C be a multicircuit with n vertices, m edges and maximum degree Δ such that*

$$\mathrm{ch}(T(C)_{(t)}) \leqslant \max\left\{ t(\Delta+1), \left\lceil \frac{tm}{\lfloor \frac{1}{2}n \rfloor} \right\rceil, \left\lceil \frac{t(m+n)}{\lfloor \frac{2}{3}n \rfloor} \right\rceil \right\} \tag{2.1}$$

for all $t \in \mathbb{N}$. *Then* (1.1) *holds*, *equality holds in* (2.1), *and all three* $(a:b)$*-choosability conjectures hold for* $T(C)$.

Proof. It is clear that the RHS of (2.1) is a lower bound for $\chi(T(C)_{(t)})$, since: $T(C)_{(t)}$ contains a clique of $t(\Delta+1)$ vertices; at most $\lfloor \frac{1}{2}n \rfloor$ of the tm vertices of $T(C)_{(t)}$ corresponding to edges of C can be given the same colour; and at most $\lfloor \frac{2}{3}n \rfloor$ of all $t(m+n)$ vertices of $T(C)_{(t)}$ can be given the same colour. Thus (2.1) implies that $\mathrm{ch}(T(C)_{(t)}) = \chi(T(C)_{(t)}) =$ the RHS of (2.1). Hence equality holds in (2.1), (1.1) holds (taking $t = 1$), and the truth of the $(a:b)$-choosability conjectures for $T(C)$ follows from Theorem 2.1. □

3. The choosability of inflations of small graphs

Let G and H be (simple) graphs such that $V(G) = \{v_1, \ldots, v_n\}$. We say that H is an *inflation* of G if $V(H)$ can be written as a disjoint union $V(H) = V_1 \cup \cdots \cup V_n$ in such a way that if $x \in V_i$ and $y \in V_j$ then $xy \in E(H)$ if and only if $i = j$ or $v_iv_j \in E(G)$. So the uniform inflation $G_{(t)}$, defined in Section 2, is an inflation of G in which $|V_i| = t$ for all i; but in general the sets V_i may be of unequal size and some may be empty, so that if F is an induced subgraph of G then any inflation of F is also an inflation of G.

In this section we shall prove that $\mathrm{ch}(H) = \chi(H)$ if H is an inflation of a graph with at most five vertices, and $\mathrm{ch}(J^2) = \chi(J^2)$ if J is an inflation of a graph G with at most seven vertices (although the proof for the case $G = C_7$ will be postponed until the next section). It follows immediately from this and Theorem 2.1 and Corollary 2.3, with no further proof needed, that all three $(a:b)$-choosability conjectures hold for all such graphs H and J^2.

We write $\omega(H)$ for the order of a largest clique of H, and $\omega_H(v)$ for the order of a largest clique containing vertex v. The following lemma, although specialized, will be very useful. (Cf. Theorem 3 of [8].)

Lemma 3.1. *Let* H *be an arbitrary graph*, *and suppose that every vertex* v *of* H *is given a list* A_v *of at least* $\omega_H(v)$ *colours*, *in such a way that nonadjacent vertices always get disjoint lists. Then* H *can be coloured from its lists.*

Proof. We use Hall's theorem on distinct representatives to show that we can give distinct colours to all the vertices. To that end, let $X \subseteq V(H)$ and let $C = \bigcup_{v \in X} A_v$. We must show that $|C| \geqslant |X|$. For each $c \in C$, let $m(c)$ be the number of $v \in X$ such that $c \in A_v$. By hypothesis, these $m(c)$ vertices form a clique (not necessarily maximal), and so $m(c) \leqslant \omega_H(v) \leqslant |A_v|$ whenever $c \in A_v$. Hence, writing $\sum'$ for the sum over all pairs (v, c) such that $v \in X$ and $c \in A_v$,

$$\left| \bigcup_{v \in X} A_v \right| = |C| = \sum{}' \, 1/m(c) \geqslant \sum{}' \, 1/|A_v| = |X|.$$

By Hall's theorem, the sets A_v: $v \in V(H)$ have a system of distinct representatives, and so we can give all vertices distinct colours from their lists. □

The following lemma will also be very useful, although its applicability is limited since its hypothesis implies that G does not contain three mutually nonadjacent vertices. (Indeed, it is not difficult to see that the hypothesis is equivalent to G containing neither $\bar{K}_3$ nor P_4 as an induced subgraph, a fact that we shall use in the proof of Lemma 3.5.)

Lemma 3.2. *Let G be a graph such that every pair of nonadjacent vertices has nonempty intersection with every maximal clique, and let H be an inflation of G. Suppose every vertex v of H is given a list A_v of at least $\omega_H(v)$ colours. Then H can be coloured from its lists.*

Proof. We prove the result by induction on $|V(H)|$, noting that it is obvious if $|V(H)| = 1$; so suppose $|V(H)| \geqslant 2$. If each two nonadjacent vertices have disjoint lists, then the result follows by Lemma 3.1. So we may suppose that some two nonadjacent vertices x, y of H have the same colour c in their lists. Give colour c to x and y, and let $H^* := H - \{x, y\}$ with c deleted from all lists. Then every $v \in V(H^*)$ still has a list of at least $\omega_{H^*}(v) = \omega_H(v) - 1$ colours, and the result follows by induction. □

We write $N(v)$ for the set of neighbours of a vertex $v \in V(G)$, and $d(v) = |N(v)|$ for its degree.

Lemma 3.3. *Let G be a graph with at most 5 vertices, other than C_5, and let H be an inflation of G. Suppose every vertex v of H is given a list A_v of at least $\omega_H(v)$ colours. Then H can be coloured from its lists.*

Proof. Suppose if possible that H is a minimal counterexample to the lemma, and that (for this H) G has as few vertices as possible. Clearly G is connected and $|V(G)| \geqslant 2$. We make two observations.

(a) If $N(v_i)$ is a clique for some $v_i \in V(G)$ (in particular, if $d(v_i) = 1$) and V_i is the subset of $V(H)$ corresponding to v_i, then we can colour all vertices of $H - V_i$ by the minimality of H, and then colour all vertices of V_i from their lists; thus G can contain no such vertex v_i.

(b) If $N(v_i) \cup \{v_i\} = N(v_j) \cup \{v_j\}$ for some $v_i, v_j \in V(G)$ (necessarily adjacent), then H is an inflation of the smaller graph $G - v_i \cong G - v_j$; thus G cannot contain two such vertices.

We know from Lemma 3.2 that G must contain two nonadjacent vertices x, y disjoint from some maximal clique K. If K is a triangle or $|V(G)| \leqslant 4$, then (a) must be violated. So K must be an edge and $|V(G)| = 5$. If $G - \{x, y\} \cong K_1 \cup K_2$ then, by (a) and (b), $G \cong C_5$, which is explicitly ruled out in the statement of the lemma. The only other possibility is that $G - \{x, y\} \cong P_3$ (the path with 3 vertices), in which case (a) and (b) imply $G \cong K_{2,3}$.

We now sketch a direct proof for $G \cong K_{2,3}$. Let the two partite sets of G be $\{u_1, u_2\}$ and $\{v_1, v_2, v_3\}$ with the corresponding subsets of $V(H)$ being denoted by the corresponding capital letters. If some colour c is present in the lists of vertices $x \in U_1$ and $y \in U_2$, then we give colour c to x, y, remove it from all other lists, and we get a contradiction because the result is assumed to hold for the smaller graph $H - \{x, y\}$. Thus no colour can be present on (vertices in) both U_1 and U_2. In a similar way, no colour can be present on all three sets V_i. Clearly every colour present on V_i must be present on U_1 or U_2, otherwise we could colour a vertex of V_i and remove it.

