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Colorings and homomorphisms of degenerate and bounded degree graphs

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Abstract

We are interested here in homomorphisms of undirected graphs and relate them to graph degeneracy and bounded degree property. Our main result reads that both in the abundance of counterexamples and from the complexity point of view the Brooks's theorem and first fit algorithm are the 'only' easy cases. © 2001 Elsevier Science B.V. All rights reserved.

1. Introduction

Given two (undirected simple) graphs $G_i = (V_i, E_i)$, $i = 1, 2$, a *homomorphism* of G_1 into G_2 is any mapping $f : V_1 \to V_2$ which satisfies $\{x, y\} \in E_1 \Rightarrow \{f(x), f(y)\} \in E_2$. This is denoted by $f : G_1 \to G_2$ and the existence (the non-existence resp.) of a homomorphism is denoted by $G_1 \to G_2$ ($G_1 \nrightarrow G_2$ resp.). Clearly the existence of a homomorphism $G \to K_k$ coincides with the existence of a k-coloring and thus homomorphisms $G \to H$ are sometimes called H-colorings.

The existence of H-colorings was thoroughly discussed in several papers (see [7,8]), and there seems to be a big gap between oriented and unoriented graphs here. However in this paper we are dealing with undirected graphs only. In this context let us quote the following two results of [8,11].

Theorem 1.1. *The H-coloring problem is polynomial when H is bipartite and NP-complete otherwise.*

This has been strengthened (see [11]) as follows.

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Theorem 1.2. *The H-coloring problem has bounded treewidth duality if and only if H is bipartite.*

(Here H-coloring problem has bounded treewidth duality if there is an integer k such that the following property holds for all graphs $G: G \nrightarrow H$ if and only if there is a partial k-tree T such that $T \rightarrow G$ and $T \nrightarrow H$.)

We are motivated here by these results and by classical Brooks's theorem (see any book of graph theory) which implies that K_3-coloring problem is easy for cubic graphs. Moreover this problem is easy for 2-degenerate graphs. One of the main results of this paper says that this is the only exception:

Theorem 1.3. *Let H be a non-bipartite triangle-free graph. Then the H-coloring problem is NP-complete for the class of* 2*-degenerate graphs.*

We provide some further evidence to show that coloring 2-degenerate (and k-degenerate) graphs is a difficult problem.

For cubic graphs the situation is different. This question was investigated in [6], where among others the following has been proved:

Theorem 1.4. *The C_5-coloring problem is NP-complete for the class of cubic graphs.*

Theorem 1.5. *For any integer b, there exists a triangle-free graph U_b such that any triangle free graph G with degree at most b is homomorphic to U_b. Consequently, the U_b-coloring problem is polynomial for triangle-free graphs with all its degrees bounded by b.*

Below, we strengthen these results in various directions and under restrictions like girth and special H. However the following problem seems to be still open.

Problem 1.6. Is it true that any cubic graph G with sufficiently large girth g is homomorphic to C_5?

We show that necessarily $g > 7$. We complement this by showing that the answer is negative for C_k, $k \geqslant 11$.

The paper is organized as follows. In Section 2 we introduce all notions and state basic results. In Section 3, we prove the complexity results for graphs with bounded odd-girth, while in Section 4 we deal with cubic graphs and in Section 5 with degenerate graphs. In Section 6, we list some open problems and comments.

2. Basic results

Definition 2.1. Let H be a graph.

(i) Let l be an integer. The (H,l)-coloring problem is the following decision problem:
Instance: A graph G, $girth(G) \geqslant l$.
Question: Is there any homomorphism from G to H?

(ii) Let l be an odd integer. The $(H, odd\text{-}l)$-coloring problem is the following decision problem:
Instance: A graph G, $C_{l-2} \nrightarrow G$ (i.e. $odd\text{-}girth(G) \geqslant l$).
Question: Is there any homomorphism from G to H?

Lemma 2.2 (Nešetřil and Rödl [10]). *Let* $\varepsilon > 0$ *be a real number*, $l > 3$ *be an integer and* I, H *be graphs such that* $I \to H$. *Then there exists a graph* F *with the following properties*:

(i) $girth(F) \geqslant l$
(ii) $V(F) = \bigcup_{i \in V(I)} F_i$ *and there is a homomorphism* $f : F \to I$ *such that for every* $i \in V(I)$, f *maps* F_i *to* i
(iii) *there is* α *such that for all* $i \in V(I)$, $|F_i| = \alpha$
(iv) *for every homomorphism* $h : F \to H$, *for every edge* $\{i,j\} \in E(I)$, *and every sets* A, B *such that* $A \subseteq F_i$, $B \subseteq F_j$, $|A| \geqslant \varepsilon\alpha$, $|B| \geqslant \varepsilon\alpha$, *there is an edge in* H *between* h/A *and* h/B.

We will call F the (l, ε, I)-majority graph for H.

Recall the concept of a core: a graph H is called a *core* if it does not admit a homomorphism to its proper subgraph. The previous lemma (in its probabilistic setting) may be strenghten to the following ('folklore') lemma (see e.g. [12]).

Lemma 2.3 (On uniquely H-colorable graphs). *Let* H *be a core and* l *be an integer. Then there exists a graph* $\hat{H}$ *of girth at least* l *such that for any two homomorphisms* f_1, $f_2 : \hat{H} \to H$, *there exists an automorphism* h *of* H *with* $f_1 = h \circ f_2$.

We shall need yet another technical refinement of this result:

Lemma 2.4. *For any two integers* l, k *and any core* H *with two specified vertices* h_1, h_2, *there exists a graph* $B = B(H, h_1, h_2, l, k)$ *of girth at least* l *with two vertices* b_1, b_2 *such that*

(i) $d_B(b_1, b_2) \geqslant k$
(ii) *there exists a homomorphism* f *from* B *to* H *with* $f(b_1) = h_1$, $f(b_2) = h_2$
(iii) *for any two homomorphisms* f_1, $f_2 : B \to H$, *there exists an automorphism* h *of* H *with* $f_1 = h \circ f_2$.

Proof. Let $\hat{H}$ be the graph satisfying Lemma 2.3 for H and l. It follows from the proof of Lemma 2.3 that $\hat{H}$ can be chosen in such a way that $|f^{-1}(h)| > 2$, for any homomorphism $f : \hat{H} \to H$ and any $h \in V(H)$. Let x be a vertex from $f^{-1}(h_1)$ and let y, z be two vertices from $f^{-1}(h_2)$. Then we have $d_{\hat{H}}(x,z) \geqslant 1$, $d_{\hat{H}}(y,z) \geqslant 2$.

