Journal of Combinatorial Theory, Series B **86**, 100–108 (2002)
doi:10.1006/jctb.2002.2116

Nilpotent Families of Endomorphisms of $(\mathscr{P}(V)^+, \cup)$

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Received September 8, 1999; published online July 8, 2002

A directed graph $G = (V, A)$ is *k-nice* if for every $u, v \in V$ (allowing $t = v$), and for every orientation of the edges of an undirected path of length k, there exists a $u - v$ walk of length k in G whose orientation coincides with that of the given path. A graph is *nice* if it is k-nice for some k. We generalize this notion using the notion of a nilpotent semigroup of endomorphisms of $(\mathscr{P}(V)^+, \cup)$, and consider two basic problems:

(1) find bounds for the nilpotency class of such semigroups in terms of their generators (in the language of graphs: provided that a graph G on n vertices is nice, find the smallest k such that G is k-nice);

(2) find a way to demonstrate non-nilpotency of such semigroups (find as simple as possible characterization of non-nice graphs).

[1] This work was partially supported by the Grants 99-01-00581 and 97-01-01075 of the Russian Foundation for Fundamental Research.
[2] This work was partially supported by the Grant Intas-Open-97-1001 and the Dutch-Russian Grant NWO-047-008-006.
[3] Partially supported by the Project LN00A056 of the Czech Ministry of Education.
[4] This work was partially supported by the BARRANDE Programme 97-137.
[5] This research was partly supported by the NATO Collaborative Research Grant 97-1519.

0095-8956/02 $35.00

1. INTRODUCTION

The notion of a nice graph was implicitly used in the papers [1, 2, 5] as a useful tool for studying oriented chromatic number of graphs. In [3], nice graphs were studied for their own sake, and some generalizations were introduced.

A directed graph G is called *k-nice* if for every two vertices u, v (allowing $t = v$), and for every orientation of the edges of the path of length k, there exists a $u - v$ walk of length k in G whose orientation of edges coincides with the given one. A directed graph G is *nice* if it is k-nice for some positive integer k (note that every k-nice graph is $(k+1)$-nice). Similarly, an undirected (multi)graph G whose edges are coloured by c colours is called *k-nice* if for every two vertices u, v (allowing $u = v$), and for every edge colouring of the path of length k with c colours, there exists a $u - v$ walk of length k in G whose colouring coincides with the given one. Such a (multi)graph is *nice* if it is k-nice for some positive integer k.

In [3], characterizations of non-nice graphs in terms of the so-called "black holes" (see the next section) were found. It was proved that in every non-nice graph there exist black holes with a simple structure (especially simple for uncoloured directed graphs). A polynomial algorithm was given to recognize nice directed graphs.

In this note we treat "niceness" of graphs (in both senses) as a special case of a more general and natural notion of nilpotency of semigroups of endomorphisms of the upper semilattice $(\mathscr{P}(V)^+, \cup)$ of all non-empty subsets of a finite set V of size n. In fact, this notion corresponds to nice edge-coloured directed (multi)graphs, and a particular case of this problem (in a different setting of matrix theory) was studied by Wielandt [6] and Holladay and Varga [4].

In Section 4 we prove that in this general case the simplest black holes (obstacles to niceness) can be more complicated than in previously considered cases.

In Section 3 we address the following question: *What is the minimum $k = k(n)$ such that every nice graph on n vertices is k-nice*? In other words, we establish some bounds for the nilpotency class of such semigroups in terms of their generators, and the size n of the ground set. We show that the nilpotency class can be exponentially large if the number of generators is n. The main result is a polynomial in n upper bound on the nilpotency class when the generators have a special form. As a corollary this gives polynomial in n upper bounds on k for uncoloured directed nice graphs and 2-edge-coloured undirected (multi)graphs.

