Discrete Mathematics 258 (2002) 373–377

DISCRETE MATHEMATICS

www.elsevier.com/locate/disc

Note

Equitable colorings of outerplanar graphs

A.V. Kostochka[a,b,*,1]

[a] *Department of Mathematics, University of Illinois, Urbana-Champaign, 1409 W. Green Street, Urbana, IL 61801, USA*

[b] *Institute of Mathematics, Novosibirsk 630090, Russia*

Received 20 February 2001; received in revised form 19 February 2002; accepted 4 March 2002

Abstract

A proper vertex coloring of a graph is *equitable* if the sizes of color classes differ by at most 1. In this note, we prove the conjecture of Yap and Zhang that every outerplanar graph with maximum degree at most Δ admits an equitable k-coloring for every $k \geqslant 1 + \Delta/2$. This restriction on k cannot be weakened.

Keywords: Graph coloring; Equitable coloring; Outerplanar graphs

1. Introduction

In many applications of graph coloring the sizes of color classes should not be too large. One of the possible formalizations of this restriction is the notion of equitable coloring. A proper vertex coloring of a graph is called *equitable* if the sizes of color classes differ by at most 1.

A graph may have an equitable k-coloring (i.e., an equitable coloring with k colors) but have no equitable $(k+1)$-coloring. For example, the complete bipartite graph $K_{2m+1,2m+1}$ for $m \geqslant 1$ has an equitable 2-coloring but has no equitable $(2m+1)$-coloring. Hajnal and Szemerédi [2] answering a question of Erdős proved that for every positive Δ and every $k \geqslant \Delta + 1$, each graph with maximum degree Δ has an equitable

* Corresponding author. Department of Mathematics, University of Illinois, Urbana-Champaign, 1409 W. Green Street, Urbana, IL 61801, USA. Tel.: +217-265-8037.

E-mail address: kostochk@math.uiuc.edu (A.V. Kostochka).

[1] This work was partially supported by the Grants 99-01-00581 and 97-01-01075 of the RFFR and by the Grant Intas-Open-97-1001.

PII: S0012-365X(02)00538-1

k-coloring. Chen et al. [1] conjectured that for an arbitrary $\Delta \geqslant 3$, every connected graph with maximum degree Δ distinct from $K_{1+\Delta}$ and $K_{\Delta,\Delta}$ has an equitable Δ-coloring.

This conjecture is proved for graphs in some classes, such as interval graphs, trees, and so on (a good survey is given in [3]). In particular, Yap and Zhang [5] proved this conjecture for outerplanar graphs. In the same paper they conjectured that every outerplanar graph with maximum degree $\Delta \geqslant 3$ is equitably k-colorable for every $k \geqslant 1 + \Delta/2$. The aim of the present note is to prove this conjecture. Notice that the star $K_{1,2k-1}$ has no equitable k-coloring and thus the restriction $k \geqslant 1 + \Delta/2$ cannot be weakened even for trees.

2. The result

Theorem 1. *Let $\Delta \geqslant 3$ and $k \geqslant 1 + \Delta/2$. Then for every outerplanar graph G with maximum degree at most Δ, there exists an equitable k-coloring of G.*

Proof. Assume that the theorem does not hold for some $k \geqslant 1 + \Delta/2$, where $\Delta \geqslant 3$. Choose among outerplanar graphs without equitable k-colorings a graph $G = (V, E)$ with the minimum number of vertices.

Lemma 2. $|V| \equiv 0 \pmod{k}$ *or* $|V| \equiv k - 1 \pmod{k}$.

Proof. Suppose $|V| = ik + j$, where $1 \leqslant j \leqslant k - 2$. Since every outerplanar graph has a vertex of degree at most 2 (see, e.g., [4, p. 240]), we can assume that $\deg_G(v) \leqslant 2$. By the minimality of G, there exists an equitable k-coloring f of $G - v$. In this coloring, exactly $k - j + 1$ color classes have cardinality i and exactly $j - 1$ classes have cardinality $i + 1$. By the choice of j, $k - j + 1 \geqslant 3$. Thus we can choose for v a color different from the colors of the neighbors of v and such that its color class has i elements. This gives an equitable k-coloring of G, a contradiction. □

