Coloring intersection graphs of geometric figures with a given clique number

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Abstract

The paper discusses the current status of bounds on the chromatic number of the intersection graphs of certain types of geometric figures with given clique number or girth. The families under consideration are boxes in $\mathbf{R}^n$, intervals in the plane, chords of a circle, and tranlates of a compact convex set. A couple of new results fitting in the picture are proved.

1 Introduction

The *intersection graph* G of a family $\mathcal{F}$ of sets is the graph with vertex set $\mathcal{F}$ where two members of $\mathcal{F}$ are adjacent if and only if they have common elements. Intersection graphs of geometric figures of special kinds can be interesting both from geometric and graph-theoretic points of view.

One of reasonable questions to ask about a family $\mathcal{F}$ of graphs is: *What is the maximum chromatic number $\chi(\mathcal{F}, k)$ over graphs in $\mathcal{F}$ with clique number at most k?* Although, as Erdős [12] observed, in general, there are graphs of arbitrarily large girth with arbitrarily high chromatic number, for several interesting families $\mathcal{F}$ of graphs the function $\chi(\mathcal{F}, k)$ is well defined. Sometimes, it makes sense to study how high the chromatic number of graphs in a family $\mathcal{F}$ with a given girth may be. In this case, somewhat artificial notation will be used: for a positive integer k, $\chi(\mathcal{F}, -k)$ will denote the maximum chromatic number of graphs in $\mathcal{F}$ with girth at least k. In particular, $\chi(\mathcal{F}, -4) = \chi(\mathcal{F}, 2)$.

Studying functions $\chi(\mathcal{F}, k)$ and $\chi(\mathcal{F}, -k)$ for families of intersection graphs of geometric figures and their complements was stimulated by seminal papers of Asplund and Grünbaum [5] and Gyárfás and Lehel [23, 25].

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The aim of this paper is to discuss the status of problems on $\chi(\mathcal{F}, k)$ and $\chi(\mathcal{F}, -k)$ for some interesting families of intersection graphs of geometric figures and their complements. For several of them, also the function $\phi(\mathcal{F}, k)$—the minimum d such that every graph $G \in \mathcal{F}$ with clique number at most k is d-degenerate—is discussed. Recall that a graph G is *d-degenerate* if every subgraph H of G has a vertex v of degree at most d in H. The reason to consider $\phi(\mathcal{F}, k)$ is that every d-degenerate graph is $(d+1)$-colorable, and moreover, 'easily $(d+1)$-colorable': the vertices of any d-degenerate graph can be ordered $v_1, v_2, \dots$ in such a way that for every i the vertex v_i has at most d neighbors among $v_1, v_2, \dots, v_{i-1}$, and we can color the vertices of such a graph one by one greedily. In particular, every d-degenerate graph is $(d+1)$-list-colorable and $\phi(\mathcal{F}, k) \geq \chi(\mathcal{F}, k) - 1$.

Analogously to $\chi(\mathcal{F}, -k)$, for a positive integer k, $\phi(\mathcal{F}, -k)$ is the minimum d such that every graph in $\mathcal{F}$ with girth at least k is d-degenerate.

Essentially, the paper discusses the intersection graphs of the following four families: boxes, intervals in the plane, chords of a circle, and equal figures (without rotation), a section for each. The final section describes applications of previous results to Ramsey type problems for the intersection graphs of geometric figures.

2 Boxes

A *box* in an n-dimensional space with given axes is a parallelepiped with sides parallel to the axes. Let $\mathcal{B}_n$ denote the family of the intersection graphs of boxes in the Euclidean n-dimensional space. Clearly, $\mathcal{B}_1$ is the family of the interval graphs. Therefore, for every k, $\chi(\mathcal{B}_1, k) = \phi(\mathcal{B}_1, k) + 1 = k$.

Bielecki [7] asked in 1948 whether $\chi(\mathcal{B}_2, 2)$ is finite. Asplund and Grünbaum [5] not only proved that $\chi(\mathcal{B}_2, k)$ is finite for every k, but found the exact value of $\chi(\mathcal{B}_2, 2)$: it is 6. The construction proving the lower bound is very nice. The proof of finiteness of $\chi(\mathcal{B}_2, k)$ implies the bound $\chi(\mathcal{B}_2, k) \leq 4k^2 - 3k$. Hendler [26] improved the bound to $\chi(\mathcal{B}_2, k) \leq 3k^2 - 2k - 1$. The best known lower bound is linear: $\chi(\mathcal{B}_2, k) \geq 3k$. The gap between the upper and lower bounds is challenging. Even improving the constant factor in the lower bound is interesting.

The situation with higher dimensions is quite different. Burling [8] constructed triangle-free graphs in $\mathcal{B}_3$ with arbitrarily high chromatic number. This means that $\chi(\mathcal{B}_n, k) = \infty$ for every $n \geq 3$ and $k \geq 2$.

Since every complete bipartite graph is in $\mathcal{B}_2$, we have $\phi(\mathcal{B}_2, 2) = \phi(\mathcal{B}_2, -4) = \infty$. But imposing stricter restrictions on girth of graphs in $\mathcal{B}_2$ leads to finite upper bounds on degeneracy. Perepelitsa and I [38] proved that $\phi(\mathcal{B}_2, -6) = 3$ and $\phi(\mathcal{B}_2, -8) = 2$. This yields $\chi(\mathcal{B}_2, -6) \leq 4$ and $\chi(\mathcal{B}_2, -8) = 3$. Then Glebov [20] proved that $\phi(\mathcal{B}_2, -7) = 3$ and $\phi(\mathcal{B}_2, -5) = 4$. The last bound implies that $\chi(\mathcal{B}_2, -5) \leq 5$ which is less than $\chi(\mathcal{B}_2, -4)$ mentioned above.

The value of $\chi(\mathcal{B}_n, -5)$ should be finite for every n, but I do not know whether anybody proved such bounds. It is likely that $\phi(\mathcal{B}_n, -5)$ is also finite for every n.

Gyárfás and Lehel [25] studied how few points are sufficient to pierce any family of boxes in the plane that has no $k+1$ pairwise disjoint members. Since every family of boxes possesses Helly property, this is equivalent to finding how few cliques are sufficient to cover the vertices of any intersection graph of boxes in the plane with the independence number at most k. In other words, this is equivalent to determining $\chi(\overline{\mathcal{B}}_2, k)$, where $\overline{\mathcal{B}}_n$ is the family of complements of the intersection graphs of boxes in the Euclidean n-dimensional space.