If no colour is present on more than one set V_i then the result holds by Lemma 3.1. So suppose w.l.o.g. colour c is present on V_1, V_2 and U_1 (only). If some colour c' is present on V_3 and U_2, then we can use c or c' to colour a vertex x_i in each set V_i, and we get a contradiction since the result holds for $H - \{x_1, x_2, x_3\}$. But if no colour is present on V_3 and U_2, then it makes no difference if we delete all edges between these two sets; then the resulting graph is an inflation of C_4 plus a pendant edge, for which the result has already been proved. □

Theorem 3.4. *Let G be a graph with at most 5 vertices, and let H be an inflation of G. Then $\mathrm{ch}(H) = \chi(H) = \omega(H)$ if $G \neq C_5$, and $\mathrm{ch}(H) = \chi(H) = \max\{\omega(H), \lceil \frac{1}{2}|V(H)| \rceil\}$ if $G = C_5$.*

Proof. This follows immediately from Lemma 3.3 if $G \neq C_5$. If $G = C_5$, then $H = L(C)$ for some multicircuit C with 5 vertices, $|V(H)|$ edges and maximum degree $\omega(H)$, and the result follows immediately from Theorem 1 of [11]. □

We now extend these results to inflations of squares of small graphs. We can use Lemma 3.2 as it stands, but we need the following analogue of Lemma 3.3.

Lemma 3.5. *Let G be a graph with at most 7 vertices, other than C_7, and let H be an inflation of G^2. Suppose every vertex v of H is given a list A_v of at least $\omega_H(v)$ colours. Then H can be coloured from its lists.*

Proof. Suppose if possible that H is a minimal counterexample to the lemma and that (for this H) G has as few vertices as possible. Clearly G is connected and $|V(G)| \geqslant 2$.

We are indebted to Fred Galvin for the following argument, which corrects errors in our proof of the lemma. Let F be the induced subgraph of G^2 with vertex-set $V(F) = \{v_i \in V(G)\colon V_i \text{ is nonempty}\}$, where V_i is the subset of $V(H)$ corresponding to v_i. Thus H is an inflation of F, which plays the role of the graph G in Lemma 3.3. We make two observations.

(a) For exactly the same reason as in Lemma 3.3(a), for all $v_i \in V(F)$, $N_F(v_i)$ is not a clique.

(b) For each $v_i \in V(G)$, $d_G(v_i) \geqslant 2$. This follows from (a) if $v_i \in V(F)$. If $v_i \notin V(F)$, then, since G is a minimal counterexample, v_i must be needed to establish some edge in F, and so $d_G(v_i) \geqslant 2$ in this case too.

By Lemma 3.2 and the remark before it, F contains either three independent vertices or an induced P_4. But if G^2 contained three independent vertices then (b) would force G to have at least nine vertices, a contradiction. Hence F contains an induced P_4, say $P: bcef$. Note that no vertex outside P can be adjacent in G to two vertices of P that do not occur consecutively in P, since otherwise P would not be an induced path in G^2. Let Q be a shortest path in G containing the vertices b, c, e, f in that order, which must exist by the previous sentence since P is a path in G^2. Note that b has only one neighbour in Q; hence, by (b), b is joined in G to some vertex a not in Q. Likewise, f is joined to some vertex g not in Q and distinct from a. Since $|V(G)| \leqslant 7$, there is at most one vertex in Q that is not in P. But bc and ce cannot both be edges in G, otherwise be would be an edge in F; and similarly ce and ef cannot both be edges in G. It follows that $|V(G)| = 7$ and c and e have a unique common neighbour d in G; then $bcdef$ is an induced P_5 in G. Furthermore, $E(G)$ consists of the 6 edges of the path $abcdefg$ together with some subset of $\{ac, ad, ag, dg, eg\}$.

By construction, $N_F(b) \subseteq \{a, c, d, g\}$. First, suppose ag is not in $E(G)$. It follows that $N_F(b) \subseteq \{a, c, d\}$. Also, by observation (b), either ac or ad is in $E(G)$. Thus $\{a, c, d\}$ induces a clique in G^2, and $N_F(b)$ induces a clique in F, contradicting observation (a) at the vertex b. Hence $ag \in E(G)$.

Next, suppose $dg \in E(G)$. Then $\{a, c, d, g\}$ induces a clique in G^2, again contradicting (a) at b. Hence dg is not in $E(G)$; by symmetry, neither is ad.

Now, if both ac and eg were in $E(G)$, that would again make $N_F(b)$ a clique, contradicting (a). Thus $E(G)$ consists of the 7 edges of the circuit $abcdefga$ together with at most one of ac and eg. Since G is not C_7, it must contain exactly one of those edges. Relabelling the vertices, we find that G consists of a C_7 with vertices $v_1, \ldots, v_7$ in that order, plus a chord $v_1 v_3$.

We now sketch a direct proof of the lemma for this case. For $i \in \{1, 2, 3\}$ let $\mathscr{K}_i$ be the family of vertex-sets of all maximal cliques in H containing V_i, and

$$\mathscr{K}_4 := \{V_1 \cup V_2 \cup V_3 \cup V_4, V_4 \cup V_5 \cup V_6\}$$

and

$$\mathscr{K}_5 := \mathscr{K}_6 := \mathscr{K}_7 := \{V_5 \cup V_6 \cup V_7\}.$$

To facilitate the induction we shall suppose only that each vertex $v \in V_i$ is given a list A_v of at least f_i colours, where

$$f_i := \max\{|K| : K \in \mathscr{K}_i\}.$$

We shall prove that in this case H can be coloured from its lists, and this will complete the proof of Lemma 3.5 since clearly $f_i \leqslant \omega_H(v)$ for each i and $v \in V_i$.

The proof now follows the arguments of Lemmas 3.1 and 3.2. Suppose first that each two nonadjacent vertices of H have disjoint lists. If $X \subseteq V(H)$ induces a clique in H, then X meets some set V_i such that $f_i \geqslant |X|$, so that $|\bigcup_{v\in X} A_v| \geqslant f_i \geqslant |X|$. (This follows directly from the definition of $\mathscr{K}_i$ if X intersects V_1, V_2 or V_3, and it is easy to see if $X \subseteq V_4 \cup V_5 \cup V_6 \cup V_7$.) If on the other hand X contains nonadjacent vertices $v_j \in V_j$ and $v_k \in V_k$ where $j < k$, then $j \in \{1,2,3,4\}$, $k \in \{5,6,7\}$ and

$$\left|\bigcup_{v\in X} A_v\right| \geqslant f_j + f_k \geqslant |V_1 \cup V_2 \cup V_3 \cup V_4| + |V_5 \cup V_6 \cup V_7| \geqslant |X|.$$

Thus by Hall's theorem the sets A_v: $v \in V(H)$ have a system of distinct representatives, and so we can give all vertices distinct colours from their lists.