Let G be drawn on the plane so that its vertices lie on a circle and every edge is a straight segment. Number the vertices of G clockwise: $v_1, \ldots, v_n$. If for every i and j with $|j - i| \equiv 0 \pmod{k}$, there is no edge $v_i v_j$, then we simply give every v_i the color $i \pmod{k}$. Since $|V| \not\equiv 1 \pmod{k}$, that would give a proper equitable coloring of G. So, we may assume that for some j, there is an edge $v_1 v_{jk+1}$ and that

$$\text{for every } j' < j \text{ and every } 1 \leqslant i \leqslant jk, \text{ there is no edge } v_i v_{i+j'k}. \tag{1}$$

Let G_1 be the subgraph of G induced by the vertex set $\{v_1, \ldots, v_{jk+1}\}$ and G_2 be the subgraph of G induced by the vertex set $\{v_{jk+1}, \ldots, v_n, v_1\}$.

Lemma 3. *Let $k \geqslant 4$. If $\deg_{G_1}(v_1) \leqslant k - 1$ and $\deg_{G_1}(v_{1+jk}) \leqslant k - 1$, then there exists a k-coloring f_1 of $V(G_1)$ with color classes $X_1, \ldots, X_k$ such that $|X_1| = |X_2| = j + 1$, $|X_3| = |X_4| = \cdots = |X_{k-1}| = j$, $|X_k| = j - 1$, $v_1 \in X_1$, and $v_{1+jk} \in X_2$.*

Proof. For $i=2,3,\ldots,jk$, color v_i with color $x_{i-1}\equiv i-1 \pmod k$, $1\leqslant x_{i-1}\leqslant k$. Because of (1), this coloring is proper. Color k will be used $j-1$ times, and every other color will be used j times. Each of v_1, v_{1+jk} is not adjacent to at least two colors. The only possibility that we cannot choose a color $\alpha<k$ not adjacent to v_1 and a color $\beta<k$, $\beta\neq\alpha$ not adjacent to v_{1+jk} is that $\deg_{G_1}(v_1)=\deg_{G_1}(v_{1+jk})=k-1$ and each of v_1, v_{1+jk} is not adjacent to exactly two colors: k and γ. In this case, let i be the largest integer less than $1+jk$ such that $v_1v_i\in E(G_1)$. Let x be the current color of v_i. Recall that $x\neq k$, since v_1 is not adjacent to vertices of color k. If there exists a color $y<x$, $y\neq\gamma$, then we swap colors k and y on all vertices v_l with $l<i$. Since the set $\{v_1,v_i\}$ separates G_1, we get a new coloring, where color k is used j times and color y is used $j-1$ times. Moreover, v_{1+jk} is still not adjacent to vertices of color k. So, after coloring v_{1+jk} with k and v_1 with γ, we are done.

Suppose that there is no such color y. This means that either $x=1$ or $x=2$ and $\gamma=1$. Since $k\geqslant 4$, in both cases there exists $z>x$ such that $z\notin\{k,\gamma\}$. Then we swap colors k and z on all vertices v_l with $l>i$. Similarly to the previous paragraph, we color v_1 with k and v_{1+jk} with γ, and are done. This proves the lemma. □

Lemma 4. *Let $k\geqslant 3$. There exists an equitable k-coloring f_1 of $V(G_1)$ with color classes $X_1,\ldots,X_k$ such that $|X_1|=j+1$, $|X_2|=|X_3|=\cdots=|X_k|=j$, and $|\{v_1,v_{1+jk}\}\cap X_1|=1$.*

Proof. For $i=1,2,\ldots,jk$, color v_i with color $x_i\equiv i \pmod k$, $1\leqslant x_i\leqslant k$. Because of (1), this coloring is proper. Every color is used j times. Let i be the largest integer less than $1+jk$ such that $v_1v_i\in E(G_1)$ and let x be the current color of v_i. If $x\geqslant 3$, then we swap colors 1 and 2 for vertices with indices greater than i. Now we can color v_{1+kj} with color 2 due to (1). That will be the required equitable coloring, where $j+1$ vertices will be colored with color 2. If $x=2$, then we swap colors 1 and k for vertices with indices greater than i. Now we can color v_{1+kj} with color k due to (1). Thus we get the required equitable coloring, where $j+1$ vertices will be colored with color 1. This proves the lemma. □

To prove the theorem, we consider several cases.