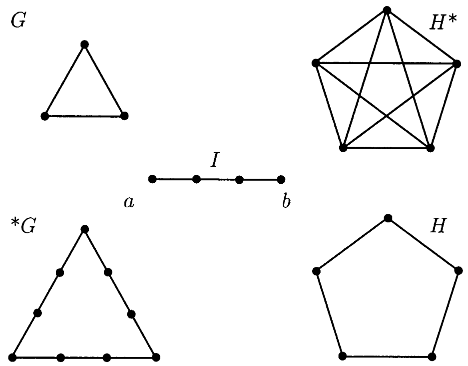

Fig. 1. The indicator construction.

Consider m disjoint copies of $\hat{H}$, where m is sufficiently large ($m \geqslant (k+1)/2$). For $j = 1, 2 \ldots m-1$, identify the vertex z from the j-copy of $\hat{H}$ with the vertex y of the $j+1$-copy of $\hat{H}$. Denote the new graph as B. Denote by b_1 the vertex x from the first copy and by b_2 the vertex z from the mth copy of $\hat{H}$. □

The first result is a technical statement which appeared in [10] and which proved to be useful in various contexts (see e.g. [12]).

Further, we shall use three type of reductions. The first two reductions are similar to those defined in [8] (which are reviewed and modified here for the sake of completeness). The third one is a new modification.

2.1. The indicator construction

Let I be a fixed graph, and let a and b be distinct vertices of I such that some automorphism of I interchanges vertices a and b. Transform a given graph H into the graph H^* defined to have the same vertex set as H and to have as the edge set all pairs $\{h, h'\}$ for which there is a homomorphism of I to H taking a to h and b to h' (see Fig. 1). Because of our assumption on I, the edges of H^* will be undirected.

Let $\mathscr{A}$ be a class of graphs. Given a graph $G \in \mathscr{A}$, let ${}^*G$ be the graph obtained from G by replacing each edge $\{g, g'\}$ by a copy of I (disjoint copies for distinct edges), identifying a with g and b with g'. Define the class $\mathscr{A} * \mathscr{I}$ by $\mathscr{A} * \mathscr{I} = \{{}^*G; G \in \mathscr{A}\}$.

Lemma 2.5. *If the H^*-coloring problem is NP-complete for class $\mathscr{A}$, then so is the H-coloring problem for class $\mathscr{A} * \mathscr{I}$.*

Proof. It follows from the definitions that there is a homomorphism ${}^*G \to H$ if and only if there is a homomorphism $G \to H^*$. □

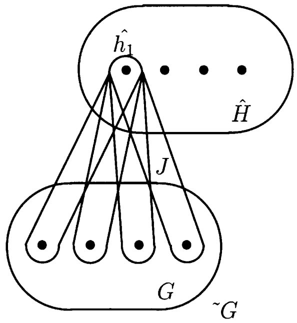

Fig. 2. The sub-indicator construction.

(In applying Lemma 2.5 we need to be careful to ensure that H^* has no loops, i.e. that no homomorphism of I to H can map a and b to the same vertex, and that H^* fails to be a bipartite graph. Otherwise, the H^*-coloring problem will not be NP-complete.)

2.2. *The sub-indicator construction*

Let J be a fixed graph with specified vertices a and $b_1, b_2 \dots b_k$. The sub-indicator construction (with respect to J, a and $b_1, b_2 \dots b_k$) transforms a given core H with specified vertices $h_1, h_2 \dots h_k$, to its subgraph $\tilde{H}$ induced by the vertex set

$$\tilde{V} = \{h' ;\ \text{there is } f : J \to H \text{ with } f(a) = h' \text{ and } f(b_1) = h_1, \dots, f(b_k) = h_k\}.$$

Let $\hat{H}$ be the graph satisfying Lemma 2.3 for H and l. Let $\hat{h}_i$ for $i = 1, 2 \dots k$, be vertices of $\hat{H}$ such that there is a homomorphism $f : \hat{H} \to H$ with $f(\hat{h}_i) = h_i$.

Let $\mathscr{A}$ be a class of graphs. Given a graph $G \in \mathscr{A}$, define $\tilde{G}$ as the graph obtained from the disjoint union of G, $\hat{H}$ and $|V(G)|$ copies of J, by identifying the vertex b_i, for $i = 1, 2 \dots k$, in each copy of J with the vertex $\hat{h}_i$ in $\hat{H}$, and by identifying each vertex g of G with the vertex a in the gth copy of J (see Fig. 2). Let $\mathscr{A}\tilde{I}$ be the class of all graphs containing $\tilde{G}$.

Lemma 2.6. *If the $\tilde{H}$-coloring problem is NP-complete for class $\mathscr{A}$, then so is the H-coloring problem for class $\mathscr{A}\tilde{I}$.*

Proof. There is a homomorphism $G \to \tilde{H}$ if and only if there is a homomorphism $\tilde{G} \to H$. $\square$

2.3. *The majority-indicator construction*

Let I be a fixed graph, and let a and b be non-adjacent distinct vertices of I which do not belong to any odd cycle of length less than $l-2$ and such that some automorphism of I maps a to b and b to a. Let H^* be a graph defined as before by applying the indicator construction with the indicator I (and vertices a and b) to a given graph H.

Lemma 2.7. *Let l be an odd integer, $l > 3$. If the $(H^*$, odd-$(l-2))$-coloring problem is NP-complete, then the $(H, odd\text{-}l)$-coloring problem is also NP-complete.*

Proof. Let $\varepsilon < 1/|V(H)|$. Let F be the (l, ε, I)-majority graph for H. Given a graph G of odd-girth at least $l-2$, let ${}^{\diamond}G$ be the graph obtained from G by replacing each vertex g by a disjoint set N_g of α vertices and by replacing each edge $\{g, g'\}$ by a copy of F (disjoint copies for distinct edges), identifying N_g with F_a and $N_{g'}$ with F_b.

Claim 1. *Odd-girth*$({}^{\diamond}G) \geqslant l$.

Proof. We know that $girth(F) \geqslant l$, $d_F(x, y) > 1$ for any two vertices $x \in F_a$, $y \in F_b$ (because a and b are non-adjacent in I), $odd\text{-}girth(G) \geqslant l-2$ and $d_F(x, x') \geqslant 2$, $odd\text{-}d_F(x, x') \geqslant l-2$ for any x, $x' \in F_a$ so as for any x, $x' \in F_b$ (because there is a homomorphism from F to I which maps both the vertices to the same vertex and the vertices a, b are not contained in I in any odd cycle of length less than $l-2$). Now it is easy to see that $odd\text{-}girth({}^{\diamond}G) \geqslant l$. □

We shall make use of the following.

Definition 2.8. Let h be a homomorphism from ${}^{\diamond}G$ to H. The majority set of a vertex $g \in V(G)$ with respect to h is the subset M_g of N_g with maximum cardinality such that $M_g \subseteq h^{-1}(v)$ for some vertex $v \in V(H)$.

Notice that for any vertex $g \in V(G)$, $|M_g| \geqslant 1/|V(H)|\alpha > \varepsilon\alpha$.