Gyárfás and Lehel [25] proved that $\lfloor 3k/2 \rfloor \leq \chi(\overline{\mathcal{B}}_2, k) \leq k(k-1)/2$. Then Beck [6] improved the upper bound to $ck\log^2 k$ and Károlyi [30] further improved and generalized the bound to $\chi(\overline{\mathcal{B}}_n, k) \leq (1+o(1))k\log^{n-1} k$. Fon-Der-Flaass and I [15] applied an idea of Gyárfás and Lehel of moving some hyperplane into an extremal position to give a simple proof of a slight refinement of the Károlyi's bound:

$$\chi(\overline{\mathcal{B}}_n, k) \leq k\log_2^{n-1} k + n - 0.5k\log_2^{n-2} k \quad \text{for } n \geq 2. \tag{1}$$

We also showed that $c\sqrt{n}/\log n \leq \chi(\overline{\mathcal{B}}_n, 2) \leq n+1$ for every n and that $\chi(\overline{\mathcal{B}}_n, 2) = n+1$ for $n \leq 4$. The asymptotic behavior of $\chi(\overline{\mathcal{B}}_n, 2)$ for $n \to \infty$ is not clear. It is possible that $\chi(\overline{\mathcal{B}}_n, 2) = o(n)$.

The lower bound $\chi(\overline{\mathcal{B}}_2, k) \geq \lfloor 3k/2 \rfloor$ was improved in [15] to $\lfloor 5k/3 \rfloor$. The main unsettled question here is whether $\chi(\overline{\mathcal{B}}_2, k)$ is superlinear or not. Even better constant factors at k in the lower bound would be interesting. Fon-Der-Flaass [14] proved that if the ratios of the length and height of boxes in a family are bounded from below and above by positive constants c_1 and c_2, then the chromatic number of a graph of this family is at most $C\,k$, where C depends on c_1 and c_2.

Similarly to $\mathcal{B}_2$, every complete bipartite graph is in $\overline{\mathcal{B}}_2$, and hence $\phi(\overline{\mathcal{B}}_2, 2) = \phi(\overline{\mathcal{B}}_2, -4) = \infty$. It seems that $\phi(\overline{\mathcal{B}}_2, -k)$ for $k \geq 5$ was not considered before. Below, the idea of an extremal hyperplane mentioned above is applied to settle this question. But first, consider the following simple lemma.

Lemma 1 *Let H be the complement of an interval graph. If the girth of H is at least five, then*
(a) H is bipartite;
(b) H does not contain any path of length 4;
(c) H is acyclic.

Proof. Let $\mathcal{F}$ be a family of closed intervals on the X-axis such that the complement H of the intersection graph of $\mathcal{F}$ has girth at least five. Then H has a transitive orientation. Any transitive orientation of a non-bipartite graph has a transitively oriented triangle. Since H has girth at least five, this proves (a).

Suppose H has a path $(x_0, x_1, x_2, x_3, x_4)$, where x_i corresponds to the interval $X_i = [l_i, r_i] \in \mathcal{F}$, $i = 0, \ldots, 4$. By (a) and the girth condition, the sets $\{x_0, x_2, x_4\}$ and $\{x_1, x_3\}$ are independent in H. Therefore, the intervals X_0, X_2, and X_4 have a common point, say p_0, and the intervals X_1 and X_3 have a common point, say p_1. We may assume that $p_0 < p_1$ and that $l_1 \leq l_3$. Since X_2 is disjoint from both X_1 and

X_3, $p_0 < l_1 \leq l_3 \leq p_1$. Since X_0 meets X_3, we have $p_0 \leq l_3 \leq r_0$. But then X_0 meets X_1, a contradiction. This proves (b). Now (a) and (b) together yield (c).

Theorem 1 *$\phi(\overline{\mathcal{B}}_2, -k) = 1$ for $k \geq 7$ and $\phi(\overline{\mathcal{B}}_2, -5) = \phi(\overline{\mathcal{B}}_2, -6) = 2$.*

Proof. First, observe that the complement of the intersection graph of the family $\mathcal{F}_0$ of the 6 squares with the sides of length 6 parallel to the axes of the Cartesian plane whose set of centers is $\{(0,5),(0,-5),(5,0),(-5,0),(2,-2),(-2,2)\}$ is the cycle of length 6. This proves that $\phi(\overline{\mathcal{B}}_2, -5) \geq \phi(\overline{\mathcal{B}}_2, -6) \geq 2$.

Now, assume that one of the upper bounds is false. This means that there exists a family $\mathcal{F} = \{B_i\}_{i=1}^s$ of closed boxes in the Cartesian plane such that the complement H of the intersection graph of $\mathcal{F}$ either is a C_k for some $k \geq 7$ or has minimum degree 3 and girth at least 5. Every B_i is defined by the quadruple $\{l_i, r_i, b_i, t_i\}$, where l_i and r_i (respectively, b_i and t_i) are the lowest and the highest X-coordinates (respectively, Y-coordinates) of the points in B_i. One can always choose an $\mathcal{F}$ such that all l_i, r_i, b_i, and t_i are distinct. Order the boxes in $\mathcal{F}$ so that $r_1 < r_2 < \ldots < r_s$. Consider the line $x = r_2$. If there are i_1 and i_2 such that $l_{i_1} > r_2$ and $l_{i_2} > r_2$, then the subgraph of H induced by the vertex set $\{B_1, B_2, B_{i_1}, B_{i_2}\}$ contains the 4-cycle $(B_1, B_{i_1}, B_2, B_{i_2})$, a contradiction. Hence the line $x = r_2$ meets all members of $\mathcal{F}$ apart from B_1 and maybe a box B_{i_1} on the right of it. Thus, B_1 and B_{i_1} (if exists) are adjacent in H.

The intersection graph of $\mathcal{F} - B_1 - B_{i_1}$ is the same as the intersection graph of the intersections of the members of $\mathcal{F} - B_1 - B_{i_1}$ with the line $x = r_2$, which is an interval graph. By Lemma 1(c), every component M of $H - B_1 - B_{i_1}$ has a vertex v_M of degree at most 1, so that if the minimum degree of H is at least 3, then both B_1 and B_{i_1} are adjacent to v_M in H. Then H has a triangle, a contradiction. Finally, if H is a cycle, then $H - B_1 - B_{i_1}$ is a path, and by Lemma 1(b), has at most 4 vertices. This proves the theorem. ■

3 Intervals and rays in the plane

Let $\mathcal{I}$ and $\mathcal{R}$ be the families of the intersection graphs of intervals and rays in the plane, respectively. Erdős (see e.g. [25]) asked whether $\chi(\mathcal{I}, -4)$ is finite. Kratochvíl and Nešetřil asked the same question for the family $\mathcal{S}$ of the intersection graphs of curves in the plane such that the intersection of every two of them is a connected curve (possibly empty of consisting of a single point). The answers to both questions are unknown. There are triangle-free graphs in $\mathcal{I}$ with chromatic number 8. It follows from recent results of McGuinness [40, 41] that $\chi(\mathcal{R}, 2) < \infty$ and that if the ratio of the longest interval to the shortest in a family of intervals in the plane is bounded by a constant c, then the chromatic number of the intersection graph of this family is bounded by a function of c.

