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Colouring Relatives of Intervals on the Plane, II: Intervals and Rays in Two Directions

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We give exact upper bounds on the chromatic number for the intersection graphs of intervals and rays in two directions on the plane in terms of clique number.

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1. INTRODUCTION

This paper is a continuation of our previous work [3]. The topic of bounds on a chromatic number of intersection graphs of intervals and their relatives on the plane was initiated by Asplund and Grünbaum [1] and Gyárfás and Lehel [2]. A number of these problems can be stated in the following framework. For a class $\mathcal{G}$ of intersection graphs and for a positive integer k, $k \geq 2$ find or bound:

(i) $f(\mathcal{G}, k)$, the maximum chromatic number of a graph in $\mathcal{G}$ with a clique number at most k;
(ii) $g(\mathcal{G}, k)$, the maximum chromatic number of a graph in $\mathcal{G}$ with a girth at least k (here we assume $k \geq 4$).

Note that $f(\mathcal{G}, 2) = g(\mathcal{G}, 4)$. In [3] we obtained bounds on $g(\mathcal{G}, k)$ for the following families:

$\mathcal{I}$: intersection graphs of intervals on the plane;
$\mathcal{R}$: intersection graphs of rays on the plane;
$\mathcal{S}$: intersection graphs of families of strings on the plane such that the intersection of any two strings is a connected subset of the plane;
$\mathcal{I}_m$: intersection graphs of intervals on the plane parallel to some m lines;
$\mathcal{R}_m$: intersection graphs of rays on the plane parallel to some m lines.

In particular, we have proved that for every integer $k \geq 6$, $g(\mathcal{R}, k) = 3$ and that $g(\mathcal{R}_2, 5) = 3$. Let us remark that the finiteness of $g(\mathcal{I}, 4) = f(\mathcal{I}, 2)$, $g(\mathcal{S}, 4) = f(\mathcal{S}, 2)$ is an open problem.

In the present paper we are concerned with $\mathcal{I}_2$ and $\mathcal{R}_2$. Obviously, $\mathcal{R}_2 \subseteq \mathcal{I}_2$ and thus $f(\mathcal{R}_2, k) \leq f(\mathcal{I}_2, k)$ for every k. The main result of this paper states that they are the same.

THEOREM 1. *Let $k \geq 2$ be an integer. Then*

$$f(\mathcal{R}_2, k) = f(\mathcal{I}_2, k) = \begin{cases} 2k, & \text{if } k \text{ is even;} \\ 2k - 1, & \text{if } k \text{ is odd.} \end{cases} \tag{1}$$

Recall that intersection graphs of line segments are perfect (which in our terminology is equivalent to $f(\mathcal{I}_1, k) = k$).

In Section 2 we introduce some notation. In Section 3 we give the upper bound on $f(\mathcal{I}_2, k)$. In the final section we present a modification of the Asplund–Grünbaum construction [1] to obtain the lower bound on $f(\mathcal{R}_2, k)$, which completes the proof of Theorem 1.

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2. Notation

Suppose that a family F of intervals, rays or lines is an *m-direction family* if there are m (straight) lines $l_1, \ldots, l_m$ such that any member of F is parallel to some l_i, $1 \le i \le m$.

For a family F of subsets of P, its *intersection graph* $G = G_F$ is the undirected graph with the vertex set F such that for $r, p \in V(G)$,

$$(r, p) \in E(G) \longleftrightarrow r \cap p \neq \emptyset.$$

Certainly, for a graph G, there could be different families F' and F'' such that $G = G_{F'} = G_{F''}$. Any such a family is called a *representation* of G. The maximal size of a clique in the graph G_F will be also called *clique number of F*

For a family F of intervals in a line l and a real $c \in l$, let $F^+(c)$ denote the subfamily of intervals in F whose left end is greater than c. Similarly, $F^-(c)$ denotes the subfamily of intervals in F whose right end is less than c and $F^0(c)$ denotes the set of intervals in F containing c.

3. The Upper Bound

An interval $I \in F$ is said to be *k-covered* (by F) if all its points belong to at least $k/2$ intervals in F (including I), and *k-uncovered* otherwise.