Claim 2. *There is a homomorphism ${}^{\diamond}G \to H$ if and only if there is a homomorphism $G \to H^*$.*

Proof. Let $h: {}^{\diamond}G \to H$ be a homomorphism. We will define a mapping $f : V(G) \to V(H^*)$ as follows. Let g be a vertex of G. Then put $f(g) = h/M_g$, where M_g is the majority set of g with respect to h. Then for any edge $\{g, g'\}$ in G, $\{f(g), f(g')\}$ is an edge in H^*.

Conversely, let $f : G \to H^*$ be a homomorphism. Put $h(w) = f(g)$ for all $w \in N_g$. Then h can be extended to the remaining vertices of ${}^{\diamond}G$ naturally. □

It follows from Claim 2 that if the H^*-coloring problem is NP-complete for the class of all graphs of odd-girth at least $l-2$, then the H-coloring problem is NP-complete for ${}^{\diamond}G$, i.e. for the class of all graphs of odd-girth at least l. This proves Lemma 2.7. □

(In applying Lemma 2.7 we need to check that H^* contains no loops and that H^* is non-bipartite, as in the former case of the indicator construction.)

3. The complexity results

3.1. $(H, odd\text{-}l)$-coloring problem

Theorem 3.1. *Let H be a non-bipartite graph. Then the $(H, odd\text{-}7)$-coloring problem is NP-complete.*

Proof. Suppose that l is the minimum odd integer such that there is a non-bipartite graph H for which the $(H, odd\text{-}l)$-coloring problem is not NP-complete. Let H be a graph with this property and such that the $(H', odd\text{-}l)$-coloring problem is NP-complete for any non-bipartite H' with

(i) fewer vertices than H, or
(ii) with the same number of vertices as H, but with more edges.

If the theorem does not hold then such an H must exist and $l \leqslant 7$. In the spirit of [8] we derive sufficiently many properties of this minimal counterexample. Observe first that H has to be a core. Now, we will prove that $\omega(H) = 3$.

Claim 3. *H contains a triangle.*

Proof. Suppose to the contrary that the shortest odd cycle C of H has $k \geqslant 5$ vertices. Then the indicator construction with a path with 3 vertices and endpoints a and b (such a graph has unbounded girth) as the indicator will transform H into the graph H^* which is undirected, has no loops and contains more edges than H (consider some chord of C). Given a graph G of odd-girth at least l. The transformed graph $^*G$ has girth at least $3l > l$. The $(H^*, odd\text{-}l)$-coloring problem is NP-complete according to the assumptions. Then by Lemma 2.5 the H-coloring problem is NP-complete on the set $\{^*G; G \text{ has odd girth at least } l\}$, i.e. $(H, odd\text{-}l)$-coloring problem is NP-complete; which is a contradiction. □

Claim 4. *H contains no K_4.*

Proof. Suppose to the contrary that H contains K_4. Let h be any vertex of H which belongs to K_4. Construct the sub-indicator J as follows:

Consider graph $B = B(H, h, h, l, (l-3)/2)$ with the vertices b_1, b_2 satisfying properties declared in Lemma 2.4. Add a new vertex a and the edge $\{b_2, a\}$, and rename the vertex b_1 to b. The sub-indicator construction (with respect to J, a and b) transforms H to the graph $H^{\sim}$ which does not contain h, but does contain a triangle in its neighborhood. Hence, $H^{\sim}$ is a non-bipartite graph with fewer vertices than H and according to the assumptions the $(H^{\sim}, odd\text{-}l)$-coloring problem is NP-complete.

Let G be a given graph of girth at least l. Then $^{\sim}G$ has girth at least l too (consider that J, $\hat{H}$, G have girth at least l and $d_J(a, b) \geqslant (l-3)/2 + 1 = (l-1)/2$). Since by Lemma 2.6 the $(H, odd\text{-}l)$-coloring problem is NP-complete, this again contradicts our assumptions. □

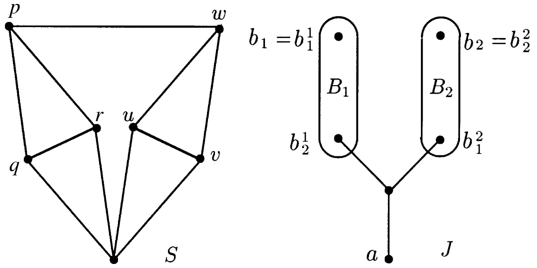

Fig. 3. The graph S and the sub-indicator J.

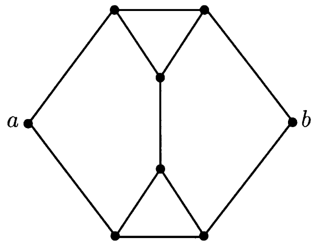

Fig. 4. The graph U.

Claim 5. *Each edge of H belongs to a triangle.*

Proof. We will show that any two vertices of H have a common neighbor. Let h be any vertex which belongs to a triangle. Consider the sub-indicator construction with the sub-indicator J defined as follows: Consider graph $B = B(H,h,h,l,(l-5)/2)$ with the vertices b_1, b_2 satisfying Lemma 2.4. Add two vertices a, c and add edges $\{a,c\}$, $\{c,b_2\}$. Rename the vertex b_1 to b. Suppose to the contrary that there exists a vertex q in H which has no common neighbor with h. Then the transformed graph $H^{\sim}$ does not contain q, but does contain a triangle containing h. Hence, $H^{\sim}$ is a non-bipartite graph with fewer vertices than H and according to the assumptions the $(H^{\sim}, odd\text{-}l)$-coloring problem is NP-complete.

Let G be a given graph of girth at least l. Then ${}^{\sim}G$ has girth at least l too (consider that J, $\hat{H}$, G have girth at least l and $d_J(a,b) \geqslant (l-5)/2 + 2 = (l-1)/2$). Then by Lemma 2.6 the $(H, odd\text{-}l)$-coloring problem is NP-complete; which is a contradiction.

Thus, it suffices to prove that each vertex belongs to a triangle. Suppose to the contrary that a vertex p does not belong to any triangle and p is joined with some vertex s in triangle. Then p, s have a common neighbor t. Hence p belongs to the triangle pst. □

Claim 6. *For the Erdős–Moser graph S given in* Fig. 3, *there is no homomorphism $S \to H$.*

Proof. If such a homomorphism exists, let $u', v' \ldots$ be the images of the vertices $u, v \ldots$ of S in H. Let $B_1 = B(H, u', u', l, (l-3)/2)$ with b_1^1, b_2^1 satisfy Lemma 2.4. Let $B_2 = B(H, v', v', l, (l-3)/2)$ with b_1^2, b_2^2 satisfy Lemma 2.4. Rename b_1^1 to b_1 and b_2^2 to b_2. The sub-indicator J of Fig. 3 with specified vertices a, b_1, b_2 yields a graph $H^{\sim}$ which contains the triangle $p'q'r'$ but does not contain the vertex w' (by Claim 4). Moreover, for any given graph G of odd-girth at least l, the transformed graph ${}^{\sim}G$ has odd-girth at least l too. We obtain a contradiction as before. □

Suppose to the contrary that $l \leqslant 7$. Consider the graph U of Fig. 4. Notice that the vertices a, b do not belong to any triangle in U and that a, b cannot be mapped to the same vertex in H (by Claim 6).