So we may suppose that some two nonadjacent vertices $x \in V_j$ and $y \in V_k$ have the same colour c in their lists. Then $\{j,k\}$ is one of the following:

$$\{1,5\}, \quad \{2,5\}, \quad \{2,6\}, \quad \{3,6\}, \quad \{4,7\}. \tag{3.1}$$

Choose $\{j,k\}$ to be $\{2,5\}$ or $\{2,6\}$ only if there is no other possibility, give colour c to x, y, and let $H^* := H - \{x,y\}$ with c deleted from all lists. It is easy to verify that if colour c was present in the list A_v of some vertex $v \in V_i$, and $f_i = |V(K)|$ where $V_i \subseteq K \in \mathscr{K}_i$, then K contains x or y, so that v still has a list of at least $f_i^* = f_i - 1$ colours. (For example, if $i{=}3$ then $f_i{=}\sum |V_h|$ where the sum is over all $h \in \{7,1,2,3\}$ or $\{1,2,3,4\}$ or $\{3,4,5\}$ or $\{3,5,7\}$ (whichever gives the largest sum), and each of these sets intersects all five pairs (3.1) except that $\{2,6\}\cap\{3,4,5\}=\{2,6\}\cap\{3,5,7\}=\emptyset$. But if $i=3$ then $\{j,k\} \neq \{2,6\}$, since if c was present on vertices in V_2, V_3 and V_6 then we would have chosen $\{j,k\} = \{3,6\}$ and not $\{2,6\}$.) Thus the result follows by induction. □

Theorem 3.6. *Let G be a graph with at most 7 vertices, and let H be an inflation of G^2. Then* $\mathrm{ch}(H){=}\chi(H){=}\omega(H)$ *if $G \neq C_7$, and* $\mathrm{ch}(H){=}\chi(H){=}\max\{\omega(H), \lceil \frac{1}{2}|V(H)|\rceil\}$ *if $G = C_7$.*

Proof. This follows immediately from Lemma 3.5 if $G \neq C_7$. For C_7, the result will be proved in the next section. □

4. Inflations of circuits of small order

The truth of the LSCC $\mathrm{ch}(G^2) = \chi(G^2)$ follows from Theorem 3.6 when G is an inflation of C_n ($n \leqslant 7$), although the proof for the case $n = 7$ has not yet been given. In this section we shall prove cases 7, 8, 10 and 11. We shall also lay the groundwork for the results about total choosability in Section 5.

Throughout this section H will be an inflation of C_n^2 with $N = |V(H)|$ vertices. Thus we can write $V(H) = \bigcup_{i=0}^{n-1} Z_i$, where $z \in Z_i$ is adjacent to $z' \in Z_j$ ($z' \neq z$) if and only if $|i - j| \leqslant 2$ (modulo n). We allow some sets Z_i to be empty, but not all. For convenience we shall refer to the direction of increasing subscripts as *clockwise*, and take it to be left-to-right in diagrams. Subscripts are to be taken modulo n throughout.

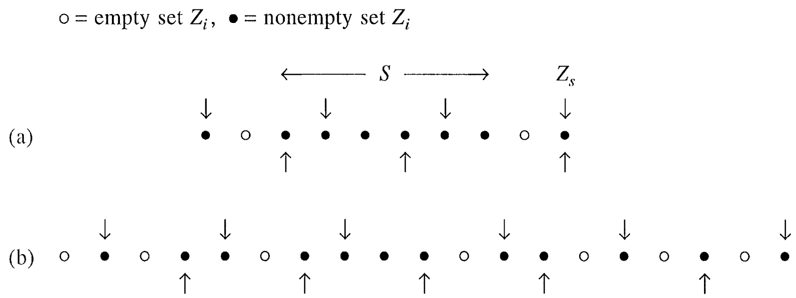

Fig. 1. The construction of kernels of D.

Now let D be the digraph with $V(D)=V(H)$ in which $z \in Z_i$ is joined by an arc to $z' \neq z$ if and only if $z' \in Z_{i-2} \cup Z_{i-1} \cup Z_i$; so each edge of H between two vertices in the same set Z_i is replaced by two oppositely oriented arcs in D, and every other edge of H is oriented anticlockwise. We refer to H and D as, respectively, an *IS-circuit* and an *IS-dicircuit* of *length n*, *IS* standing for *inflated squared.*

If $Z_i = Z_{i+l+1} = \emptyset$ and $Z_{i+1}, \ldots, Z_{i+l}$ are all nonempty, then we refer to $\{Z_{i+1}, \ldots, Z_{i+l}\}$ as a *segment* of D (or of H) of *length l*. For $j \in \{0, 1, 2\}$, a *j-segment* is a segment of length $l \equiv j$ (modulo 3). Recall that a *kernel* of a digraph D is a set K of nonadjacent vertices such that every vertex in $V(D) \backslash K$ is joined by an arc to at least one vertex in K. The following lemma is fundamental.

Lemma 4.1. *Suppose $Z_i = \emptyset$ for at least one i. Then D fails to have a kernel if and only if at least one of Z_i and Z_{i+1} is nonempty for each i* (mod n), *and D has no* 0-*segments and an odd number of* 1-*segments.*

Proof. Consider the following algorithm, which attempts to construct a kernel $K = \{k_1, \ldots, k_r\}$ of D. Choose k_1 to be a vertex in some nonempty set Z_s. Having chosen $k_1, \ldots, k_j$, where $k_j \in Z_p$, say, choose $k_{j+1} \in Z_q$ where Z_q is the first set after Z_{p+2} in clockwise order such that $Z_q \neq \emptyset$. If k_{j+1} is adjacent or equal to k_1, set $r := j$ and stop. We note that if this happens and $k_{j+1} \in Z_q$ then $q = s-2$ or $s-1$ or s (mod n), and K is a kernel if and only if $q = s$.

If $Z_i = Z_{i+1} = \emptyset$ then the above algorithm constructs a kernel if we take Z_s to be the next nonempty set after Z_i in clockwise order, since it is then clear that we must have $q = s$ when the algorithm terminates. So let us assume that at least one of Z_i and Z_{i+1} is nonempty, for each i. If D has a 0-segment S, then take Z_s to be the next nonempty set after the end of S; Fig. 1(a) shows that no matter how the algorithm returns to the start of S, it must terminate with $q = s$. (The arrows in Fig. 1 point to the sets Z_i that contain a vertex of K, and there are two different possibilities, as shown by the down-arrows and the up-arrows, respectively.)

So let us assume that every segment of D is a 1-segment or a 2-segment. Fig. 1(b) shows two different ways in which the algorithm can pass along a section of D. Note that in every segment $\{Z_{i+1}, \ldots, Z_{i+l}\}$, K includes a vertex from every set Z_{i+j} such that $j \equiv h \pmod 3$, where $h=1$ or 2 depending on the segment. (If $l=1$ and $h=2$ then K includes no vertex from that segment.) Moreover, h changes from 1 to 2 or *vice versa* every time the algorithm leaves a 1-segment. Thus it will return to a suitably chosen starting-point Z_s if and only if the number of 1-segments is even. □

If every vertex of D is given a list of colours, we say that a colour is *present on* Z_i if it belongs to the list of at least one vertex in Z_i. For a colour c, let D_c denote the subdigraph of D induced by the set of all vertices with colour c in their lists. If D_c has a kernel K, then we call K *a kernel of* (*the colour-class of*) c. For $0 \leqslant i \leqslant n-1$, let $f_i := |Z_{i-2}| + |Z_{i-1}| + |Z_i|$, which is one more than the outdegree in D of the vertices in Z_i.

Lemma 4.2. *If each vertex in Z_i $(0 \leqslant i \leqslant n-1)$ is given a list of at least f_i colours, and for every colour c, every induced subdigraph of D_c has a kernel, then H can be coloured from its lists.*

Proof. This is straightforward to prove by induction: pick a colour c, let K be a kernel of D_c, colour all vertices of K with c, remove c from all lists, and apply the induction hypothesis to $H - K$. The details are left to the reader. □

We now deal with circuits of orders 7, 8 and 11. The significance of these orders is that, apart from values $n \leqslant 5$ for which the result has already been proved, these are precisely the orders n for which $\lceil \frac{1}{4}(n-2) \rceil = \lfloor \frac{1}{3}n \rfloor$; this equality is used in the proof of Theorem 4.4. Recall that $N = |V(H)|$.