Similarly to the situation with the boxes, the graphs in $\mathcal{S}$ with girth at least 5 have bounded degeneracy. It is proved in [36] that $\phi(\mathcal{S}, -k) < 2(k-2)/(k-4)$ for

$k > 4$. In particular, $\phi(\mathcal{S}, -5) \leq 5$ and $\phi(\mathcal{S}, -8) = 2$. Since $\mathcal{R} \subset \mathcal{I} \subset \mathcal{S}$, the upper bounds on $\phi(\mathcal{S}, -k)$ above are also bounds on $\phi(\mathcal{I}, -k)$ and $\phi(\mathcal{R}, -k)$. But for $\mathcal{R}$, we can say more. It is proved in [36] that $\phi(\mathcal{R}, -5) \leq 3$ and $\phi(\mathcal{R}, -6) = 2$.

For a positive integer m, let $\mathcal{I}_m$ and $\mathcal{R}_m$ be the families of the intersection graphs of intervals and rays in the plane, respectively, parallel to m given directions. Clearly, $\chi(\mathcal{R}_m, k) \leq \chi(\mathcal{I}_m, k) \leq mk$ for every m and k. It is a bit surprising that for $m = 2$ we have here equality for even k and that the values for $\mathcal{I}_2$ and $\mathcal{R}_2$ are the same. It is proved in [37] that

$$\chi(\mathcal{R}_2, k) = \chi(\mathcal{I}_2, k) = \begin{cases} 2k, & \text{if } k \text{ is even;} \\ 2k-1, & \text{if } k \text{ is odd.} \end{cases}$$

The lower bound uses a modification of the Asplund–Grünbaum construction [5]. Again, all complete bipartite graphs are in $\mathcal{R}_2$, and hence $\phi(\mathcal{R}_2, 2) = \infty$.

4 Circular arc graphs and circle graphs

Let $\mathcal{A}$ be the family of the intersection graphs of arcs of a circle. Although, every graph in $\mathcal{A}$ is not far from an interval graph (deleting from a graph $G \in \mathcal{A}$ a clique corresponding to arcs containing a given point on the circle leaves a subgraph of G that is an interval graph), finding chromatic number of graphs in $\mathcal{A}$ is an NP-complete problem [17]. Tucker [45] conjectured that $\chi(\mathcal{A}, k) = \lfloor 3k/2 \rfloor$, and this was proved by Karapetian [27]. With the family $\overline{\mathcal{A}}$ of the complements to graphs in $\mathcal{A}$ the situation is simpler. Gavril [19] proved that $\chi(\overline{\mathcal{A}}, k) = k + 1$ for $k \geq 2$.

A graph is a *circle graph*, if it is the intersection graph of a family of chords of a circle. Circle graphs are also known as *overlap graphs*. They arise in many combinatorial problems ranging from sorting problems to studying planar graphs to continous fractions (see, e.g. [21, 16]). In particular, for a given permutation P of $\{1, 2, \ldots, n\}$, the problem of finding the minimum number of stacks needed to obtain the permutation $\{1, 2, \ldots, n\}$ from P reduces to finding the chromatic number of a corresponding circle graph [13, 21].

Let $\mathcal{X}$ be the family of the circle graphs, and $\overline{\mathcal{X}}$ be the family of their complements. The clique number and the independence number of a circle graph can be found in polynomial time [18, 21], but the problem of finding the chromatic number of such graphs is NP-complete [17], and the complexity of finding the chromatic number of their complements is unknown. This adds more attraction to finding $\chi(\mathcal{X}, k)$ and $\chi(\overline{\mathcal{X}}, k)$.

The situation with $\chi(\overline{\mathcal{X}}, k)$ is more or less clear. It is proved in [34] that

$$\chi(\overline{\mathcal{X}}, k) \sim k \ln k, \tag{2}$$

and the upper bound on $\chi(\overline{\mathcal{X}}, k)$ differs from the lower one by less than $5k/12$. Certainly, one can try to find the exact values of the function, but the formula is likely a bit complicated.

The problem of finding $\chi(\mathcal{X}, k)$ seems more difficult. Karapetian [28, 29] proved that $4 \le \chi(\mathcal{X}, 2) \le 8$. Then it was proved in [34] that $\chi(\mathcal{X}, 2) \le 5$ and Ageev [2] constructed triangle-free circle graphs with chromatic number equal to 5. For general k, Gyárfás [24] proved that $\chi(\mathcal{X}, k) \le 2^k(2^k - 2)k^2$. His idea was elaborated in [34] to prove that $\chi(\mathcal{X}, k) \le 2^k k^2(k-1)$, and then in [35] to prove that $\chi(\mathcal{X}, k) \le 50 \cdot 2^k$. The only known non-linear lower bound (see [34]) is $\chi(\mathcal{X}, k) \ge 0.5k(\ln k - 2)$. Since there were claims (e.g., [46]) that $\chi(\mathcal{X}, k)$ grows linearly, and the construction proving the lower bound was published only in Russian, this construction is described in the last section. Anyway, the gap between the exponential upper bound and the barely super-linear lower bound is truly challenging.

As in most of the previous cases, the complete bipartite graphs are in $\mathcal{X}$, and so $\phi(\mathcal{X}, -4) = \infty$. Ageev [3] proved that $\phi(\mathcal{X}, -k) = 2$ for every $k \ge 5$.

In fact, the upper bound $50 \cdot 2^k$ was proved in [35] for chromatic number of *polygon-circle* graphs, i.e., intersection graphs of polygons inscribed in a circle. The class $\mathcal{PC}$ of such graphs includes all circular arc graphs and all minors of circle graphs. No lower bound on $\chi(\mathcal{PC}, k)$ better than $k \log k$ is known.

5 Translates of a compact convex set

Recently, the problem of coloring intersection graphs of translates of a plane figure attracted some attention. Akiyama, Hosono, and Urabe [4] considered $\chi(\mathcal{C}_n, k)$, where $\mathcal{C}_n$ is the family of the intersection graphs of unit cubes in the n-dimensional Euclidean space with sides parallel to the axes. This family is a part of the family $\mathcal{B}_n$ of the intersection graphs of boxes in R^n discussed above. They proved that $\chi(\mathcal{C}_2, 2) = 3$ and asked about $\chi(\mathcal{C}_2, k)$, and more generally about $\chi(\mathcal{C}_n, k)$.