Consider the majority-indicator construction with the indicator U. It transforms H into the graph H^* which is undirected (by the symmetry of I), has no loops (by Claim 6) and is non-bipartite (each edge of H belongs to a triangle by Claim 5, and U admits a homomorphism onto a triangle which maps a and b to an edge; thus H^* contains all edges of H). Then $(H^*, odd\text{-}(l-2))$-coloring problem is NP-complete. According to Lemma 2.7 the $(H, odd\text{-}l)$-coloring problem is also NP-complete; this is a contradiction with the choice of H. Hence, Theorem 3.1 is proved. □

3.2. Symmetric graphs

In this section we show that for many graphs H we can prove a result analogous to Theorem 3.1 in the full generality.

Definition 3.2. We say that a graph H satisfies the symmetry condition if there is an odd $k > 1$ and the vertices $v_1, v_2 \dots v_k$ of H with the following properties: for $i = 1, 2 \dots k$,

(1) there is an automorphism f_i of H with

$$f_i(v_i) = v_{(i+1)}, \quad f_i(v_{(i+1)}) = v_{(i+2)},$$

(2) there is an automorphism g_i of H with

$$g_i(v_i) = v_{(i-1)}, \quad g_i(v_{(i-1)}) = v_i,$$

where the indices are counted modulo k.

Remark 3.3. For any l and any non-bipartite graph H which satisfies the symmetry condition, the (H, l)-coloring problem is NP-complete.

Proof. Let l be the minimum integer such that there is a non-bipartite graph H for which the (H, l)-coloring problem is not NP-complete. That is, we assume that for any non-bipartite graph H' and any $l' < l$, the (H', l')-coloring problem is NP-complete. We know that $l > 3$ according to Theorem 1.1. Let $v_1, v_2 \dots v_k$ be the vertices of H which guarantee the symmetry condition for H. Consider the indicator construction with the indicator $B = B(H, v_1, v_2, l, 2)$ (the graph satisfying Lemma 2.4), and with specified vertices $a = b_1$, $b = b_2$. Then the new graph H^* is undirected, has no loops and is non-bipartite (contains the odd cycle $v_1, v_2 \dots v_k$). It follows easily from the symmetry condition. Thus, the $(H^*, l-1)$-coloring problem is NP-complete according to the choice of l. For any graph G of girth at least $l-1$, the transformed graph $^*G$ has girth at least l. Thus the (H, l)-coloring problem is NP-complete, which is a contradiction. □

However let us note that most graphs H are asymmetric.

4. Cubic graphs

4.1. (n,r)-configurations

For $r \geqslant 3$ and $n > r$, let $\mathbf{G}(n, r\text{-}reg)$ denote the set of all r-regular graphs with vertex set $V = \{1, 2, \ldots, n\}$. We always assume that $rn = 2k$ is an even number, and so k is the number of edges in a graph. We say that *almost all r-regular graphs have a certain property Q* if the portion of graphs in $\mathbf{G}(n, r\text{-}reg)$ not possessing Q is $o(|\mathbf{G}(n, r - reg)|)$. It is not too easy to calculate $|\mathbf{G}(n, r - reg)|$ (see e.g. [3]). In order to facilitate studying $\mathbf{G}(n, r - reg)$, Bollobás [2] (for a more detailed description see [3]) introduced a very convenient model of (n,r)-configurations.

Let $W = \bigcup_{j=1}^{n} W_j$ be a fixed set of $2k = rn$ labeled vertices, where $|W_j| = r$ for each j. An *(n,r)-configuration F* is a partition of W into k pairs of vertices, called *edges* of F. Let Φ be the *set of (n,r)-configurations*. Clearly,

$$|\Phi| = N(k) = (2k-1)!!.$$

(Recall that for any positive odd integer $m, m!! = m(m-2)\cdots 3 \times 1$.) For $F \in \Phi$, let $\phi(F)$ be the multigraph with vertex set $V = \{1, 2, \ldots, n\}$ in which each i and j are joined by the same number of edges as W_i and W_j are joined in F. In other words, $\phi(F)$ is obtained from F by merging each W_i into a vertex i. Clearly, $\phi(F)$ is an r-regular multigraph on V (sometimes, with loops). The most important fact is that the portion of $F \in \Phi$ such that $\phi(F)$ is a simple graph is at least c_r, where $c_r > 0$, for every sufficiently large n, and each simple graph on V corresponds to the same number of (n,r)-configurations (namely, to $(r!)^n$). Thus, if we prove that almost all (n,r)-configurations have a certain property Q, then almost all r-regular graphs have Q, as well.

4.2. Short cycles

We shall start with a well-known lemma.

Lemma 4.1. *The portion of $(n,3)$-configurations F such that $\phi(F)$ contains more than* $10(3l)^l$ *cycles of length l or less is at most* 0.1.

Proof. The number of different possible cycles of length k on n vertices is $\binom{n}{k}(k-1)!/2$. For every such a cycle, 6^k different sets of k edges in an $(n,3)$-configuration can form this cycle. Then the number of $(n,3)$-configurations containing a given set of k edges is $(3n-1-2k)!!$. It follows that the expected number of 'short' cycles per an $(n,3)$-configuration is at most

$$\sum_{k=1}^{l}\binom{n}{k}\frac{(k-1)!}{2}6^k\frac{(3n-1-2k)!!}{(3n-1)!!}$$

$$\leqslant \sum_{k=1}^{l}\frac{(6n)^k}{2k}(3n-2k+1)^{-k}\leqslant \sum_{k=1}^{l}(3k)^k/2k<(3l)^l.$$

Hence, at most 10% of all $(n,3)$-configurations can have at least $10(3l)^l$ 'short' cycles each. □

4.3. Almost bipartite subgraphs

Lemma 4.2. *For at most half of $(n,3)$-configurations F, there exists a bipartition (V_1,V_2) of V such that the total number of edges induced by $\phi(F)$ in both V_1 and V_2 is less than $0.091n$.*