2. NILPOTENT FAMILIES

Let V be a finite set, $|V| = n$. By $\mathscr{P}(V)^+$ we denote the set of all non-empty subsets of V. We consider endomorphisms of the semi-lattice $(\mathscr{P}(V)^+, \cup)$, that is, mappings $\varphi : \mathscr{P}(V)^+ \to \mathscr{P}(V)^+$ satisfying the identity

$$\varphi(X \cup Y) = \varphi(X) \cup \varphi(Y).$$

Every such mapping φ is uniquely determined by the values it takes on one-element subsets:

$$\varphi(X) = \bigcup_{x \in X} \varphi(\{x\}).$$

It is often convenient to view such mappings as directed graphs on the vertex set V in which (x, y) is an arc if and only if $y \in \varphi(\{x\})$. Occasionally, we shall abuse the notation by writing $\varphi(x)$ instead of $\varphi(\{x\})$. Thus endomorphisms of $(\mathscr{P}(V)^+, \cup)$ are in one-to-one correspondence with directed graphs on V.

The set $M(V)$ of all endomorphisms of the semi-lattice $(\mathscr{P}(V)^+, \cup)$ forms a semigroup under composition of mappings; as usual in this context, we shall refer to composition as *product*. The semigroup has a right zero element Ω, $\Omega(X) = V$ for all $X \in \mathscr{P}(V)^+$, In fact, Ω is the unique two-sided zero of $M_\Omega(V)$, the sub-semigroup of those elements φ for which $\varphi(V) = V$.

Let $\Lambda = \{\varphi_1, \ldots, \varphi_r\} \subseteq M_\Omega(V)$ be any collection of endomorphisms. By Λ^k we denote the set of all products $x_1 \cdots x_k$ of k elements from Λ. The collection Λ is called *nilpotent of class k*, or k-nilpotent, if $\Lambda^k = \{\Omega\}$ for some $k > 0$. This terminology is in accordance with traditional usage of the word "nilpotent." Obviously, if Λ is k-nilpotent then it is m-nilpotent for every $m > k$.

Observe that $\Lambda = \{\varphi_1, \ldots, \varphi_r\}$ is nilpotent if the directed graphs corresponding to endomorphisms $\varphi_1, \ldots, \varphi_r$ form together a k-nice coloured directed multigraph, that is a directed multigraph G with arcs coloured with r colours such that for every $u, v \in V(G)$ and every arc colouring of the oriented path of length k with r colours, there exists an oriented $u - v$ walk with the given colouring of arcs.

Following [3], we say that $\emptyset \neq X \subseteq V$ is a *black hole* for φ if $\varphi(X) = X$; if $X \neq V$ then the black hole is called *non-trivial*. Note that if for some X we have $\varphi(X) \subseteq X$ then some non-empty subset of $\varphi(X)$ is a black hole. In particular, if $\varphi(V) \neq V$ then φ has a non-trivial black hole.

It is easy to see that Λ is nilpotent if and only if no composition of mappings from Λ has a black hole. If Λ is nilpotent then, starting from any non-empty subset $X \subseteq V$ and applying mappings from Λ in any sequence, we eventually shall reach the whole set V, and in not more than $2^n - 2$ steps (because each proper subset of V can appear at most once; otherwise it would be a black hole for some composition of mappings from Λ). Thus, every nilpotent family is $2^n - 2$-nilpotent.

For some nilpotent collections, $2^n - 2$ steps may be necessary. Here is an example. Let $X_1, \ldots, X_{2^n-1}$ be any linear extension of $\mathscr{P}(V)^+$ ordered by inclusion. For $i = 1, \ldots, 2^n - 2$ the mapping φ_i is defined by $\varphi_i(x) = X_{i+1}$ if $x \in X_i$, $\varphi_i(x) = V$ if $x \notin X_i$. Then the system $\Lambda = \{\varphi_1, \ldots, \varphi_{2^n-2}\}$ is nilpotent, and it takes exactly $2^n - 2$ steps to transform X_1 into $X_{2^n-1} = V$ if we take the mappings in their order. Later, in Example 3, we shall see that by being more careful we can obtain the same number of steps for a nilpotent family of only n different mappings.