Lemma 4.3. *If $n=7$, 8 or 11, and some induced subdigraph of D has no kernel, then there exists an i such that* $f_i < N/\lfloor \frac{1}{3}n \rfloor$ and $f_{i+1} < N/\lfloor \frac{1}{3}n \rfloor$.

Proof. We prove the contrapositive: we assume that there is no such i and deduce that every induced subdigraph of D has a kernel. We are indebted to one of the referees for the following argument, which is simpler than our own. It relies on the fact that if there is an r such that $Z_r = \emptyset$ (if $n=7$) or $Z_r = Z_{r+4} = \emptyset$ (if $n=8$ or 11), then every induced subdigraph of D has a kernel. This holds because it is easy to see that the existence of such an r implies that either two consecutive sets Z_h are empty, or the number of 1-segments is even, and in either case the conclusion follows from Lemma 4.1. For brevity write $I_j := \{j, j-1, j-2\}$.

If $n=7$ or 8, choose j and k so that $f_j \geqslant \frac{1}{2}N$, $f_k \geqslant \frac{1}{2}N$, and I_j and I_k are disjoint; this is possible with $k = j-3$ or $j-4$. Then $Z_r = \emptyset$ for each $r \notin I_j \cup I_k$. If $n=7$ this is all we need. If $n=8$ then either $Z_r = \emptyset$ for two consecutive values of r, or else $Z_r = Z_{r+4} = \emptyset$ for some r, and the result again follows.

If $n = 11$, choose h, j, k so that $f_h \geqslant \frac{1}{3}N$, $f_j \geqslant \frac{1}{3}N$, $f_k \geqslant \frac{1}{3}N$, and the sets I_h, I_j and I_k are pairwise disjoint. Then $Z_r = \emptyset$ for each $r \notin I_h \cup I_j \cup I_k$. Now it is again easy to see that either $Z_r = \emptyset$ for two consecutive values of r, or else $Z_r = Z_{r+4} = \emptyset$ for some r, and the result follows as before. □

Theorem 4.4. *If* $n = 7$, 8 *or* 11, *and each vertex in* Z_i $(0 \leqslant i \leqslant n-1)$ *is given a list of at least* $\max\{f_i, \lceil N/\lfloor \frac{1}{3}n \rfloor \rceil\}$ *colours, then the vertices of H can be coloured from their lists.*

Proof. We prove the result by induction on N. There are three cases.

Case 1: for some colour c, D_c has no kernel. Then, by Lemma 4.3, there exists an i such that $f_i < N/\lfloor \frac{1}{3}n \rfloor$ and $f_{i+1} < N/\lfloor \frac{1}{3}n \rfloor$. Since D_c has no kernel, D_c contains at least one vertex of $Z_j \cup Z_{j+1}$, for each j. Starting with $k_1 \in Z_s$, where $s := i+2$ if $D_c \cap Z_{i+2} \neq \emptyset$ and $s := i+3$ otherwise, construct a kernel K of $D_c - (Z_{s-2} \cup Z_{s-1})$ by the algorithm in the proof of Lemma 4.1. At each stage in the construction, if $k_j \in Z_p$ then $k_{j+1} \in Z_{p+3}$ or Z_{p+4}, and so $\frac{1}{4}(n-2) \leqslant |K| \leqslant \frac{1}{3}n$. Therefore $|K| = \lceil \frac{1}{4}(n-2) \rceil = \lfloor \frac{1}{3}n \rfloor$. Colour the elements of K with colour c, and let $H^* := H - K$ with c deleted from all lists. Then, in an obvious terminology, $N^*/\lfloor \frac{1}{3}n \rfloor = N/\lfloor \frac{1}{3}n \rfloor - 1$ and $f_j^* = f_j - 1$ whenever c was present on Z_j in H, except possibly if $j \in \{i, i+1\}$, when $f_j^* \leqslant f_j < N/\lfloor \frac{1}{3}n \rfloor$ so that $f_j^* \leqslant \lceil N^*/\lfloor \frac{1}{3}n \rfloor \rceil$. It follows that H^* satisfies the hypotheses of the theorem. We may therefore suppose inductively that H^* can be coloured from its lists, and hence so can H.

Case 2: for some colour c, D_c has a kernel but some induced subdigraph D_c' of D_c does not. Then D_c', and hence D_c, must contain at least one vertex of $Z_j \cup Z_{j+1}$, for each j. So, by the argument of Case 1, every kernel K of D_c satisfies $\frac{1}{4}n \leqslant |K| \leqslant \frac{1}{3}n$, so that $|K| = \lceil \frac{1}{4}n \rceil = \lfloor \frac{1}{3}n \rfloor$. We can now colour the elements of some kernel K with colour c, and proceed exactly as in Case 1, but without the need for any exceptional values of j.

Case 3: for every colour c, every induced subdigraph of D_c has a kernel. Then the result follows immediately from Lemma 4.2. □

We cannot expect to get such a simple result for other values of n. We now consider the case $n = 10$, which is the only other case we have been able to deal with. Let $I := \{0, 1, \ldots, 9\}$, $I_0 := \{0, 2, 4, 6, 8\}$, $I_1 := \{1, 3, 5, 7, 9\}$, $Y_0 := \bigcup_{i \in I_0} Z_i$, $Y_1 := \bigcup_{i \in I_1} Z_i$, $N_0 := |Y_0|$ and $N_1 := |Y_1|$, so that $N_0 + N_1 = N$. Suppose lists of colours are assigned to the vertices of H. For a colour c, let $I(c)$ denote the set of i such that c is present on Z_i. Using Lemma 4.1, it is not difficult to see that if c is a colour such that D_c has no kernel then $I(c) = I$ or I_0 or I_1; it is this small number of possibilities that makes the case $n = 10$ tractable. We shall say that H has *type* (x, y), where x is

0 if there is no colour c such that $I_0 \subseteq I(c)$,
1 if there is a colour c such that $I_0 = I(c)$ but none such that $I_0 \subsetneq I(c)$,
2 if there is a colour c such that $I_0 \subsetneq I(c)$,

and y is defined similarly with I_1 in place of I_0.

Theorem 4.5. *Suppose that* $n = 10$ *and that each vertex in* Z_i $(0 \leqslant i \leqslant 9)$ *is given a list of at least* l_i *colours, where* $l_i \geqslant f_i$. *Suppose also that*

(i) *if there is a colour* c *such that* $I_0 \subseteq I(c)$, *then*

$$\sum_{i \in I_0} l_i \geqslant \frac{5}{2} N_0; \tag{4.1}$$

(ii) *if there is a colour* c *such that* $I_1 \subseteq I(c)$, *then*

$$\sum_{i \in I_1} l_i \geqslant \frac{5}{2} N_1; \tag{4.2}$$

(iii) *if there are colours* c_0 *and* c_1 (*not necessarily distinct*) *such that* $I_0 \subseteq I(c_0)$ *and* $I_1 \subseteq I(c_1)$, *where at least one of these inclusions is strict, then*

$$l_i \geqslant \frac{1}{3} N \quad \textit{for each } i. \tag{4.3}$$

Then the vertices of H *can be coloured from their lists.*

Before proving Theorem 4.5, it will be convenient to prove a lemma.