Proof. Let (V_1,V_2) be a bipartition of V with $|V_1|=0.5n-k$. If V_1 induces s edges of a cubic graph G on V, then exactly $3(0.5n-k)-2s$ edges of G connect V_1 with V_2, and hence V_2 induces $3k+s$ edges. In particular, the total number of edges induced by both V_1 and V_2 is even if and only if k is even. Let $f(n,m,k)$ be the number of pairs $[F,(V_1,V_2)]$, such that F is an $(n,3)$-configuration and (V_1,V_2) is a partition of V with $|V_1|=0.5n-k$ and the total number of edges induced by $\phi(F)$ in both V_1 and V_2 is equal to m. By the above, if $f(n,m,k)>0$, then $m+k$ is even. For such m and k we have

$$f(n,m,k)=\binom{n}{0.5n-k}\binom{1.5n-3k}{m-3k}(m-3k-1)!!$$
$$\times\binom{1.5n+3k}{m+3k}(m+3k-1)!!(1.5n-m)!. \tag{1}$$

(Here $\binom{n}{0.5n-k}$ is the number of choices of V_1, $\binom{1.5n-3k}{m-3k}$ is the number of choices of $m-3k$ ends of $(m-3k)/2$ edges induced by V_1, $(m-3k-1)!!$ is the number of perfect matchings on these $m-3k$ ends, $\binom{1.5n+3k}{m+3k}$ and $(m+3k-1)!!$ are analogous numbers for V_2, and $(1.5n-m)!$ is the number of ways to put the remaining $1.5n-m$ edges so to connect V_1 with V_2.)

We will show that the function

$$f(n)=\frac{1}{(3n-1)!!}\sum_{m=0}^{\lfloor 0.091n\rfloor}\sum_{k=0}\lfloor m/3\rfloor f(n,m,k) \tag{2}$$

tends to 0 when n tends to infinity. That would imply the validity of the lemma.

Claim 7. *For any $k<m/3<n/30$, $f(n,m,k)\geqslant f(n,m,k+2)$.*

Proof. If $f(n,m,k+2)=0$, we are done. Let $f(n,m,k+2)\neq 0$. Then

$$\frac{f(n,m,k)}{f(n,m,k+2)}$$
$$=\frac{0.5n+k+1}{0.5n-k}\frac{(1.5n-3k)(1.5n-3k-1)\dots(1.5n-3k-5)}{(m-3k)(m-3k-1)\dots(m-3k-5)}$$
$$\times(m-3k-1)(m-3k-3)(m-3k-5)$$
$$\times\frac{(m+3k+6)(m+3k+5)\dots(m+3k+1)}{(1.5n+3k+6)(1.5n+3k+5)\dots(1.5n+3k+1)}$$
$$\times\frac{1}{(m+3k+5)(m+3k+3)(m+3k+1)}$$
$$=\frac{0.5n+k+1}{0.5n-k}\frac{(1.5n-3k)(1.5n-3k-1)\dots(1.5n-3k-5)}{(1.5n+3k+6)(1.5n+3k+5)\dots(1.5n+3k+1)}$$
$$\times\frac{(m+3k+6)(m+3k+4)(m+3k+2)}{(m-3k)(m-3k-2)(m-3k-4)}$$
$$\geqslant\frac{0.5n+k+1}{0.5n-k}\left(1-\frac{6k+6}{1.5n+3k+1}\right)^6\left(1+\frac{6k+6}{m-3k}\right)^3.$$

Under conditions of the claim, the last expression is greater than 1. This proves the claim. □

Claim 8. *For any odd $m<n/10$, $f(n,m,1)\leqslant f(n,m+1,0)$.*

Proof. Indeed,

$$\frac{f(n,m,1)}{f(n,m+1,0)}$$
$$=\frac{\binom{n}{0.5n-1}\binom{1.5n-3}{m-3}(m-4)!!\binom{1.5n+3}{m+3}(m+2)!!(1.5n-m)!}{\binom{n}{0.5n}\binom{1.5n}{m+1}m!!\binom{1.5n}{m+1}m!!(1.5n-m-1)!}$$
$$=\frac{0.5n\cdot(1.5n-3)!(m+1)!^2(1.5n-m-1)!^2(1.5n+3)!(m+2)(1.5n-m)!}{(0.5n+1)(m-3)!(1.5n-m)!^2(m+3)!m(m-2)\cdot(1.5n-m-1)!}$$
$$=\frac{0.5n\cdot(m-2)(m-1)m(m+1)(m+2)(1.5n+3)(1.5n+2)(1.5n+1)}{(0.5n+1)(1.5n-m)1.5n(1.5n-1)(1.5n-2)m(m-2)(m+2)(m+3)}$$
$$=\frac{(m-1)(m+1)(1.5n+2)(1.5n+1)}{(1.5n-m)(1.5n-1)(1.5n-2)(m+3)}<\frac{2(m-1)}{1.5n-m}<1. \qquad \square$$

Claim 9. *For any even $m<n/10$, $f(n,m,0)\leqslant 0.1f(n,m+2,0)$.*

Proof. Indeed,

$$\frac{f(n,m,0)}{f(n,m+2,0)} = \frac{\binom{n}{0.5n}\binom{1.5n}{m}^2((m-1)!!)^2(1.5n-m)!}{\binom{n}{0.5n}\binom{1.5n}{m+2}^2((m+1)!!)^2(1.5n-m-2)!}$$

$$= \frac{(m+2)^2(m+1)^2(1.5n-m)(1.5n-m-1)}{(1.5n-m)^2(1.5n-m-1)^2(m+1)^2}$$

$$= \frac{(m+2)^2}{(1.5n-m)(1.5n-m-1)} < 0.1. \qquad \square$$

In view of the claims above, we derive from (2) that

$$f(n) \leqslant \frac{1}{(3n-1)!!} n f(n, \lfloor 0.091n \rfloor, 0).$$

Denoting $\alpha = \lfloor 0.091n \rfloor / n$, we get

$$f(n) \leqslant n \frac{\binom{n}{0.5n}\binom{1.5n}{\alpha n}^2((\alpha n-1)!!)^2(3n/2-\alpha n)!}{(3n-1)!!}$$

$$< n \frac{2^n(1.5n)!^2}{(3n/2-\alpha n)!2^{\alpha n}(\alpha n/2)!^2(3n-1)!!}.$$

Using Stirling formula we have for large n:

$$f(n) \leqslant n \frac{2^{n-\alpha n}(3n/2e)^{3n}}{((3n-2\alpha n)/2e)^{3n/2-\alpha n}(\alpha n/2e)^{\alpha n}(3n/e)^{1.5n}}.$$

Dividing both numerator and denominator by $(n/e)^{3n}$, we get

$$f(n) \leqslant n \left(\frac{2^{1-\alpha}(3/2)^3}{(1.5-\alpha)^{1.5-\alpha}(\alpha/2)^{\alpha}3^{1.5}} \right)^n = \left(\frac{3^{1.5}2^{-2}}{(1.5-\alpha)^{1.5-\alpha}(\alpha)^{\alpha}} \right)^n.$$

Observe that for $\alpha = 0.091$,

$$\phi(\alpha) = \frac{3^{1.5} \times 2^{-2}}{(1.5-\alpha)^{1.5-\alpha}(\alpha)^{\alpha}} = \frac{0.25 \times 3^{1.5}}{1.409^{1.409} \times 0.091^{0.091}} \leqslant 0.997$$

and that the derivative

$$(\ln \phi(\alpha))' = (-(1.5-\alpha)\ln(1.5-\alpha) - \alpha \ln \alpha)' = \ln \frac{1.5-\alpha}{\alpha}$$

is positive for $0 < \alpha < 0.1$. This proves that $f(n) \to_{n \to \infty} 0$. Hence the lemma holds. $\square$

4.4. C_{11} fails to be universal

Now, we can prove the following fact.