LEMMA 1. *For every family F of intervals in a line with clique number k, there exists a k-colouring of G_F such that the colour k is used only for colouring of vertices corresponding to k-covered intervals.*

PROOF. By induction on $|F|$. Let F be a smallest family contradicting the lemma.

CASE 1. There exists $c \in l$ such that $|F^0(c)| < k/2$, $F^-(c) \neq \emptyset$ and $F^+(c) \neq \emptyset$. By the minimality of F, there are required k-colourings f^- and f^+ of $F^-(c) \cup F^0(c)$ and $F^+(c) \cup F^0(c)$, respectively. Note that $|f^-(F^0(c))| = |f^+(F^0(c))| = |F^0(c)|$ and the colour k is not in $f^-(F^0(c)) \cup f^+(F^0(c))$. Hence we can rename some colours in f^+ (not touching k) so that $f^-(I) = f^+(I)$ for every $I \in F^0(c)$. Then $f^- \cup f^+$ is a required colouring for F, a contradiction.

CASE 2. For every $c \in l$, if $|F^0(c)| < k/2$, then either $F^-(c) = \emptyset$ or $F^+(c) = \emptyset$. This means that there are at most $k-1$ k-uncovered intervals. We take any proper k-colouring of F and rename the colours so that none of the (at most) $k-1$ k-uncovered intervals gets colour k. This proves the lemma. □

LEMMA 2. *For every 2-direction family F of intervals with clique number k, there exists a $2k$-colouring of G_F. Moreover, if k is odd, then there exists a $(2k-1)$-colouring of G_F.*

PROOF. The first statement is trivial: we simply colour first with k colours all the intervals in the first direction, and colour with colours $k+1, \ldots, 2k$ all the intervals in the second direction.

Let k be odd. By Lemma 2, we can colour all the intervals in the first direction with colours $1, \ldots, k$ so that colour k will be used only for some k-covered intervals. Similarly, we can colour with colours $k, \ldots, 2k-1$ all the intervals in the second direction so that colour k also will only be used for some k-covered intervals. Since no k-covered interval in the first direction meets any k-covered interval in the second direction, we obtain a proper $(2k-1)$-colouring of G_F. □

4. CONSTRUCTION

We modify the construction of Asplund and Grünbaum [1]. The two directions in question will be horizontal and vertical.

4.1. Building blocks. A horizontal *k-bunch* with *centre a* is a set of k horizontal rays with a common origin O having x-coordinate a, such that $\lceil k/2 \rceil$ of these rays are oriented to the right and the remaining $\lfloor k/2 \rfloor$ rays are oriented to the left. The rays oriented to the right are called *the right part* of the bunch, and the rest are *the left part* of the bunch.

A horizontal (k, l)*-bundle* is a set of 2^l disjoint horizontal k-bunches with different centres. Similarly, we can define vertical bunches and bundles (where the role of the 'right' plays 'up'). A *k-bundle* is a (k, l)-bundle with $l = l(k) = \lfloor \log_2 k \rfloor$.

Given a horizontal (k, l)-bundle B with centres $a_1, \dots, a_{2^l}$ (where $a_1 < a_2 < \dots < a_{2^l}$), the 2^l*-zone* is the open half-plane to the right of the line $y = a_{2^l}$ and for $i = 1, \dots, 2^l - 1$, the *i-zone* is the open vertical strip whose left boundary is the line $y = a_i$ and right boundary is the line $y = a_i$. Let B^i denote the set of rays in B intersecting the i-zone. Since no i-zone contains a centre, for every i the clique number of the intersection graph of B^i is at most $\lceil k/2 \rceil$.

LEMMA 3. *Let B be a horizontal (k, l)-bundle with centres $a_1, \dots, a_{2^l}$, where $a_1 < a_2 < \dots < a_{2^l}$. For every proper colouring of G_B, there exists an i such that on the rays in B^i at least $k - \frac{k}{2^{l+1}}$ colours are present.*

PROOF. Let $l = 0$. Then B consists of $2^0 = 1$ k-bunch and B^1 comprises the right part R of this bunch. For any proper colouring of G_B, exactly colours $\lceil k/2 \rceil$ are used on R.