In connection with the channel assignment problem in broadcast networks, Clark, Colbourn, and Johnson [10] and Gräf, Stumpf, and Weißenfels [22] considered colorings of graphs in the class $\mathcal{U}$ of intersection graphs of unit disks in the plane. They proved that finding the chromatic number of graphs in $\mathcal{U}$ is an NP-complete problem. In [22, 44], and [43] polynomial approximation algorithms are given implying that $\chi(\mathcal{U}, k) \le 3k - 2$. Perepelitsa [44] also considered intersection graphs of translates of triangles and boxes in the plane as well as the more general family $\mathcal{T}$ of intersection graphs of translates of a fixed convex compact figure in the plane. She proved that $\phi(\mathcal{T}, k) \le 8k - 8$ which implies that $\chi(\mathcal{T}, k) \le 8k - 7$.

Kim, Nakprasit, and I [32] improved Perepelitsa's bound to

$$\phi(\mathcal{T}, k) \le 3k - 3 \quad \text{for} \quad k \ge 2. \tag{3}$$

The bound (3) on degeneracy is sharp. In [32], for every $k \ge 2$ we present the intersection graph G of a family of unit circles in the plane with $\omega(G) = k$ that is not $(3k-4)$-degenerate. It is not clear whether the bound $\chi(\mathcal{T}, k) \le 3k - 2$ implied by (3) is sharp or not. The best construction known to us gives only $\chi(\mathcal{T}, k) \ge \lceil 5k/4 \rceil$. My feeling is that $3k - 2$ is not a sharp bound for $\chi(\mathcal{T}, k)$.

We made use of the following old result of Minkowski.

Lemma 2 (Minkowski [42]) *Let K be a convex set in the plane. Then $(x+K)\cap(y+K)\neq\emptyset$ if and only if $(x+\frac{1}{2}[K+(-K)])\cap(y+\frac{1}{2}[K+(-K)])\neq\emptyset$.*

Since $\frac{1}{2}[K+(-K)]$ is always a centrally symmetric set, it is enough to prove the upper bound only for centrally symmetric sets.

The idea of the proof of (3) also allows to estimate the maximum degree of any graph in $\mathcal{T}$ with a given clique number. Namely, it is proved in [32] that the maximum degree of each graph in $\mathcal{T}$ with clique number k, $k\geq 2$, is at most $6k-7$. This bound is also sharp and it helps to prove the bound

$$\phi(\mathcal{D},k)\leq 6k-7 \tag{4}$$

for the more general family $\mathcal{D}$ of homothetic copies of a fixed convex compact set in the plane. In other words, every graph in $\mathcal{D}$ is an intersection graph of planar figures obtained by translating, shrinking, and blowing up of a given convex compact set in the plane. We do not know whether the bound $6k-7$ is sharp.

One can consider the more general families $\mathcal{T}_n$ of intersection graphs of translates of a convex compact figure in $\mathbf{R}^n$. It is probably harder to find the exact values of $\phi(\mathcal{T}_n,k)$ if $n\geq 3$, but one can get linear upper bounds for every fixed n. One of the possible approaches is described in the rest of this section. In fact, after this survey was submitted, Kim and Nakprasit [33] found linear (in k) upper bounds (depending on n) for $\chi(\overline{\mathcal{T}}_n,k)$, where $\overline{\mathcal{T}}_n$ is the class of complements of graphs in $\mathcal{T}_n$. In particular, they proved that $\chi(\overline{\mathcal{T}}_2,k)\leq 3k-2$.

The next lemma is so simple that it does not need a proof, but is quite helpful.

Lemma 3 *If $\mathcal{M}=\{M_i\}_{i=1}^s$ is a family of compact convex centrally symmetric subsets of $\mathbf{R}^n$ and f is a non-singular linear transformation of $\mathbf{R}^n$, then the family $\{f(M_i)\}_{i=1}^s$ is also a family of compact convex centrally symmetric subsets of $\mathbf{R}^n$, and the intersection graphs of $\mathcal{M}$ and $\{f(M_i)\}_{i=1}^s$ are the same.*

For $M\subset\mathbf{R}^n$, let $r(M)$ denote the maximum radius of a ball inscribed into M.

Lemma 4 *If $M\subset\mathbf{R}^n$, centrally symmetric with respect to the origin O, has $r(M)>0$, then there exists a non-singular linear transformation f of $\mathbf{R}^n$ such that $r(f(M))=1$ and the boundary of $f(M)$ contains n linearly independent points $v_1,\ldots,v_n\in\mathbf{R}^n$ with $|v_1|=|v_2|=\ldots=|v_n|=1$.*

Proof. Let $M_1=\frac{1}{r(M)}M$. Then $r(M_1)=1$ and the boundary of M_1 (by the definition of $r(M)$) contains a point v_1 with $|v_1|=1$.

Suppose that for some m, $1\leq m<n$, there is a non-singular linear transformation f_m of $\mathbf{R}^n$ such that $r(f_m(M))=1$ and the boundary of $f_m(M)$ contains m linearly independent points $v_1,\ldots,v_m\in\mathbf{R}^n$ with $|v_1|=|v_2|=\ldots=|v_m|=1$. A way to

construct f_{m+1} is as follows. If any vector v on the boundary of $f_m(M)$, linearly independent of $v_1, \dots, v_m$, has length 1, then we let $v_{m+1} = v$ and $f_{m+1} = f_m$. Otherwise, choose an arbitrary basis $\{v_1, \dots, v_m, w_{m+1}, \dots, w_n\}$ in $\mathbf{R}^n$ containing $v_1, \dots, v_m$. For $\alpha > 0$, let h_α be the linear transformation of $\mathbf{R}^n$ moving every vector with coordinates $(x_1, \dots, x_{n-1}, x_n)$ into the vector with coordinates $(x_1, \dots, x_{n-1}, \alpha x_n)$. Let α_0 be the minimum $\alpha > 0$ such that $r(h_\alpha(f_m(M))) = 1$. By the definition, h_α does not change vectors that are linear combinations of $v_1, \dots, v_m$. Therefore, $\alpha_0 < 1$ and there is some v on the boundary of $f_m(M)$ linearly independent of $v_1, \dots, v_m$ such that $|h_{\alpha_0}(v)| = 1$. Thus, one can take $f_{m+1} = h_{\alpha_0}(f_m)$ and $v_{m+1} = v$.

Repeating this procedure until $m = n$ yields the lemma. ■

The next statement is auxiliary here, but maybe can be used in other situations, so it is stated as a theorem.

Theorem 2 *For every family $\mathcal{M}$ of translates of a compact convex set M in $\mathbf{R}^n$ with $r(M) > 0$, there exists a family $\mathcal{F}$ of translates of a compact convex set F in $\mathbf{R}^n$ with the same intersection graph and such that*
(a) F is centrally symmetric w.r.t. the origin O;
(b) F contains the ball of radius 1 with the center O;
(c) F is contained in the cube $\{(x_1, \dots, x_n) : -1 \le x_i \le 1, i = 1, \dots, n\}$.