Case 1: $\deg_{G_2}(v_1)\leqslant k-1$ and $\deg_{G_1}(v_{jk+1})\leqslant k-1$. By the minimality of G, there exist an equitable k-coloring f_1 of G_1-v_{jk+1} and an equitable k-coloring f_2 of G_2-v_1. Let $X_1,\ldots,X_k$ (respectively, $Y_1,\ldots,Y_k$) be the color classes of f_1 (respectively, f_2). Recall that $|X_1|=\cdots=|X_k|=j$. We may assume that $v_1\in X_1$ and $v_{jk+1}\in Y_1$. Under conditions of the case, we can choose some m and r such that Y_m is not adjacent to v_1 and X_r is not adjacent to v_{jk+1}. Since v_1v_{jk+1} is an edge, $m\neq 1$ and $r\neq 1$. We may assume $m=2$ and $r=2$. Let $Z_1=X_1\cup Y_2$, $Z_2=X_2\cup Y_1$, and for $s=3,\ldots,k$, let $Z_s=X_s\cup Y_s$. Since $\{v_1,v_{jk+1}\}$ separates G, every Z_s is an independent set. This contradicts the choice of G.

Case 2: $\deg_{G_1}(v_1)\leqslant k-1$ and $\deg_{G_2}(v_{jk+1})\leqslant k-1$. The same proof.

Since $\deg_{G_1}(v_1)+\deg_{G_2}(v_1)\leqslant\Delta+1\leqslant 2k-1$ and $\deg_{G_1}(v_{jk+1})+\deg_{G_2}(v_{jk+1})\leqslant\Delta+1\leqslant 2k-1$, there remain the following two cases:

Case 3: $\deg_{G_1}(v_1) \geqslant k$ and $\deg_{G_1}(v_{jk+1}) \geqslant k$. By Lemma 4, there exists an equitable k-coloring of G_1 such that either the color class of v_1 or that of v_{jk+1} (say, of v_1) contains $j+1$ vertices.

If $|V| \equiv k-1 \pmod{k}$, then by the minimality of G, there exists an equitable k-coloring f_2 of G_2, and all color classes of f_2 have the same cardinality. Thus, renaming colors in f_2 so that $f_2(v_1) = f_1(v_1)$ and $f_2(v_{jk+1}) = f_1(v_{jk+1})$, we get an equitable k-coloring of G in which the only smaller class contains v_{jk+1}.

Suppose now that $|V| \equiv 0 \pmod{k}$. Let G'' be obtained from G_2 by contracting v_1 with v_{jk+1} into a new vertex v^* (and deleting parallel edges, if needed). Since $\deg_{G''}(v^*) \leqslant 2(\Delta - k) < \Delta$ and G'' is outerplanar, by the minimality of G there exists an equitable k-coloring f_2 of G''. Again, all color classes of f_2 have the same cardinality. Let v^* be colored with color α in f_2. Since v_{jk+1} is adjacent to at most $k-2$ vertices in $G'' - v^*$, there is a color $\beta \neq \alpha$ such that v_{jk+1} is not adjacent to vertices of color β in $G'' - v^*$. Thus, renaming colors in f_1 so that $f_1(v_1) = \alpha$ and $f_1(v_{jk+1}) = \beta$, we get an equitable k-coloring of G.

Case 4: $\deg_{G_2}(v_1) \geqslant k$ and $\deg_{G_2}(v_{jk+1}) \geqslant k$.

By the minimality of G, there exists an equitable k-coloring f_2 of G_2. Let $Y_1, \ldots, Y_k$ be the color classes of f_2. Depending on whether $|V|$ is $k-1$ or 0 modulo k, all Y_i's have the same cardinality, or exactly one of them (say, Y_k) is slightly larger. Due to the symmetry between v_1 and v_{jk+1}, we may assume that $v_1 \notin Y_k$.

We distinguish two subcases.

Subcase 4.1: $v_{jk+1} \in Y_k$. Note that this covers the case $|V| \equiv k-1 \pmod{k}$.

Let G' be obtained from G_1 by contracting v_1 with v_{jk+1} into a new vertex v'. Since $\deg_{G'}(v') \leqslant 2(\Delta - k) < \Delta$ and G' is outerplanar, by the minimality of G, there exists an equitable k-coloring f' of G'. Let $X_1, \ldots, X_k$ be the color classes of f'. Each of X_i has cardinality j. We may assume that $v' \in X_k$. Then by construction, no vertex in $X_k - v'$ is adjacent to $V(G_2)$. Thus, $Z_k = (X_k - v') \cup Y_k$ is an independent set of cardinality $\lfloor |V|/k \rfloor$. Since $\deg_{G_2}(v_1) \geqslant k$, at most $k-2$ color classes of f' are adjacent to v_1. So, we may assume that X_1 is not adjacent to v_1 and $v_1 \in Y_1$. In this case letting $Z_i = X_i \cup Y_i$ for $i = 1, \ldots, k-1$, we obtain an equitable k-coloring of G, a contradiction.