Lemma 4.6. *Suppose that all the hypotheses of Theorem* 4.5 *are satisfied except that* (ii) *and* (iii) *are replaced by*

(ii′) *if there is a colour* c *such that* $I_1 \subseteq I(c)$ *then*

$$\sum_{i \in I_1} (l_i - f_i) \geqslant N_1 - 2. \tag{4.4}$$

Suppose moreover that there is no colour c *such that* $I_0 \subsetneq I(c)$. *Then the vertices of* H *can be coloured from their lists.*

Proof. Let H be a minimal counterexample to the Lemma. Suppose that H has type (x, y). We first deal with the possibility that $x = 0$.

Suppose first that $(x, y) = (0, 2)$. Let c be a colour such that $I_1 \subsetneq I(c)$. Then D_c has a kernel K, necessarily with $|K| = 3$, and we can choose K so that $|K \cap Y_0| = 1$ and $|K \cap Y_1| = 2$. Give colour c to the vertices of K, and let H^* be the graph obtained from H by deleting these three vertices and removing colour c from every list. Then each remaining vertex $v \in Z_i$ has a list of at least l_i^* colours, where $l_i^* := l_i - 1$ for each $i \in I(c)$ and $l_i^* := l_i$ for each other i. In an obvious notation, $f_i^* = f_i - 1$ for each $i \in I(c)$ by the definition of a kernel, and $f_i^* \leqslant f_i$ for each other i, and so

$$l_i^* - f_i^* \geqslant l_i - f_i \geqslant 0 \quad \text{for each } i \in I. \tag{4.5}$$

Thus $l_i^* \geqslant f_i^*$ for each i, and (4.4) remains satisfied since $N_1^* \leqslant N_1$. Hence H^* satisfies the hypotheses of the lemma. (Note that (4.1) does not apply, since there is no colour c with $I_0 \subseteq I(c)$.) Since H^* is not a counterexample to the lemma, H^* can be coloured from its lists, and hence so can H. This contradiction shows that $(x, y) \neq (0, 2)$.

If $(x, y) = (0, 0)$ then, for every colour c, every induced subdigraph of D_c has a kernel, and the result follows immediately from Lemma 4.2. If $(x, y) = (0, 1)$, then we relabel the sets Z_i so as to interchange I_0 and I_1, so that H has type $(1,0)$; the hypotheses of the lemma remain satisfied because the fact that H has type $(0,1)$ implies $N_1 \geqslant 5$ and so, by (4.4),

$$\sum_{i\in I_1} l_i \geqslant \sum_{i\in I_1} f_i + N_1 - 2 = N + 2N_1 - 2 \geqslant 3N_1 - 2 > \frac{5}{2}N_1;$$

hence (4.2) holds, and turns into (4.1) after the relabelling.

Thus from now on we may suppose that $x = 1$; that is, there exists a colour c_0 such that $I(c_0) = I_0$.

Suppose if possible that $N_1 = 0$. Since $\sum_{i\in I_0} f_i = N + N_0 = 2N_0$, it follows from (4.1) that $f_i < l_i$ for some i, say $i = 0$. Give colour c_0 to one vertex in each of the sets Z_2 and Z_6, and let H^* be the graph obtained from H by deleting these two vertices and removing colour c_0 from every list, with $l_i^* := l_i - 1$ for each i. Then, in an obvious notation, $f_0^* = f_0 \leqslant l_0 - 1 = l_0^*$, $f_i^* = f_i - 1 \leqslant l_i - 1 = l_i^*$ for all other even i, and $N_0^* = N_0 - 2$. So (4.1) holds for H^*. Thus H^* can be coloured from its lists, and hence so can H. This contradiction shows that $N_1 \neq 0$.

Therefore there is a colour c_1 such that $I(c_1) \cap I_1 \neq \emptyset$. We shall prove several claims about c_1.

Claim 1. *If $j \in I(c_1) \cap I_1$, then $I(c_1) \cap \{j-1, j+1\} \neq \emptyset$.*

Proof. If $I(c_1) \cap \{j-1, j+1\} = \emptyset$, then give colour c_1 to a vertex in Z_j and colour c_0 to vertices in Z_{j-3} and Z_{j+3}. Let H^* be the graph obtained from H by removing these three vertices and removing colours c_0 and c_1 from the lists of vertices adjacent to removed vertices of the same colour. Then every remaining vertex $v \in Z_i$ has at least l_i^* colours in its list, where $l_i^* := l_i$ if $i \in \{j-4, j+4\}$ and $l_i^* := l_i - 1$ otherwise. (Recall that colour c_0 was not present on any vertex in Z_{j-4} or Z_{j+4}, since $I(c_0) = I_0$.) Also $f_i^* = f_i$ if $i = j-4$, $f_i^* = f_i - 1$ otherwise, $N_0^* = N_0 - 2$, and $N_1^* < N_1$. Thus (4.5) holds, and (4.1) and (ii′) (in the statement of Lemma 4.6) hold for H^*. Thus H^* can be coloured from its lists, and hence so can H. This contradiction proves Claim 1. □

Claim 2. *If $j \in I(c_1) \cap I_0$, then $I(c_1) \cap \{j-2, j+2\} \neq \emptyset$.*

Proof. If $I(c_1) \cap \{j-2, j+2\} = \emptyset$, then give colour c_1 to a vertex in Z_j and colour c_0 to vertices in Z_{j-2} and Z_{j+4}. Let H^* be the graph obtained from H by removing these three vertices and removing colours c_0 and c_1 from the lists of vertices adjacent to removed vertices of the same colour. Then everything works with $l_j^* := l_j - 2$, $l_i^* := l_i$ if $i \in \{j-3, j+3, j+5\}$, and $l_i^* := l_i - 1$ otherwise. Also $f_j^* = f_j - 2$, $f_i^* = f_i$ if $i \in \{j-3, j+3\}$, $f_i^* = f_i - 1$ otherwise, $N_0^* = N_0 - 3$, and $N_1^* = N_1$. Thus (4.5) holds, and (4.1) and (ii′) hold for H^*. This contradiction proves Claim 2. □

Claim 3. *If $I(c_1) \cap \{2j, 2j+1\} = \emptyset$, then $2j+2 \notin I(c_1)$.*

Proof. If $2j+2 \in I(c_1)$, give colour c_1 to a vertex in Z_{2j+2}. Everything works with $l_i^* := l_i - 1$ and $f_i^* = f_i - 1$ if $i \in \{2j+2, 2j+3, 2j+4\}$, $l_i^* := l_i$ and $f_i^* = f_i$ otherwise, $N_0^* = N_0 - 1$, and $N_1^* = N_1$. □

Claim 4. *If $j \in I(c_1) \cap I_0$ and $j+2 \notin I(c_1)$, then $j+6 \notin I(c_1)$.*

Proof. Suppose $\{0,6\} \subset I(c_1)$ and $2 \notin I(c_1)$.

Case 1: $3 \in I(c_1)$. Give colour c_1 to vertices in Z_0, Z_3 and Z_6 and colour c_0 to vertices in Z_4 and Z_8. Everything works with $l_i^* := l_i - 2$ and $f_i^* = f_i - 2$ if $i \in \{0,4,6,8\}$, $l_i^* := l_i - 1$ and $f_i^* \leqslant f_i - 1$ for each other i, $N_0^* = N_0 - 4$, and $N_1^* < N_1$.