Theorem 4.3. *For any $l \geqslant 3$, there exists a cubic graph of girth at least $l+1$ which has no homomorphism to C_{11}.*

Proof. By the above lemmas, for every sufficiently large n, there exists an $(n,3)$-configuration F such that:

(a) $\phi(F)$ contains at most $10(3l)^l$ cycles of length l or less;

(b) for every bipartition (V_1,V_2) of V, the total number of edges induced by $\phi(F)$ in both V_1 and V_2 is at least $0.091n$.

Let n be so large that $10(3l)^l < 10^{-5}n$. Let $H = \phi(F)$. Let M be a smallest set of vertices in H such that it has a vertex from every short cycle in H. By (a), $|M| < 10^{-5}n$. Let $H_1 = H - M$. By the construction, H_1 has no short cycles. Suppose that it has a homomorphism to C_{11}. Then $V(H_1)$ has a tripartition (W_1,W_2,W_3) such that every W_i is independent and $|W_3| \leqslant |V(H_1)|/11$. Each vertex $w \in W_3$ is adjacent either to at most one vertex in W_1, or to at most one vertex in W_2. It follows that we can move each vertex from W_3 either to W_1 or to W_2 so that for the resulting bipartition (W_1',W_2') of $V(H_1)$, the total number of edges induced by both W_1' and W_2' is at most $|V(H_1)|/11 \leqslant n/11$. Now, in the same manner, we can move each vertex from M to W_1' or W_2' so that on each step at most one edge is added to the subgraphs induced by W_1' and W_2'. For the resulting bipartition (V_1,V_2), the total number of edges induced by both V_1 and V_2 is at most

$$n/11 + |M| \leqslant n(1/11 + 10^{-5}) < n(0.09091 + 0.00001) < 0.091n,$$

a contradiction to (b).

Thus, H_1 is a graph of girth at least $l+1$ with the maximum degree at most 3 which has no homomorphism to C_{11}. There are many ways to construct a cubic graph with the same girth containing H_1. One such a way is as follows. Let $k = 3|V(H_1)| - 2|E(H_1)|$. There exists a k-regular graph G_1 of girth $l+1$. Replacing each vertex in G_1 by a copy of H_1 and using for the edges of G_1 free valencies in the corresponding copies of H_1, we get the desired cubic graph G. □

Note that Theorem 4.3 improves a consequence of a theorem of Bollobás ([3], see also [9]) who found the best known upper bound on the independence number of almost all cubic graphs on n vertices.

4.5. Homomorphisms to Petersen's graph and C_5

It has been conjectured [6] that any cubic graph of sufficiently large girth is homomorphic to C_5. In the previous sections we proved that a similar conjecture for C_{11} (instead of C_5) does not hold. Here we prove that there are graphs of girth 7 which are not homomorphic to the Petersen's graph (and thus also to C_5; independently, M. Albertson and E. Moore recently constructed a cubic graph of girth 7 which is not homomorphic to C_5). We shall use the following concept.

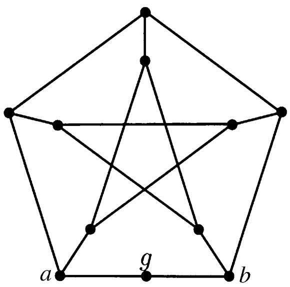

Fig. 5. The graph G, girth$(G) = 5$, $G \nrightarrow P_{10}$.

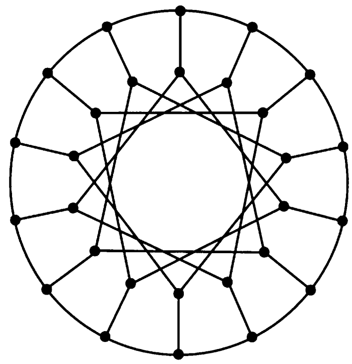

Fig. 6. The graph G, girth$(G) = 7$, $G \nrightarrow P_{10}$.

Definition 4.4. The independence ratio of a graph G is

$$i(G) = \frac{\alpha(G)}{|V(G)|}.$$

We shall apply the following 'no homomorphism lemma'.

Lemma 4.5 (Albertson and Collins [1]). *Let G, H be graphs such that H is vertex-transitive and $G \to H$. Then $i(G) \geqslant i(H)$.*

It is easy to construct a cubic graph of girth 5 which has no homomorphism to the Petersen's graph. We give an example as a warm up. This example consists of two copies of the graph G depicted in Fig. 5 connected by an edge between their vertices g. There is just one homomorphism from $G - \{g\}$ to the Petersen's graph up to the automorphism. This homomorphism maps the vertices a, b to the vertices which are joined with an edge. Since the Petersen's graph contains no triangle there is no homomorphism from G to the Petersen's graph.

Proposition 4.6. *There exists a cubic graph of girth* 7 *which has no homomorphism to the Petersen's graph.*

Proof. The Petersen's graph P_{10} is vertex-transitive and the maximum number of independent vertices in it is 4. Then $i(P_{10}) = \frac{4}{10}$. It suffices to show that there is a cubic graph G of girth 7 with $i(G) < \frac{4}{10}$. Hence $G \nrightarrow H$ according to Lemma 4.5. The desired graph G is depicted in Fig. 6.

It is easy to see that G is cubic and that G has girth 7. With respect to the fact that G has 28 vertices, it suffices to show that G does not have independent set with 12 vertices. (Then $i(G) \leqslant \frac{11}{28} < \frac{4}{10}$.) The graph G is depicted with 14 vertices on the outside and 14 vertices on the inside. We will call the outside vertices *external* and the inside vertices *internal vertices* of G. The internal vertices constitute two cycles with 7 vertices. Then at most 6 of them are independent of each other, 3 in each of the cycles. The external vertices constitute also a cycle. Then at most 7 of them are independent. Let S be an independent set with seven external vertices. If it contains some internal vertices then all of them are from one 7-cycle. Thus it contains at most three internal vertices and at most 10 vertices together.

Let S be an independent set which contains six internal vertices. Number the external vertices according to their order in the 14-cycle. Then no three successive odd vertices and no three successive even vertices are in S all at once. It follows that at most five external vertices are in S. Hence S has at most 11 vertices. No other case is possible. □

Lemma 4.7 (Galluccio et al. [6]). *Let C_k be an odd cycle with k vertices. The C_k-coloring problem is NP-complete for the class of all cubic graphs of girth k.*

We strengthen this result as follows.