Proof. Let $\mathcal{M}$ be a family of translates of a compact convex set M in $\mathbf{R}^n$ with $r(M) > 0$. The set $M_1 = 0.5(M - M)$ is centrally symmetric and, by Lemma 2, the family $\mathcal{M}_1$ of translates of M_1 by the same shifts has the same intersection graph. Also, if $r(M) > 0$, then $r(M_1) > 0$.

Now, by Lemmas 3 and 4, there exists a family $\mathcal{M}_2$ with the same intersection graph, where $\mathcal{M}_2$ is the family of translates of an $M_2 \subset \mathbf{R}^n$, centrally symmetric with respect to the origin O, such that $r(M_2) = 1$ and the boundary of M_2 contains n linearly independent points $v_1, \dots, v_n \in \mathbf{R}^n$ with $|v_1| = |v_2| = \dots = |v_n| = 1$. Applying a linear transformation of $\mathbf{R}^n$ that maps vectors $v_1, \dots, v_n$ into the vectors of an orthonormal basis of $\mathbf{R}^n$, we come to a family $\mathcal{M}_3$ that is the family of translates of an $M_3 \subset \mathbf{R}^n$ satisfying (a) and (b) and such that in an Euclidean system of coordinates in $\mathbf{R}^n$, the vectors $(1, 0, 0, \dots, 0)$, …, $(0, 0, \dots, 0, 1)$ belong to the boundary of M_3.

Assume that a point $(x_1, \dots, x_n)$ with $x_1 > 1$ is in M_3. Consider the plane P containing $(x_1, \dots, x_n)$, $(1, 0, 0, \dots, 0)$, and O. Since $r(M_3) = 1$, the circle C of radius 1 with the center O in P is a part of M_3 and hence the convex hull of $C \cup \{(x_1, \dots, x_n)\}$ is also a part of M_3. Since $x_1 > 1$, some of the two tangents to C containing $(x_1, \dots, x_n)$ crosses the line $x_2 = x_3 = \dots = x_n = 0$ at a point $(\beta, 0, 0, \dots, 0)$ with $\beta > 1$. Then $(\beta, 0, 0, \dots, 0) \in M_3$ and hence the point $(1, 0, 0, \dots, 0)$ is an inner point of M_3. This contradicts the fact that $(1, 0, 0, \dots, 0)$ is on the boundary of M_3. Thus, (c) is proved. ■

The message of Theorem 2 is that to bound $\phi(\mathcal{T}_n, k)$ from above, it is enough to consider only translates of sets satisfying the conditions (a)–(c) of this theorem.

The following folklore lemma is instrumental.

Lemma 5 *Let M be a centrally symmetric convex subset of $\mathbf{R}^n$. Then for any translates M_1, M_2, and M_3 of M, if the centers of M_2 and M_3 are in M_1, then M_2 and M_3 have a common point. In other words, the intersection graph of translates of M with centers in M_1 is a clique.*

Proof. Let c_i be the center of M_i, $i = 1, 2, 3$. Let $s_j = c_j - c_1$, $j = 2, 3$. Since $c_j \in M_1$, $j = 2, 3$, we have $c_i + s_j \in M_i$ for $i = 1, 2, 3$ and $j = 2, 3$. But $c_1 + s_2 + s_3 = c_2 + s_3 = c_3 + s_2$ and hence $c_1 + s_2 + s_3 \in M_2 \cap M_3$. ■

Now we are ready to prove

Theorem 3 *Let $\mathcal{M}$ be a family of translates of a compact convex set M in $\mathbf{R}^n$ such that the clique number of the intersection graph H of $\mathcal{M}$ is k. Then the maximum degree of H is at most $\lceil 2\sqrt{n}\rceil^n(k-1)$. The minimum degree of H is at most $\lceil \sqrt{n}\rceil \lceil 2\sqrt{n}\rceil^{n-1}(k-1)$.*

Proof. If $r(M) = 0$, then since M is convex, the problem simply reduces to a smaller dimension. Thus, we may assume that $r(M) > 0$ and hence consider only families $\mathcal{F}$ satisfying conditions (a)–(c) of Theorem 2. Let F be a member of $\mathcal{F}$. For convenience, we assume that the center of F is the origin O. Let L be the set of centers of members of $\mathcal{F}$ intersecting F. By Theorem 2(c), every point of L belongs to the cube $Y = \{(x_1, \ldots, x_n) : -2 \le x_i \le 2, i = 1, \ldots, n\}$.

By Lemma 5, to prove the first statement of the theorem, it is enough to cover Y with $\lceil 2\sqrt{n}\rceil^n$ translates of F. Clearly, Y can be covered by that many cubes with side $2/\sqrt{n}$. Every cube with side $2/\sqrt{n}$ is a part of a ball of radius 1, and by Theorem 2(b), F contains such a ball. This proves the first statement of the theorem.

To prove the second statement, observe that if F_0 is a member of $\mathcal{F}$ whose center has the largest first coordinate, then the centers of the members of $\mathcal{F}$ intersecting F_0 are all in a box of size $2 \times 4 \times 4 \times \ldots \times 4$. Thus, it is enough to cover this box by $\lceil \sqrt{n}\rceil \lceil 2\sqrt{n}\rceil^{n-1}(k-1)$ cubes with side $2/\sqrt{n}$. ■

Note that the theorem implies $\phi(\mathcal{T}_n, k) \le \lceil \sqrt{n}\rceil \lceil 2\sqrt{n}\rceil^{n-1}(k-1)$. In particular, we get $\phi(\mathcal{T}_2, k) \le 6(k-1)$ which is worse than the bound in [32], but better than that in [44]. On the other hand, the idea in [32] deriving a bound on the degeneracy of intersection graphs of homothetic copies of a convex plane figure from the bound on the maximum degree of intersection graphs of translates of a convex plane figure applies here. Thus, Theorem 3 yields the following fact.

Theorem 4 *Let $\mathcal{M}$ be a family of homothetic copies of a compact convex set M in $\mathbf{R}^n$ such that the clique number of the intersection graph H of $\mathcal{M}$ is k. Then the minimum degree of H is at most $\lceil 2\sqrt{n}\rceil^n(k-1)$.*

Certainly, the maximum degree of the graph H in Theorem 4 is not bounded.

6 Ramsey questions

For a positive integer n and a class $\mathcal{F}$ of graphs, let $\rho(\mathcal{F}; N)$ denote the maximum m such that each graph $G \in \mathcal{F}$ on N vertices has either a clique or an independent set of size at least m. Because of the symmetry between cliques and independent sets, if $\overline{\mathcal{F}}$ is the family of complements of graphs in $\mathcal{F}$, then

$$\rho(\overline{\mathcal{F}}; N) = \rho(\mathcal{F}; N) \quad \text{for every } N. \tag{5}$$

The classical bounds of Erdős say that $c_1 \log_2 N \le \rho(\mathcal{G}; N) \le c_2 \log_2 N$ for the class $\mathcal{G}$ of all finite graphs. For many classes $\mathcal{F}$ of intersection graphs of geometric figures, the order of magnitude of $\rho(\mathcal{F}; N)$ is larger than logarithmic. Larman, Matoušek, Pach, and Törőcsik [39] proved that for the family $\mathcal{P}$ of intersection graphs of convex compact sets in the plane, $\rho(\mathcal{P}, N) \ge N^{0.2}$. On the other hand, Károlyi, Pach, and Tóth [31] showed that $\rho(\mathcal{P}, N) \le N^{0.4207}$.