Suppose that the lemma is true for every $l' < l$. Let f be a proper colouring of G_B. Consider the $(k, l-1)$-bundle B_1 with centres $a_1, a_3, a_5, \dots, a_{2^l - 1}$. By the induction assumption, there is an i such that on B_1^i at least $k - \frac{k}{2^l}$ colours are present. Suppose that the set of colours present on B_1^i is S and $|S| = s$. The x-borders of the i-zone in B_1^i are a_{2i-1} and a_{2i+1}. Then at least $k - s$ colours not in S are present on the k-bunch with the centre a_{2i}. Therefore, either B^{2i-1} or B^{2i+1} meets at least $s + 0.5(k - s)$ colours. This proves the lemma. □

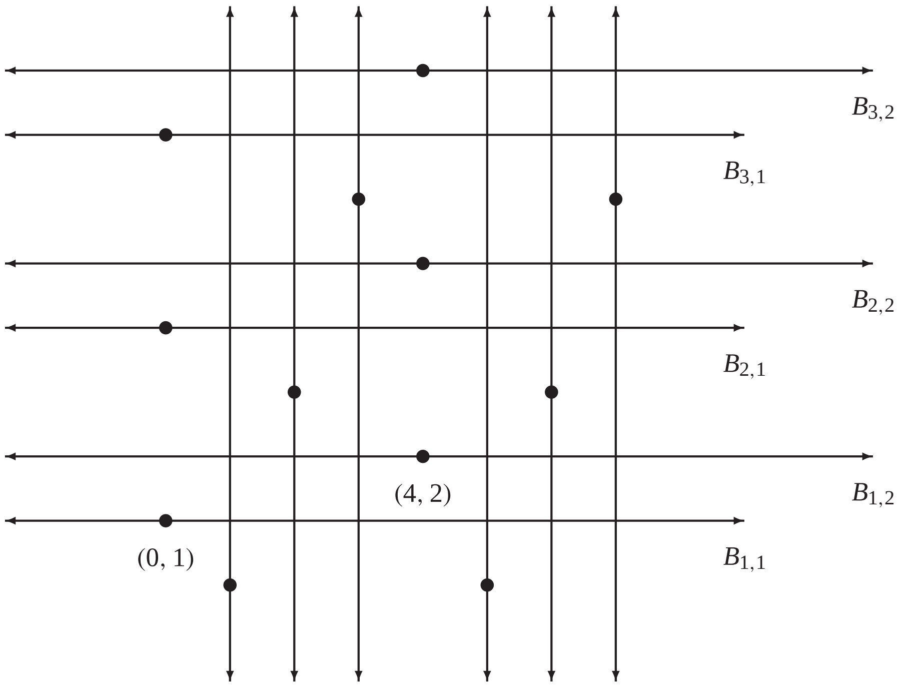
COROLLARY 1. *Let B be a horizontal k-bundle with centres $a_1, \dots, a_{2^{l(k)}}$, where $a_1 < a_2 < \dots < a_{2^{l(k)}}$. For every proper colouring of the intersection graph of B, there exists an i such that on the rays in B^i at least k colours are present.*

PROOF. By Lemma 3, on the rays in some B^i at least $k - \frac{k}{2^{l(k)+1}} = k - \frac{k}{2^{\lfloor \log_2 k \rfloor + 1}} > k - 1$ colours are present. □

For a k-bundle B and a colouring f of G_B, we say that an i-zone is *f-bad*, if f uses at least k colours on B^i. Then Corollary 1 simply states that for every k-bundle B and every proper colouring f of G_B, there is an f-bad i-zone.

4.2. The idea and an example. The construction (for an even k) consists of a family U of horizontal k-bundles and a family T of vertical k-bunches such that for every proper colouring f of the whole family, we can choose $l(k)$ vertical k-bunches forming a k-bundle B_0 and $l(k)$ horizontal k-bundles such that every i-zone of B_0 intersects an f-bad zone of some k-bundle in U. Then an f-bad zone of B_0 (which exists by Corollary 1), intersects an f-bad zone of some k-bundle in U. This implies that we need at least $2k$ colours. Since each of the left and right parts of every k-bunch contains exactly $k/2$ rays, the clique number of $G_{U \cup T}$ will be k.