Theorem 4.8. *Let $k \geqslant 5$ be an odd integer. Then the C_k-coloring problem is NP-complete for the class of all cubic graphs of girth $k+2$.*

Proof. Let G be a given graph of girth at least k. We will define a new graph $^*G$ with the following properties:

(a) $\Delta(^*G) \leqslant 3$,
(b) $\text{girth}(^*G) = k+2$,
(c) $^*G \to C_k$ if and only if $G \to C_k$,
(d) $|V(^*G)| \leqslant c|V(G)|$, where c depends only on k and $|V(H)|$.

The C_k-coloring problem is NP-complete for the class of all cubic graphs of girth at least k according to Lemma 4.7. Then the C_k-coloring problem must be NP-complete for $^*G$ (i.e. for cubic graphs of girth at least $k+2$).

Construction of the graph $^*G$ (see Fig. 7 for illustration): First we shall construct a graph $I(k)$. Consider two disjoint sets of vertices $\{a_1, a_2 \ldots a_{k+2}\}$ and $\{b_1, b_2 \ldots b_{k+2}\}$. Add the edges $\{a_i, b_i\}$ and $\{a_i, a_{(i+2) \bmod (k+2)}\}$ for $i = 1, 2 \ldots k+2$. For every even $i \in \{2, 4 \ldots k+1\}$ join the vertices b_i and b_{i+1} by the path $P_{i,i+1}$ with $(k-1)/2$ edges (b_i is initial and b_{i+1} terminal vertex of $P_{i,i+1}$). Join the vertices b_1, b_2; b_3, b_4 respectively; by the path $P_{1,2}$; $P_{3,4}$ respectively; both with $(k-1)/2$ edges. Denote the first edge of the path P_{12} by e and the last edge of the path P_{45} as f. Denote the arising graph by $I(k)$. Call as a touch vertex one of the vertices of $I(k)$ which is not incident with e, nor with f, and which has degree 2.

Let v be a vertex of G. Take $\deg(v)$ copies of $I(k)$, for every odd $i \in \{1, 3 \ldots \deg(v)-1\}$ identify the edge f of ith and $(i+1)$th copy and for every even $i \in \{2, 4 \ldots \deg(v)-1\}$ identify the edge e of ith and $(i+1)$th copy. Call the arising graph as $I(k, \deg(v))$. Denote by $T(v)$ the set of all touch vertices in $I(k, \deg(v))$. We call the vertices of degree 2 in $T(v)$ as free touch vertices of v. Replace every vertex v of G by the graph $I(k, \deg(v))$. If $\{u, v\}$ is an edge of G then connect one of the free touch vertices of v with one of the free touch vertices of u. (These used touch vertices become non-free.)

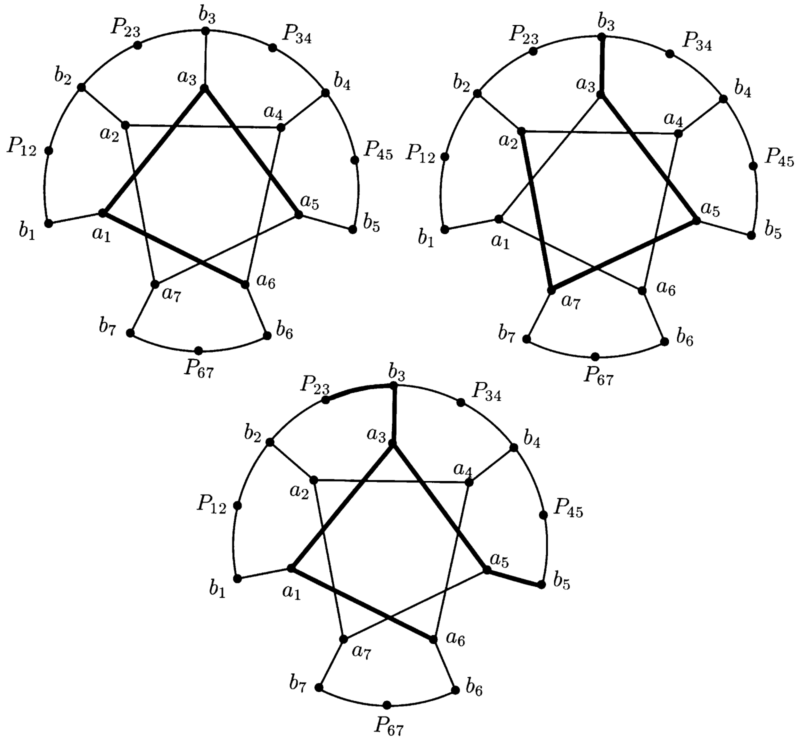

Fig. 7. C_5-coloring of $I(5)$.

Now we will verify the four properties of $^*G$ announced above.

(a) It follows clearly from the construction of $^*G$. Moreover, $^*G$ can be easily transformed to the cubic graph with the core $^*G$ (by hanging a copy of a cubic graph of girth at least $k+2$ which is homomorphic to C_k on the vertices of degree 2 in $^*G$ and making some changes).

(b) The graph $I(k)$ is of girth $k+2$ and length of the shortest path between 2 of the touch vertices of any vertex v of G is 6. Thus $\text{girth}({}^*G) = \min\{7 \times \text{girth}(G), k+2\} \geqslant \min\{7k, k+2\} = k+2$.

(c) We shall show that each homomorphism of $^*G$ into C_k maps all the touch vertices of some vertex $v \in V(G)$ to the same vertex. Number the vertices of C_k by $1, 2 \ldots k$ in such a way that the successive vertices are joined with an edge. Let f be a homomorphism from $I(k)$ to C_k. We claim that in every cycle of lengh $k+2$ in $I(k)$ just 3 edges are mapped to the same edge and these 3 edges are successive. This is now the main rule. Images of the remaining edges differ from each other. Consider all the neighbours of the vertex a_3 in $I(k)$ (i.e. a_1, a_5 and b_3). At least two of them have the same image. We will consider the 3 cases. In Fig. 7 we mark with a bold line the edges $\{u, v\}$ in $I(k)$ which are mapped in f to the edge $\{f(a_3), f(a_5)\}$. We call the edges $\{a_i, a_{(i+1) \bmod (k+2)}\}$, $i = 1, 2 \ldots k+2$, the inner edges of $I(k)$.

(1) $f(a_1) = f(a_5) \neq f(b_3)$:

Then $\{a_1, a_3\}$, $\{a_3, a_5\}$ need to be bold and $\{a_3, b_3\}$ cannot be bold. By the main rule applied to the cycle $\{a_1, a_3 \ldots a_{k+1}\}$ just one of the edges $\{a_1, a_{k+1}\}$ and $\{a_5, a_7\}$ is bold. Without loss of generality, let it be $\{a_1, a_{k+1}\}$. None of the other inner edges is bold. Then there are only 2 successive bold edges in the cycle $\{b_3 \ldots b_4, a_4, a_6 \ldots a_1, a_3\}$. This contradicts the rule.