There is an interplay between bounds on $\rho(\mathcal{F}; N)$ and on $\chi(\mathcal{F}, k)$ and $\chi(\overline{\mathcal{F}}, k)$. For example, suppose that $\chi(\mathcal{F}, k) \le f(k)$. Then every graph $G \in \mathcal{F}$ on N vertices with clique number k has an independent set of size at least $N/f(k)$. Therefore,

$$\rho(\mathcal{F}; N) \ge \min_{1 \le k \le N} \max\{k, \frac{N}{f(k)}\}. \tag{6}$$

Thus, the bound (1) together with (6) and (5) gives

$$\rho(\mathcal{B}_n; N) \ge \min_{1 \le k \le N} \max\{k, \frac{N}{k \log_2^{n-1} k}\} \ge \sqrt{2N} (\log_2 N)^{-(n-1)/2}.$$

Similarly, (2) yields

$$\rho(\mathcal{X}; N) \ge \min_{1 \le k \le N} \max\{k, \frac{N}{(1+o(1))k \ln k}\} \ge (1+o(1))\sqrt{2N/\ln N},$$

and (4) yields

$$\rho(\mathcal{D}; N) \ge \min_{1 \le k \le N} \max\{k, \frac{N}{6k-6}\} \ge \sqrt{6N}.$$

On the other hand, if one proves that $\rho(\mathcal{F}; N) \le \sqrt{N/g(N)}$, for some monotone non-decreasing function $g(N) \ge 1$, then $\chi(\mathcal{F}, k) \ge k\, g(k^2)$. In the rest of the section, for every $m \ge 2$, a circle graph $G(m)$ will be constructed such that

(i) $|V(G(m))| \ge 2m(2m+1)(-2 + \ln 4(m+1))$;
(ii) the independence number, $\alpha(G(m))$, of $G(m)$ is at most $2m$;
(iii) the clique number, $\omega(G(m))$, of $G(m)$ is at most $4m-2$.

This series of graphs witnesses that for infinitely many N, $\rho(\mathcal{X}; N) \le 4\sqrt{N/\ln N}$, and for infinitely many k, $\chi(\mathcal{X}, k) \ge 0.5k(\ln k - 2)$ and $\chi(\overline{\mathcal{X}}, k) \ge 0.5k(\ln k - 1.5)$.

It was mentioned that a graph G is a circle graph if and only if it is an *overlap graph*, i.e., a graph whose vertex set is a family $\mathcal{F}$ of intervals on the real line and two intervals are adjacent in G iff they have a common point but none of them contains the other. We will consider the overlap graph of the family of open intervals described in the next paragraph[1].

The open (respectively, closed) interval with ends a and b will be denoted by $]a, b[$ (respectively, $[a, b]$). For $i = 1, \dots, m$, let

$$\mathcal{F}(i,m) = \{\,]j, j+2im+1[:\ j = 0, i, 2i, \dots, (\lfloor 2m(2m+1)/i \rfloor - 2m)i\},$$

and let $\mathcal{F}(m) = \bigcup_{i=1}^{m} \mathcal{F}(i,m)$. Then

$$|\mathcal{F}(m)| = \sum_{i=1}^{m} \left(\left\lfloor \frac{2m(2m+1)}{i} \right\rfloor - 2m + 1 \right) \geq \sum_{i=1}^{m} \left(-2m + \frac{2m(2m+1)}{i} \right) \geq$$

$$\geq 2m(2m+1) \left(-\frac{1}{2} + \sum_{i=1}^{m} \frac{1}{i} \right) \geq 2m(2m+1) \left(-0.5 + \ln(m+1) \right) \geq$$

$$\geq 2m(2m+1) \left(-0.5 + \ln 4(m+1) - \ln 4 \right).$$

This proves (i).

Recall that independent sets in the overlap graph of a family $\mathcal{F}$ correspond to *non-overlapping subfamilies* of $\mathcal{F}$, i.e. to the families of intervals where no two members overlap.

Lemma 6 *Let j be a non-negative integer and s be a positive integer. Let $\mathcal{F}'$ be a non-overlapping subfamily of $\mathcal{F}(m)$ such that every member of $\mathcal{F}'$ is contained in $[j, j+s]$. Then*
(a) $|\mathcal{F}'| \leq \frac{s-1}{2m}$;
(b) if $]j, j+s[\notin \mathcal{F}'$ and $s \geq 2$, then $|\mathcal{F}'| < \frac{s-1}{2m}$.

Proof. We use induction on s. If $s \leq 2m+1$, then the statement is evident. Suppose that the lemma is proved for all $s < s_0$ and for an arbitrary j consider a non-overlapping subfamily $\mathcal{F}'$ of the family $\mathcal{F}(m)$ such that all members of $\mathcal{F}'$ are contained in $[j, j+s_0]$. Let j_1 be the leftmost left end of an interval in $\mathcal{F}'$ and let $I_1 =]j_1, j_1 + 2mi_1 + 1[$ be the longest interval in $\mathcal{F}'$ with the left end j_1. If $j_1 > j$, then by the choice of s_0,

$$|\mathcal{F}'| \leq \frac{(j+s_0) - j_1 - 1}{2m} < \frac{s_0 - 1}{2m}.$$

Thus, $j_1 = j$.

[1] All the figures considered before were closed figures, but for every finite family of open intervals, there exists another family of closed intervals with the same overlap graph.

CASE 1. $s_0 > 2mi_1 + 1$. Since $j + 2mi_1 + 1$ is the right end of I_1, it is not an internal point of any interval in $\mathcal{F}'$. Hence $\mathcal{F}'$ is the disjoint union of $\mathcal{F}'_1$ and $\mathcal{F}'_2$, where

$$\mathcal{F}'_1 = \{]a, b[\in \mathcal{F}' \,:\, b \leq j + 2mi_1 + 1\} \text{ and } \mathcal{F}'_2 = \{]a, b[\in \mathcal{F}' \,:\, a \geq j + 2mi_1 + 1\}.$$

By the induction hypothesis, $|\mathcal{F}'_1| \leq \frac{2mi_1}{2m}$ and $|\mathcal{F}'_2| \leq \frac{s_0-2mi_1-1-1}{2m}$. Hence $|\mathcal{F}'| \leq \frac{s_0-2}{2m} < \frac{s_0-1}{2m}$.