Subcase 4.2: $v_{jk+1} \notin Y_k$. Note that this implies $|V| \equiv 0 \pmod{k}$. We may assume $v_1 \in Y_1$, $v_{jk+1} \in Y_2$.

First, let $k \geqslant 4$. By Lemma 3 there exists a k-coloring f_1 of $V(G_1)$ with color classes $X_1, \ldots, X_k$ such that $|X_1| = |X_2| = j+1$, $|X_3| = |X_4| = \cdots = |X_{k-1}| = j$, $|X_k| = j-1$, $v_1 \in X_1$ and $v_{1+jk} \in X_2$. Then the partition $(X_1 \cup Y_1, X_2 \cup Y_2, \ldots, X_k \cup Y_k)$ of $V(G)$ is an equitable k-coloring of G, a contradiction.

Now, let $k = 3$. Consider G_2' obtained from G_2 by adding two vertices: w adjacent to v_{3j+1} and v_1, and u adjacent only to w. Clearly, G_2' is outerplanar. Suppose that $|V(G_2')| < |V(G)|$. Then there exists an equitable 3-coloring f_2' of G_2'. Since $|V(G_2')| \equiv 0 \pmod{3}$, all color classes of f_2' have the same size. But then deleting vertices w and u, we get an equitable 3-coloring of G_2 satisfying conditions of Subcase 4.1. This contradiction shows that $|V(G_2')| = |V(G)|$ and therefore $j = 1$. Moreover, if neither of v_2 and v_3 is adjacent to both v_1 and v_4, then f_2 can be easily extended to v_2 and v_3 so that we get an equitable 3-coloring of G. Therefore, we may assume that v_2 is adjacent to v_1 and v_4, and v_3 can be adjacent only to v_2 (since each of v_1 and v_4 has already 3 neighbors in G_2).

Since Case 3 does not hold, there exist some $l \geqslant 4$ and $j' \geqslant 1$ such that $v_l v_{l+3j'} \in E(G_2)$. For the same reasons as in the previous paragraph, we may assume that $j' = 1$, v_{l+1} is adjacent to v_l and v_{l+3}, and v_{l+2} is adjacent only to v_{l+1}. Let $G_0 = G - v_2 - v_3 - v_{l+2}$. By the minimality of G, there exists an equitable 3-coloring f_0 of G_0 with colors 1, 2, and 3. Since $|V(G_0)| \equiv 0 \pmod 3$, all color classes of f_0 have the same size. We can now color v_2 with a color $\alpha \in \{1,2,3\} - f_0(v_1) - f_0(v_4)$, then color v_{l+2} with a color $\beta \in \{1,2,3\} - f_0(v_{l+1}) - f_0(v_2)$, and finally color v_3 with a color $\gamma \in \{1,2,3\} - f_0(v_2) - f_0(v_{l+2})$. This produces an equitable 3-coloring of G, a contradiction.

Remark. As one of the referees pointed out, one can extract from the proof an efficient algorithm for equitable k-coloring of outerplanar graphs with maximum degree Δ whenever $k \geqslant 1 + \Delta/2$.

Acknowledgements

I thank Douglas West for interesting discussions.

References

[1] B.-L. Chen, K.-W. Lih, P.-L. Wu, Equitable coloring and the maximum degree, European J. Combin. 15 (1994) 443–447.

[2] A. Hajnal, E. Szemerédi, Proof of a conjecture of Erdős, in: A. Rényi, V.T. Sós (Eds.), Combinatorial Theory and its Applications, Vol. II, North-Holland, Amsterdam, 1970, pp. 601–623.

[3] K.-W. Lih, The equitable coloring of graphs, in: D.-Z. Du, P. Pardalos (Eds.), Handbook of Combinatorial Optimization, Vol. 3, Kluwer, Dordrecht, 1998, pp. 543–566.

[4] D.B. West, Introduction to Graph Theory, 2nd Edition, Prentice-Hall, Upper Saddle River, 2001.

[5] H.P. Yap, Y. Zhang, The equitable Δ-coloring conjecture holds for outerplanar graphs, Bull. Inst. Math. Acad. Sinica 25 (1997) 143–149.