(2) $f(a_1) \neq f(a_5) = f(b_3)$:

Then $\{a_3, b_3\}$, $\{a_3, a_5\}$ are bold, $\{a_1, a_3\}$ is not bold. By the main rule applied to the cycle $\{b_2 \ldots b_3, a_3, a_5 \ldots a_{k+1}, a_2\}$ either $\{a_5, a_7\}$ or the last edge of $P_{2,3}$ is bold. But it could not be the last edge of $P_{2,3}$, because it would make a contradiction with the main rule applied to the cycle $\{b_1 \ldots b_2 \ldots b_3, a_3, a_1\}$. Thus $\{a_5, a_7\}$ is bold. Now we will apply the main rule to the cycle $\{a_1, a_3 \ldots a_{k+1}\}$. We get that $\{a_7, a_{9 \bmod k}\}$ is bold. Then there are 4 succesive bold edges in the cycle $\{b_2 \ldots b_3, a_3, a_5 \ldots a_{k+2}, a_2\}$. This contradicts the main rule.

(3) $f(a_1) = f(a_5) = f(b_3)$:

Then $\{a_3, b_3\}$, $\{a_1, a_3\}$ and $\{a_3, a_5\}$ need to be bold. By the main rule applied to the cycle $\{a_1, a_3 \ldots a_{k+1}\}$ just one of the edges $\{a_1, a_{k+1}\}$ and $\{a_5, a_7\}$ is bold. Without loss of generality, let it be $\{a_1, a_{k+1}\}$. No one of the other inner edges is bold. According to the rule applied to the cycle $\{b_3 \ldots b_4, a_4, a_6 \ldots a_1, a_3\}$ no other edge of this cycle is bold. We get that $\{a_5, b_5\}$ is bold from the cycle $\{b_3 \ldots b_4 \ldots b_5, a_5, a_3\}$ and that the last edge of p_{23} is bold from the cycle $\{b_2 \ldots b_3, a_3, a_5 \ldots a_{k+1}, a_2\}$. Hence we get the unique coloring (up to symmetry of $I(k)$) of the path $\{b_1 \ldots b_2 \ldots b_3 \ldots b_4 \ldots b_5\}$ which is increasing or decreasing. For us, the most important fact is that the colors of the vertices of the edge e or f give unambiguously the color of the touch vertex (and vice versa).

Let g be a homomorphism from ${}^*G$ to C_k. We know that the touch vertices of any vertex v has the same color. For every edge $\{u, v\}$ of G there is an edge between a touch vertex of u and a touch vertex of v. Thus, g gives a homomorphism from G to C.

Conversely let f be a homomorphism from G to C. For any $w \in V({}^*G)$, put $g(w) = f(v)$, $w \in T(v)$. Then, g can be similarly extended to the rest of vertices (these vertices are the in $I(k, \deg(v))$). □

(d) It follows easily from the construction of ${}^*G$. □

5. Degenerate graphs

If we replace the cubic graphs by k-degenerate and even 2-degenerate graphs then we have a nearly complete solution.

Theorem 5.1. *Let H be a non-bipartite graph. Then the H-coloring problem is NP-complete for the class of all $\omega(H)$-degenerate graphs.*

Proof. Let H be a non-bipartite graph. We will use the indicator construction for $l=3$. Let $k=\omega(H)$. Let I be the graph obtained from two copies of K_{k+1} in the following way: denote a, c two arbitrary vertices of one copy of K_{k+1} and b, d two arbitrary vertices of the other copy of K_{k+1}. Delete the edges $\{a,c\}$ and $\{b,d\}$ and add the edges $\{a,d\}$ and $\{c,b\}$. Then I is homomorphic to H because it is homomorphic to K_k. The indicator construction with I (and vertices a and b) transforms H into the graph H^* which is undirected by the symmetry of I. We will show that H^* has no loops.

Assume to the contrary that there is a homomorphism $f: I \to H$ which maps a and b to the same vertex v. The vertex a is mapped to a vertex of a k-clique in H, because f maps a clique of I to a clique of H with the same number of vertices. The vertex $f(c)$ must be joined with each vertex of this k-clique, except $f(a)$. But $f(a)=f(b)$ and $f(c)$ is joined with $f(b)$. Since $f(c)$ is joined with each vertex of a k-clique in H and it constitutes with this k-clique a $(k+1)$-clique in H. This is a contradiction.

For proving that H^* is not bipartite we will consider the two following cases: If $k=2$ then H^* contains all the edges of H. Hence, it must be non-bipartite. If $k\geqslant 3$ then the indicator I is homomorphic to H in such a way that a, b are mapped to an arbitrary edge of a k-clique. Hence H^* contains K_k. Thus H^* is non-bipartite.

Let G be an arbitrary graph. The H^*-coloring problem is NP-complete for G according to Theorem 1.1. Then the H-coloring problem is NP-complete for the transformed graph $G*$ according to Lemma 2.5. It remains to verify that the new graph $^*G$ is k-degenerate. We will define the numbering of vertices of $^*G$ in the following way: first we number the original vertices of G arbitrarily (this set of vertices is independent in $^*G$), afterwards we number the rest of vertices arbitrarily (these vertices are the in copies of I – they have degree just k). This numbering demonstrates that $^*G$ is k-degenerate, i.e. $\omega(H)$-degenerate. □

6. Concluding remarks

A related problem was considered in [11], where the following statement was proved: Let k, m be two positive integers. If G is a graph of girth $g > 2^{k+2}(4km)^{4km-1}+2(k+1)$ then any partial k-tree homomorphic to G is also homomorphic to the odd cycle C_{2m+1}.

Let us note that the cubic graphs which do not map to C_{11} have circular chromatic number (see [13] for the definition) at least $\frac{11}{5}=2+\frac{1}{5}$. One can check that the circular chromatic number of a graph G is at most $2+1/m$ if and only if there is a homomorphism $G \to C_{2m+1}$.

Circulant graphs are symmetric in the sense of Definition 3.2. Seeing that for computation of the circular chromatic number of a given graph G is necessary to decide whether G is homomorphic to a circulant graph, we can conclude from Remark 3.3 that the problem of finding the circular chromatic number of a given graph G with girth at least l is NP-complete for any l.

It is not clear how to prove that Petersen's graph-coloring problem is NP-hard for cubic graphs of girth 5 (a problem mentioned already in [6]).

On the positive side it follows from [4,5] that the following problem is polynomial for any k and any graph H:

Instance: *a partial k-tree G.*

Question: *does there exists a homomorphism $G \to H$?*

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