CASE 2. $s_0 = 2mi_1 + 1$. Consider $\mathcal{F}'' = \mathcal{F}' - \{I_1\}$. Then we have Case 1, and hence by (b),

$$|\mathcal{F}''| < \frac{s_0 - 1}{2m} = i_1.$$

It follows that $|\mathcal{F}''| \leq i_1 - 1$ and thus $|\mathcal{F}'| \leq i_1 = \frac{s-1}{2m}$. ■

Lemma 7 *The cardinality of each non-overlapping subfamily of $\mathcal{F}(m)$ is at most $2m$. In other words, $\alpha(G(m)) \leq 2m$, that is, (ii) holds.*

Proof. Let $\mathcal{F}'$ be a non-overlapping subfamily of $\mathcal{F}(m)$. Observe that the union of all members of $\mathcal{F}(m)$ is the interval $]0, 2m(2m + 1) + 1[$ and this interval is not in $\mathcal{F}(m)$. Hence by Lemma 6(b),

$$|\mathcal{F}'| < \frac{2m(2m + 1) + 1 - 1}{2m} = 2m + 1.$$

This yields the lemma. ■

Lemma 8 *If every two members of a family $\mathcal{F}' \subset \bigcup_{i=t}^{m} \mathcal{F}(i, m)$ overlap and the right ends of all members of $\mathcal{F}'$ belong to $]a, b]$, then $b - a \geq t|\mathcal{F}'| - (t - 1)(m - t + 1)$.*

Proof. Let $\mathcal{F}' = \{]a_i, b_i[: i = 1, \ldots, s\}$, where $b_i = a_i + 2ml_i + 1$, $i = 1, \ldots, s$. Since every two members of $\mathcal{F}'$ overlap, we can number them so that

$$a < a_1 < a_2 < \ldots < a_s < b_1 < \ldots < b_s \leq b.$$

We need to estimate $b_s - a$. By induction on i, we show that

$$b_i - a \geq it - (t - 1)(m - l_i + 1). \tag{7}$$

For $i = 1$, (7) becomes $b_1 - a \geq t - (t - 1)(m - l_1 + 1)$. Since $l_i \leq m$ for every i,

$$t - (t - 1)(m - l_1 + 1) \leq t - (t - 1) = 1.$$

But $b_1 - a \geq 1$ since $b_1 \in]a, b]$. Suppose now that (7) is proved for every $i \leq i_0 - 1$.

CASE 1. $l_{i_0} \leq l_{i_0-1} - 1$. Then $b_{i_0} - a \geq 1 + b_{i_0-1} - a \geq 1 + (i_0 - 1)t - (t - 1)(m - l_{i_0-1} + 1) \geq 1 - t + i_0t - (t - 1)(m - (l_{i_0} + 1) + 1) = i_0t - (t - 1)(m - l_{i_0} + 1)$.

CASE 2. $l_{i_0} = l_{i_0-1}$. By the construction, $b_{i_0} \geq b_{i_0-1} + l_{i_0} \geq b_{i_0-1} + t$. Therefore,

$$b_{i_0} - a \geq t + b_{i_0-1} - a \geq t + (i_0-1)t - (t-1)(m - l_{i_0-1} + 1) = i_0 t - (t-1)(m - l_{i_0} + 1).$$

CASE 3. $l_{i_0} \geq l_{i_0-1} + 1$. Since $a_{i_0} \geq 1 + a_{i_0-1}$, we have $b_{i_0} - b_{i_0-1} \geq 1 + 2m(l_{i_0} - l_{i_0-1})$. Hence,

$$b_{i_0} - a = (b_{i_0} - b_{i_0-1}) + (b_{i_0-1} - a) \geq 1 + 2m(l_{i_0} - l_{i_0-1}) + (i_0-1)t - (t-1)(m - l_{i_0-1} + 1) =$$
$$= i_0 t - (t-1)(m - l_{i_0} + 1) + 1 + 2m(l_{i_0} - l_{i_0-1}) - t - (t-1)(l_{i_0} - l_{i_0-1}) =$$
$$= i_0 t - (t-1)(m - l_{i_0} + 1) + (m - t + 1)(l_{i_0} - l_{i_0-1}) + m(l_{i_0} - l_{i_0-1}) - t + 1 \geq$$
$$\geq i_0 t - (t-1)(m - l_{i_0} + 1) + (m - t + 1)(l_{i_0} - l_{i_0-1} + 1).$$

This proves (7). Now, by (7) for $i = s$, since $l_s \geq t$,

$$b - a \geq b_s - a \geq st - (t-1)(m - l_s + 1) \geq st - (t-1)(m - t + 1).$$

■

Lemma 9 *The cardinality of each overlapping subfamily of $\mathcal{F}(m)$ is at most $4m-2$. In other words, $\omega(G(m)) \leq 4m-2$, that is, (iii) holds.*

Proof. Let $\mathcal{F}'$ be an overlapping subfamily of $\mathcal{F}(m)$ and t_0 be the minimum t such that $\mathcal{F}' \subset \bigcup_{i=t}^{m} \mathcal{F}(i,m)$. Then for some j there exists $I_0 =]j, j+2mt_0+1[\in \mathcal{F}'$. Let $\mathcal{F}'_1 = \{\,]a,b[\in \mathcal{F}' \,:\, b \in I_0\}$ and $\mathcal{F}'_2 = \{\,]a,b[\in \mathcal{F}' \,:\, a \in I_0\}$. Let $b' = \min\{b :]a,b[\in \mathcal{F}'_1\}$ and $a' = \max\{a :]a,b[\in \mathcal{F}'_2\}$. Since all members of $\mathcal{F}'$ overlap, $a'+1 \leq b'$. By Lemma 8,

$$b' - j > (j + 2mt_0) - a' \geq t_0|\mathcal{F}'_1| - (t_0-1)(m - t_0 + 1).$$

By the analog of the lemma for left ends,

$$a' - j \geq t_0|\mathcal{F}'_2| - (t_0-1)(m - t_0 + 1).$$

Summing the last two inequalities, one gets

$$2mt_0 \geq t_0(|\mathcal{F}'| - 1) - 2(t_0-1)(m - t_0 + 1).$$

Hence $|\mathcal{F}'| - 1 \leq 2m + 2(1 - 1/t_0)(m - t_0 + 1) = 4m - 2t_0 + 2 - 2(m+1)/t_0 + 2$, and

$$|\mathcal{F}'| \leq 4m + 5 - 2\left(t_0 + \frac{m+1}{t_0}\right) \leq 4m + 5 - 4\sqrt{m+1}.$$

Since $m \geq 2$, the last expression is less than $4m-1$. This proves the lemma and the whole result. ■

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References

[1] P.K. Agarwal and J. Pach, *Combinatorial Geometry*, John Wiley & Sons. Inc., 1995.