Case 2: $3 \notin I(c_1)$. By Claim 3, $4 \notin I(c_1)$. Give colour c_1 to vertices in Z_0 and Z_6 and colour c_0 to vertices in Z_4 and Z_8. Everything works with $l_i^* := l_i - 2$ and $f_i^* = f_i - 2$ if $i \in \{0,6,8\}$, $l_3^* := l_3$ and $f_3^* = f_3$, $l_i^* := l_i - 1$ and $f_i^* \leqslant f_i - 1$ for each other i, $N_0^* = N_0 - 4$, and $N_1^* = N_1$. □

We are now in a position to complete the proof of the lemma. By Claim 1 and the hypotheses of lemma, $1 \leqslant |I(c_1) \cap I_0| \leqslant 4$. Thus there is a $j \in I(c_1) \cap I_0$ such that $j+2 \notin I(c_1)$. By Claim 4, $j+6 \notin I(c_1)$ and $j+4 \notin I(c_1)$ (since $j+4 \in I(c_1)$ and $j+6 \notin I(c_1)$ would imply $j \notin I(c_1)$). By Claim 2, $j-2 \in I(c_1)$. By Claim 3, $j-3 \in I(c_1)$, and by Claim 1, $I(c_1) \subseteq \{j-3, j-2, j-1, j, j+1\}$. Give colour c_1 to vertices in Z_{j-3} and Z_j. Everything works with $l_i^* := l_i - 1$ and $f_i^* = f_i - 1$ if $i \in I(c_1)$, $l_i^* := l_i$ and $f_i^* \leqslant f_i$ otherwise, $N_0^* \leqslant N_0 - 1$ and $N_1^* < N_1$. This completes the proof of Lemma 4.6. □

Proof of Theorem 4.5. Let H be a minimal counterexample to the Theorem. If H has type $(0,0)$ or $(1,0)$, then it satisfies the hypotheses of Lemma 4.6, and so is not a counterexample to the Theorem. If H has type $(0,1)$ then relabel the sets Z_i so that H has type $(1,0)$, and then the same applies.

Suppose H has type $(1,1)$. Let c_0 and c_1 be colours such that $I(c_0)=I_0$ and $I(c_1)=I_1$. Give colour c_0 to vertices in Z_0 and Z_6 and colour c_1 to vertices in Z_3 and Z_9. Forming H^* in the obvious way, we find that all lists decrease in size by at most 1, each f_i decreases by at least 1, N_0 and N_1 each decrease by 2, and N decreases by 4, so that all conditions are still satisfied. The consequent contradiction shows that H does not have type $(1,1)$.

It follows that H has type $(2,0)$, $(2,1)$, $(2,2)$, $(1,2)$ or $(0,2)$. Suppose w.l.o.g. $N_0 \geqslant N_1$.

Suppose if possible that some colour c is present on every set Z_i. Since $\sum_{i \in I_1} f_i = N + N_1 \leqslant \frac{3}{2}N$, there exists a $j \in I_1$ such that $f_j \leqslant \frac{3}{10}N < \frac{1}{3}N$, so that $f_j < l_j$ by (4.3). Colour with c one vertex in each of the sets Z_{j+1}, Z_{j+4} and Z_{j+7}, and let H^* be the graph obtained from H by deleting these three vertices and removing colour c from every list. Then everything works with $l_i^* := l_i - 1$ for each i, since $f_j^* = f_j \leqslant l_j - 1 = l_j^*$, $f_i^* = f_i - 1 \leqslant l_i - 1 = l_i^*$ if $i \neq j$, $N_0^* = N_0 - 2$, $N_1^* = N_1 - 1$, and $N^* = N - 3$. Note that (4.2) holds because $I(c)=I$ implies $N_1\geqslant 5>3$, and so, by (4.3),

$$\sum_{i\in I_1} l_i^* = \sum_{i\in I_1}(l_i-1)\geqslant \frac{5}{3}N-5\geqslant \frac{10}{3}N_1-5>\frac{5}{2}N_1-\frac{5}{2}=\frac{5}{2}N_1^*. \tag{4.6}$$

So H^* satisfies the hypotheses of the theorem. Thus H^* can be coloured from its lists, and hence so can H. This contradiction shows that no colour is present on every set Z_i.

Suppose that there exists a colour c such that $I_0 \subsetneq I(c)$. Since we have just shown that $I(c)\neq I$, it follows that D_c has a kernel K, necessarily with $|K|=3$, and we can choose K so that $|K\cap Y_0|=2$ and $|K\cap Y_1|=1$. We now obtain a contradiction almost exactly as in the previous paragraph, since K contains vertices in the sets Z_{j+1}, Z_{j+4} and Z_{j+7} for some $j\in I_1\setminus I(c)$, and we can take $l_j^*:=l_j\geqslant f_j=f_j^*$ and $l_i^*:=l_i-1\geqslant f_i-1=f_i^*$ for each other i. (Note that there is no need to check (4.2) unless the hypothesis of (ii) is satisfied, which implies that $N_1\geqslant 5$ and that (4.3) holds; thus (4.6) holds with the first $=$ replaced by $>$.) This contradiction shows that there is no such colour c, so that H does not have type $(2,0)$, $(2,1)$ or $(2,2)$.

Hence H must have type $(0,2)$ or $(1,2)$, which means that there is a colour c such that $I_1 \subsetneq I(c)$. If H has type $(0,2)$ then we get a contradiction exactly as in the previous paragraph, with I_0 and I_1 interchanged. (There is no need to check (4.1).) So suppose H has type $(1,2)$. Then (4.1), (4.2) and (4.3) all hold. If $N_1\leqslant \frac{1}{3}N+1$, then, by (4.3),

$$\sum_{i\in I_1}(l_i-f_i)\geqslant \frac{5}{3}N-N-N_1\geqslant N_1-2,$$

and so the result follows from Lemma 4.6. So we may suppose that $N_1>\frac{1}{3}N+1$, which implies $N_0<\frac{2}{3}N-1$ and so $\frac{1}{3}N-1>\frac{1}{2}N_0-\frac{1}{2}$. Now we can repeat the argument of the previous paragraph but with N_0 and N_1 interchanged. The only possible problem would be (4.1), but this holds (when $N_0\geqslant 5>3$) because, by (4.3),

$$\sum_{i\in I_0} l_i^* \geqslant \sum_{i\in I_0}(l_i-1)\geqslant \frac{5}{3}N-5>\frac{5}{2}N_0-\frac{5}{2}=\frac{5}{2}N_0^*.$$

Thus we get the same contradiction, and this completes the proof of Theorem 4.5. □

In the next two theorems we summarize the implications of these results for the LSCC and the LTCC, respectively.

Theorem 4.7. *Let G be an inflation of C_n ($n\in\{7,8,10,11\}$), $H:=G^2$, $N:=|V(H)|$, and (if $n=10$) let N_0 and N_1 have the meanings in Theorem 4.5. Then*

$$\mathrm{ch}(H)=\chi(H)=\begin{cases}\max\{\omega(H),\lceil\frac{1}{2}N\rceil\} & \text{if } n=7 \text{ or } 8,\\ \max\{\omega(H),\lceil\frac{1}{2}N_0\rceil,\ \lceil\frac{1}{2}N_1\rceil,\ \lceil\frac{1}{3}N\rceil\} & \text{if } n=10,\\ \max\{\omega(H),\lceil\frac{1}{3}N\rceil\} & \text{if } n=11\end{cases}$$

and all three $(a:b)$-choosability conjectures hold for H.