FIGURE 1. A 4-chromatic ray family.



EXAMPLE. Let $k = 2$, then $l(k) = 2$. The family U consists of three 2-bundles B_1, B_2 and B_3. The bundle B_i consists of two 2-bunches: $B_{i,1}$ with the centre at point $a_{i,1} = (0, 3i - 2)$ and $B_{i,2}$ with the centre at point $a_{i,2} = (4, 3i - 1)$. Note that j-zones ($j = 1, 2$) are the same for all three 2-bundles.

The family T consists of six vertical 2-bunches $W_{i,j}$ ($i \in \{1, 2, 3\}$, $j \in \{1, 2\}$), where the centre of $W_{i,j}$ is located at $w_{i,j} = (3(j - 1) + i, 3(i - 1))$ (see Figure 1).

Assume that there exists a 3-colouring f of $G_{U \cup T}$. By Corollary 1, either 1-zone or 2-zone is f-bad for at least two 2-bundles among B_1, B_2 and B_3. Let, for definiteness, 1-zone be f-bad for 2-bundles B_1 and B_3. Consider the 2-bundle W formed by the 2-bunches $W_{1,1}$ and $W_{3,1}$ (here index j was chosen to be 1, since 1-zone is f-bad, and the index i was chosen to be 1 and 3 since the zone is f-bad for B_1 and B_3). Now, the 1-zone for W crosses an f-bad zone for B_1 and the 2-zone for W crosses an f-bad zone for B_3. Thus in one of these spots, we need at least $2 + 2 = 4$ different colours, a contradiction.

4.3. The construction for even k. Let $k \geq 2$ be even and $l = l(k) = \lfloor \log_2 k \rfloor$. The family U of horizontal rays in the construction consists of $l(l - 1) + 1$ k-bundles $B_1, \ldots, B_{l(l-1)+1}$. The bundle B_i consists of l k-bunches $B_{i,j}$, where the centre of $B_{i,j}$ is at point $a_{i,j} = ((j - 1)l^2, (l + 1)(i - 1) + j)$. Note that j-zones ($j = 1, \ldots, 2^l$) are the same for all k-bundles in U.

The family T of vertical rays in the construction consists of k-bunches $W_{i,j}$ ($i \in \{1, \ldots, l(l - 1) + 1\}$, $j \in \{1, \ldots, l\}$), where the centre of $W_{i,j}$ is located at $w_{i,j} = (l^2(j - 1) + i, (l + 1)(i - 1))$.

Assume that there exists a proper $(2k-1)$-colouring f of $G_{U\cup T}$. By Corollary 1, every of B_i, $i = 1, \ldots, l(l-1)+1$, contains an f-bad $j(i)$-zone for some $j(i) \in \{1, \ldots, l\}$. Then there exists a j_0 such that j_0-zone is f-bad for at least l B_i-s, say, for $i_1, \ldots, i_l$. Denote by W the k-bundle formed by k-bunches $W_{i_1, j_0}, \ldots, W_{i_l, j_0}$. By Corollary 1, for some s, s-zone of W is f-bad. Let R denote the rectangle $[l^2(j-1), l^2 j] \times [(l+1)(i_s - 1), (l+1)i_s]$. By the above, the interior of R is crossed by horizontal rays of at least k colours and by vertical rays of at least k colours. This contradicts the choice of f.

4.4. The case of odd k. If we simply repeat the construction from Section 4.3, then the resulting graph will have clique number $k+1$, since the right parts of horizontal k-bunches and up parts of vertical k-bunches have $(k+1)/2$ rays each. To avoid this, consider the construction $(U \cup T)_{k-1}$ for $k-1$ and replace every $(k-1)$-bunch in U by a k-bunch. Since for an odd k, $l(k) = l(k-1)$, every former $(k-1)$-bundle B_i becomes a k-bundle. Now, the clique number of $G_{U\cup T}$ is k, and repeating the argument of Section 4.3, we see that at some spot, vertical rays of at least $k-1$ colours cross horizontal rays of at least k colours. Therefore, for every odd k we get a $(2k-1)$-chromatic construction.

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