[2] A. A. Ageev, A triangle-free circle graph with chromatic number 5, *Discrete Math.* **152** (1996), 295–298.

[3] A. A. Ageev, Every circle graph of girth at least 5 is 3-colourable, *Discrete Math.* **195** (1999), 229–233.

[4] J. Akiyama, K. Hosono, and M. Urabe, Some combinatorial problems, *Discrete Math.* **116** (1993), 291-298.

[5] E. Asplund and B. Grünbaum, On a coloring problem, *Math. Scand.* **8** (1960), 181-188.

[6] J. Beck, Oral communication.

[7] A. Bielecki, Problem 56, *Colloq. Math.*, **1** (1948), 333.

[8] J. P. Burling, *On coloring problems of families of prototypes*, Ph.D. Thesis, Univ. of Colorado, 1965.

[9] G. D. Chakerian and S. K. Stein, Some intersection properties of convex bodies, *Proc. Amer. Math.Soc.*, **18** (1967), 109–112.

[10] B. N. Clark, C. J. Colbourn, and D. S. Johnson, Unit disk graphs, *Discrete Math.* **86** (1990), 165–177.

[11] A. Dumitrescu, G. Tóth, Ramsey-type results for unions of comparability graphs. *Graphs and Combinatorics*, **18** (2002), 245–251.

[12] P. Erdős, Graph theory and probability, *Canadian J. of Mathematics*, **11** (1959), 34–38.

[13] S. Even, A. Itai, Queues, stacks and graphs, *Theory of Machines and Computations*, Acad. Press, N.Y., 1971, 71–86.

[14] D. G. Fon-Der-Flaass, Oral communication.

[15] D. G. Fon-Der-Flaass and A. Kostochka, Covering boxes by points, *Discrete Mathematics*, **120** (1993), 269–275.

[16] H. de Fraysseix, A characterization of circle graphs, *European Journal of Combinatorics*, **5** (1983), 223–238,

[17] M. Garey, D. Johnson, G. Miller, and C. Papadimitriou, The complexity of coloring circular arcs and chords, *SIAM J. of Alg. Disc. Methods*, **1** (1980), 216–227.

[18] F. Gavril, Algorithms for a maximum clique and a maximum independent set of a circle graph, *Networks*, **3** (1973), 261–273.

[19] F. Gavril, Algorithms on circular-arc graphs, *Networks*, **4** (1974), 357–369.

[20] A. Glebov, Bounds on the degeneracy of intersection graphs of boxes in the plane depending on the girth, *Diskret. Analiz i Issl. Oper.*, **9** (2002), # 2, 3–20 (In Russian).

[21] M. Golumbic, *Algorithmic graph theory and perfect graphs.* Academic Press, 1980.

[22] A. Gräf, M. Stumpf, and G. Weißenfels, On Coloring Unit Disk Graphs, *Algorithmica* **20** (1998), 277-293.

[23] A. Gyárfás, Problems from the world surrounding perfect graphs. Proceedings of the International Conference on Combinatorial Analysis and its Applications (Pokrzywna, 1985). *Zastos. Mat.* **19** (1987), 413–441.

[24] A. Gyárfás, On the chromatic number of multiple interval graphs and overlap graphs, *Discrete Math.* **55** (1985), 161–166.

[25] A. Gyárfás, J. Lehel, Covering and coloring problems for relatives of intervals, *Discrete Math.* **55** (1985), 167–180.

[26] C. Hendler, Schranken für Färbungs- und Cliquenüberdeckungszahl geometrisch represäntierbarer Graphen, Master Thesis, 1998.

[27] I. Karapetian, On coloring of circular arc graphs, *Doklady AN ArmSSR (Notes of the Armenian Acad. of Sci.)*, **70** (1980), # 5, 306–311 (in Russian).

[28] I. Karapetian, On perfect, circular arc and circle graphs, Ph.D. Thesis, Novosibirsk, 1984 (in Russian).

[29] I. Karapetian, Chordal graphs, *Matematicheskie voprosi kibernetiki i vichislitelnoi tehniki*, No. 14 (1985), 6-10, Erevan (in Russian).

[30] Gy. Károlyi, On point covers of parallel rectangles, *Period. Math. Hungar.* **23** (1991), 105–107.

[31] Gy. Károlyi, J. Pach, and G. Tóth, Ramsey-type results for geometric graphs, I, *Discrete and Comp. Geometry*, **18** (1997), 247–255.

[32] S.-J. Kim, A. Kostochka, and K. Nakprasit, On the chromatic number of intersection graphs of convex sets in the plane, submitted.

[33] S.-J. Kim and K. Nakprasit, Coloring the complements of intersection graphs of geometric figures, submitted.

[34] A. Kostochka, On upper bounds on the chromatic numbers of graphs, *Transactions of the Institute of Mathematics (Siberian Branch of the Academy of Sciences of USSR)*, **10** (1988), 204-226 (in Russian).

[35] A. Kostochka and J. Kratochvíl, Covering and coloring polygon-circle graphs *Discrete Mathematics*, **163** (1997), 299–305.

[36] A. Kostochka and J. Nešetřil, Coloring relatives of intervals on the plane, I: chromatic number versus girth, *European Journal of Combinatorics*, **19** (1998), 103–110.

[37] A. Kostochka and J. Nešetřil, Colouring relatives of intervals on the plane, II: intervals and rays in two directions, *European Journal of Combinatorics*, **23** (2002), 37–41.

[38] A.V. Kostochka and I.G. Perepelitsa, Coloring intersection graphs of boxes on the plane, *Discrete Math.* **220** (2000), 243-249.

[39] D. Larman, J. Matoušek, J. Pach, and J. Törőcsik, A Ramsey-type result for convex sets, *Bull. London Math. Soc.* **26** (1994), 132-136.

[40] S. McGuinness, Colouring arcwise connected sets in the plane. I. *Graphs Combin.*, **16** (2000), 429–439.

[41] S. McGuinness, Colouring arcwise connected sets in the plane. II. *Graphs Combin.*, **17** (2001), 135–148.

[42] H. Minkowski, Dichteste gitterformige Lagerung kongruenter Körper, *Nachr. der K. es. der Wiss. zu Göttingen, Math. Phys. Kl.*, (1904), 311-355.

[43] R. Peeters, On coloring j-unit sphere graphs, FEW 512, Departement of Economics, Tilburg University, 1991.

[44] I.G. Perepelitsa, Bounds on the chromatic number of intersection graphs of sets in the plane, *Discrete Math.*, **262** (2002), 221-227.

[45] A. Tucker, Coloring a family of circular arcs, *SIAM J. Appl. Math.*, **29** (1975), 493–502.

[46] W. Unger, On the k-colouring of circle-graphs, *Lecture Notes in Comput. Sci.*, **294** (1988), Springer, Berlin, 61–72.