Proof. If $n \in \{7, 8, 11\}$, then it is clear that $\mathrm{ch}(H) \geqslant \chi(H) \geqslant \max\{\omega(H), \lceil N/\lfloor \frac{1}{3}n \rfloor \rceil\} \geqslant \mathrm{ch}(H)$, the final inequality following from Theorem 4.4. If $n = 10$ then the analogous statement with the more complicated expression follows from Theorem 4.5. For all these values of n, since $H_{(t)} = G^2_{(t)}$ and $G_{(t)}$ is also an inflation of C_n, we can deduce that $\mathrm{ch}(H_{(t)}) = \chi(H_{(t)})$ for all $t \in \mathbb{N}$. The truth of the $(a:b)$-choosability conjectures now follows from Theorem 2.1. □

Theorem 4.8. *If C is a multicircuit of order $n = 3$, 4 or 5 with m edges and maximum degree Δ, and $t \in \mathbb{N}$, then* (2.1) *holds. Hence* (1.1) *holds, and all three* $(a:b)$*-choosability conjectures hold for $T(C)$.*

Proof. It suffices to prove (2.1), since the rest then follows immediately by Corollary 2.4.

If $n = 3$, then taking $H := T(C)_{(t)}$ in Theorem 3.6 gives $\mathrm{ch}(H) = \chi(H) = \omega(H)$; and it is easy to see that $\omega(H) = tm$, which equals the RHS of (2.1) since $m \geqslant \frac{1}{2}(m+n)$.

If $n = 4$, then taking $H := T(C)_{(t)}$ and $n := 8$ in Theorem 4.7 gives $\mathrm{ch}(H) = \chi(H) = \max\{\omega(H), \lceil \frac{1}{2}N \rceil\}$; and it is easy to see that this equals the RHS of (2.1), since here $n = 4$, $\omega(H) = t(\Delta + 1)$, $N = t(m + n)$ and $\frac{1}{2}(m+n) > \frac{1}{2}m$.

If $n = 5$, then taking $H := T(C)_{(t)}$ and $n := 10$ in Theorem 4.7 gives $\mathrm{ch}(H) = \chi(H) = \max\{\omega(H), \lceil \frac{1}{2}N_0 \rceil, \lceil \frac{1}{2}N_1 \rceil, \lceil \frac{1}{3}N \rceil\}$; and it is easy to see that this equals the RHS of (2.1), since here $n = 5$, $\omega(H) = t(\Delta + 1)$, $N_0 = 5t$, $N_1 = tm$ and $N = t(m + n)$. The result follows. □

5. The total choosability of multicircuits

In this section we prove (2.1) for a reasonably wide range of multicircuits of even order. However, to begin with we consider multicircuits of odd order as well. So throughout this section C will be a multicircuit of order n and H will be an induced subgraph of $T(C)_{(t)}$ for some $t \in \mathbb{N}$. Then H is an IS-circuit of length $2n$ with $V(H) = Z_0 \cup \cdots \cup Z_{2n-1}$ (in the terminology of the previous section). Assume that the even-numbered sets Z_i correspond to the vertices of C, so that $|Z_i| \leqslant t$ for each even i.

Suppose we are given nonnegative integers f_i, g_i and h_i $(0 \leqslant i \leqslant 2n-1)$ with the following properties. As in the previous section, $f_i = |Z_{i-2}| + |Z_{i-1}| + |Z_i|$ for each i. Suppose $g_i = \max\{|Z_{i-3}| - t, 0\}$ if i is even, $g_i = 0$ if i is odd and n is even, and if n is odd then $\sum_{i=1}^{(n-1)/2} g_{4i+1} \geqslant |Z_1| - t$. Finally, suppose that for each odd j there exist $2t$ vectors $v_{j,1}, \ldots, v_{j,2t}$ of length $2n - 2$ of the form $(0, \ldots, 0, 1, 1, 1, 1, 0, \ldots, 0)$ (that is, four consecutive 1's and the rest 0's) such that

$$(h_{j+2}, h_{j+3}, \ldots, h_{2n-1}, h_0, \ldots, h_{j-1}) \geqslant \sum_{i=1}^{2t} v_{j,i} \tag{5.1}$$

(meaning that each coordinate on the LHS is at least as large as the corresponding coordinate on the RHS), where subscripts on the LHS are taken modulo $2n$.

Lemma 5.1. *Suppose that every vertex in Z_i ($0 \leqslant i \leqslant 2n-1$) is given a list of at least $f_i + g_i + h_i$ colours. Then the vertices of H can be coloured from their lists.*

Proof. Let H be a counterexample such that $N = |V(H)|$ is as small as possible. As in the previous section, let D be the IS-dicircuit corresponding to H, and if c is any colour, let D_c be the subdigraph of D induced by the vertices that have colour c in their lists. There are two cases to consider.

Case 1: $|Z_i| \geqslant t+1$ for every odd value of i. Let c be any colour that is present on at least one odd-numbered set. If K is an independent set of vertices in D_c, we call a set Z_i *K-defective* if there is a vertex $v \in D_c \cap Z_i$ that is neither in K nor joined by an arc to a vertex of K. If the only K-defective sets are even-numbered sets Z_i such that $K \cap Z_{i-3} \neq \emptyset$, then we call K a *pseudokernel*. Suppose that D_c has a pseudokernel K. Give colour c to the vertices of K, and define $H^* := H - K$ with c removed from all lists. Let $f_i^* := f_i - 1$ if $K \cap (Z_{i-2} \cup Z_{i-1} \cup Z_i) \neq \emptyset$, $g_i^* := g_i - 1$ if Z_i is K-defective, and $f_i^* := f_i$, $g_i^* := g_i$ and $h_i^* := h_i$ in all other cases. Then it is easy to see that H^* satisfies the hypotheses of the lemma, and so can be coloured from its lists by the minimality of H. Thus so can H, and this contradiction shows that D_c cannot contain a pseudokernel.

Suppose first that c is not present on every odd-numbered set; then we will get a contradiction by showing that D_c must contain a pseudokernel. There is an odd number d such that c is present on Z_d but not on Z_{d+2} (subscripts modulo $2n$); hence c is present on Z_{d+1} and Z_{d+3}, since otherwise D_c would have a kernel by Lemma 4.1. Take $s = d+3$ and construct a set K by the algorithm in the proof of Lemma 4.1. Since c is present on Z_d, the algorithm must terminate by putting into K an element of Z_{d-2}, Z_{d-1} or Z_d. In the first case K is a pseudokernel (the only K-defective set being Z_{d+1}, since $D_c \cap Z_{d+2} = \emptyset$), and in the last two cases K is a kernel. In all cases we have a contradiction.

It follows that c must in fact be present on every odd-numbered set. If n is even, then we can easily form a pseudokernel by choosing a vertex from every set Z_{i+1} with i divisible by 4. So we may suppose that n is odd. If c were present on any even-numbered set Z_e, then we could form a pseudokernel by choosing a vertex from $D_c \cap Z_i$ for $i \in \{e, e+3, e+7, \ldots, e-7, e-3\}$ (modulo $2n$); this contradiction shows that c cannot be present on any even-numbered set. Since $\sum_{i=1}^{(n-1)/2} g_{4i+1} \geqslant |Z_1| - t > 0$, we can choose a j such that $g_{4j+1} > 0$. Let $I := \{1, 5, \ldots, 4j-3, 4j+3, 4j+7, \ldots, 2n-3\}$. For each $i \in I$, choose a vertex in $D_c \cap Z_i$ and colour it c. Let K be the set of vertices chosen. The only K-defective set is Z_{4j+1}. Define $H^* := H - K$ with c removed from all lists. Let $g_{4j+1}^* := g_{4j+1} - 1$, $f_i^* := f_i - 1$ if $K \cap (Z_{i-2} \cup Z_{i-1} \cup Z_i) \neq \emptyset$, and $f_i^* := f_i$, $g_i^* := g_i$ and $h_i^* := h_i$ in all other cases. Then it is easy to see that H^* satisfies the hypotheses of the lemma, and so can be coloured from its lists by the minimality of H. Thus so can H, and this contradiction completes the discussion of Case 1.

Case 2: $|Z_d| \leqslant t$ for some odd value of d. In this case we replace the n inequalities (5.1) by the single weaker inequality

$$(h_{d+2}, h_{d+3}, \dots, h_{2n-1}, h_0, \dots, h_{d-1}) \geqslant \sum_{i=1}^{z} v_{d,i}, \tag{5.2}$$

where $z := |Z_d| + |Z_{d+1}| \leqslant 2t$. Let the four consecutive 1's in $v_{d,z}$ be in positions corresponding to sets $Z_a, \dots, Z_{a+3}$. Let c be any colour that is present on Z_r, where $r = d$ if $Z_d \neq \emptyset$ and $r = d + 1$ otherwise. As before, we may suppose that D_c does not have a kernel.

Let a *quasikernel* be an independent set K of vertices of D_c such that there are no K-defective sets (as defined in Case 1) except possibly for some or all of the sets Z_{a+i}, $i \in \{0, 1, 2, 3\}$. Suppose that D_c has a quasikernel that meets $Z_d \cup Z_{d+1}$. Give colour c to all vertices in K, and define $H^* := H - K$ with c removed from all lists. Let $z^* := z - 1$, $h_i^* := h_i - 1$ if Z_i is K-defective, $f_i^* := f_i - 1$ if $K \cap (Z_{i-2} \cup Z_{i-1} \cup Z_i) \neq \emptyset$, and $f_i^* := f_i$, $g_i^* := g_i$ and $h_i^* := h_i$ in all other cases. Then it is easy to see that H^* satisfies the hypotheses of the lemma with (5.2) in place of (5.1), and so H^* can be coloured from its lists. Thus so can H, and this contradiction shows that D_c cannot have a quasikernel that meets $Z_d \cup Z_{d+1}$.

If Z_s is any set on which c is present, let K_s be formed by applying the algorithm in the proof of Lemma 4.1 to D_c, starting with Z_s. If c is present on *every* set Z_i, then it is easy to see that K_{a+2}, K_{a+3} and K_{a+4} are all quasikernels, and two of them meet $Z_d \cup Z_{d+1}$, a contradiction. So we may assume that there is at least one set Z_i on which c is not present.

Suppose that c is present on sets $Z_{p+1}, \dots, Z_r$ but not on Z_p, where $r = p + q$ $(q \geqslant 1)$. If $r = d + 1$ then $p = d$ and $q = 1$. So if $q \equiv 0 \pmod 3$ then $r = d$ and c is present on Z_{d+1} (since, by Lemma 4.1, there are no 0-segments in D_c); in this case we choose $s = p + 1$. If $q \equiv 1$ or $2 \pmod 3$ then we choose $s = p + 1$ or $p + 2$, respectively. In every case, K_s meets $Z_d \cup Z_{d+1}$. (But K_s may not be a quasikernel.)

By Lemma 4.1 and its proof, every segment is a 1-segment or a 2-segment, and, in each segment $\{Z_{i+1}, \dots, Z_{i+l}\}$, K_s includes a vertex from every set Z_{i+j} such that $j \equiv h \pmod 3$, where $h = 1$ or 2 depending on the segment. Let us say that this segment is of *type* h. Choose b minimal, $a \leqslant b \leqslant a + 3$, such that $Z_b \cap K_s \neq \emptyset$, and let S be the segment containing Z_b; clearly b exists, and if $b = a + 3$ then $D_c \cap Z_{a+2} = \emptyset$ and S has type 1. Let $K' := K_s \cap \{Z_s \cup Z_{s+1} \cup \cdots \cup Z_{b-1}\}$. If S is of type 1, then form K from K' by skipping Z_b and continuing the construction with the first nonempty set $Z_j \cap D_c$ with $j \geqslant b + 1$ (exactly as if S were of type 2). Then the only K-defective set is Z_b. If S is of type 2, then $b \leqslant a + 2$. In this case, form K from K' by skipping Z_b and Z_{b+1} and continuing the construction with the first nonempty set $Z_j \cap D_c$ with $j \geqslant b + 2$ (exactly as if S were of type 1). Then the only K-defective sets are Z_b and (possibly) Z_{b+1}. Whatever the type of S, K is a quasikernel that meets $Z_d \cup Z_{d+1}$, and this contradiction finally completes the proof of Lemma 5.1. □

We can now deduce the LTCC for a reasonably large class of multicircuits of even order.

Theorem 5.2. *Let C be a multicircuit with n vertices, m edges and maximum degree Δ, where n is even. Suppose that one of the following holds*:

(i) *C contains a set X of six vertices in three disjoint consecutive pairs, $X = \{v_p, v_{p+1}, v_q, v_{q+1}, v_r, v_{r+1}\}$, such that, for each $v \in X$, $d(v) \leqslant \Delta - 1$;*
(ii) *C contains a set X of four vertices in two disjoint consecutive pairs, $X = \{v_p, v_{p+1}, v_q, v_{q+1}\}$, such that, for each $v \in X$, $d(v) \leqslant \Delta - 2$.*
Then (2.1) *holds. Hence* (1.1) *holds, and all three* $(a:b)$*-choosability conjectures hold for* $T(C)$.

Proof. It suffices to prove (2.1), since the rest then follows immediately by Corollary 2.4. So let $H := T(C)_{(t)}$ for some $t \in \mathbb{N}$. Assume C has vertices $v_0, \ldots, v_{n-1}$ in cyclic order, with corresponding sets $Z_0, \ldots, Z_{2n-2}$ in H. Define f_i and g_i as in the second paragraph of this Section. If i is even, say $i = 2j$, then $|Z_i| + g_i = t + g_i = |Z_{i-3}|$, and so $f_i + g_i = f_{i-1} = t(d(v_{j-1}) + 1)$. If i is odd, say $i = 2j-1$, then $f_i + g_i = f_i = t(d(v_{j-1}) + 1)$ (the same).

Suppose first that (i) holds. Define $h_i := t$ if $i = 2j+1$ or $2j+2$ and $v_j \in X$, and $h_i := 0$ otherwise. Then $f_i + g_i + h_i \leqslant t(\Delta + 1)$ for each i. It is easy to see that numbers $v_{j,i}$ can be defined so that all the hypotheses of Lemma 5.1 are satisfied, since for each odd j the LHS of (5.1) has 10 or 12 coordinates equal to t, eight of which occur in two blocks of four. By Lemma 5.1, $\mathrm{ch}(H) \leqslant t(\Delta + 1)$. This proves (2.1), and the rest immediately follows.

Suppose now that (ii) holds. Define $h_i := 2t$ if $i = 2j+1$ or $2j+2$ and $v_j \in X$, and $h_i := 0$ otherwise. Then $f_i + g_i + h_i \leqslant t(\Delta + 1)$ for each i. As in (i), numbers $v_{j,i}$ can be defined so that all the hypotheses of Lemma 5.1 are satisfied, since for each odd j the LHS of (5.1) now has 6 or 8 coordinates equal to $2t$, four of which occur consecutively. So all the same conclusions follow as before. □

Acknowledgements

We are greatly indebted to an unnamed referee, who read the first version of this paper extremely carefully and made many helpful comments.

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