3. BOUNDS FOR NILPOTENCY CLASS

In this section, we obtain polynomial upper bounds for the nilpotency class when the collection Λ is small and/or satisfies some additional properties.

A mapping ψ is called *opposite to* φ if for all $x, y \in V$, $y \in \varphi(\{x\})$ exactly when $x \in \psi(\{y\})$; and is called *symmetric* if it is opposite to itself. These notions naturally arise in the graph-theoretical setting: taking the opposite of a mapping corresponds to inverting all arcs of a directed graph; and symmetric mappings correspond to simple (undirected) graphs.

The case of Λ consisting of only one mapping was considered by Wielandt [6] and Holladay and Varga [4] in the language of matrices. In graph language, this case corresponds to graphs with uncoloured edges.

We start from two examples of nilpotent mappings. In all examples, we let $V = \{0, 1, \ldots, n-1\}$ with its natural order.

Example 1. Let the mapping a correspond to the path $0 - 1 - \cdots - (n-1)$ with a loop at the vertex $n-1$; this mapping is symmetric. The single-element collection $\{a\}$ is nilpotent, and if we start with the set $X = \{0\}$, we reach V only after $2n - 2$ steps.

Example 2. Let a correspond to the directed graph formed from the oriented cycle $0 \to 1 \to \cdots \to n-1 \to 0$ and an extra arc $(n-1) \to 1$. Again, $\{a\}$ is nilpotent, and if we start with the set $X = \{0\}$, we reach V only after $n(n-2)$ steps.

We now give an easy proof of the results of [4, 6] that these examples are extremal.

Proposition 1. *Let* $n \geqslant 4$ *and* $\Lambda = \{\varphi\}$. *If* Λ *is nilpotent then it is* $n(n-2)$*-nilpotent. If, moreover,* φ *is symmetric then* Λ *is* $(2n-2)$*-nilpotent.*

Proof. Take a longest sequence $X_0, X_1, \ldots, X_m$ such that $X_{i+1} = \varphi(X_i)$ and $X_m \neq V$, and suppose, by way of contradiction, that $m \geqslant n(n-2)$. We may assume that $X_0 = \{x\}$ is a one-element set. The directed graph G corresponding to φ must be strongly connected (otherwise there is a non-trivial black hole for φ). Let C be a shortest oriented cycle in G and $k = |C|$. Let t be the length of a shortest (oriented) path from x to C and $y \in V(C)$ be the end vertex of such a path (if $x \in V(C)$, then $y = x$). Define $Y_t = \{y\}$ and $Y_{i+1} = \varphi(Y_i)$ for $i \geqslant t$. Then $y \in Y_{t+k}$, so we have $Y_t \subseteq Y_{t+k}$ and $Y_t \subseteq X_t$. The mapping φ preserves the relation of one set being a subset of another; therefore, applying it several times, we obtain that $Y_{t+j} \subseteq X_{t+j}$ for every $j \geqslant 0$, and that

$$Y_t \subseteq Y_{t+k} \subseteq Y_{t+2k} \subseteq \cdots \subseteq Y_{t+(n-1)k}.$$

All inclusions in this chain are strict (otherwise we would have a black hole); and so $|X_{t+(n-1)k}| \geqslant |Y_{t+(n-1)k}| \geqslant n$. Thus, $X_{t+(n-1)k} = V$. If $k \leqslant n-2$, then $t + (n-1)k \leqslant (n-k) + (n-1)k \leqslant 2 + (n-1)(n-2) \leqslant n(n-2)$, contrary to our assumption about m. If $k = n$, then G is an oriented cycle which corresponds to a non-nilpotent mapping. Let $k = n-1$ and $z \in V(G) - C$. Let v be an in-neighbour of z and w be an out-neighbour of z. Let s be the distance on C from v to w. By the choice of C, $s \leqslant 2$. If $s = 1$, then G contains the graph of Example 2 as a subgraph and thus Λ is $n(n-2)$-nilpotent. The only unexplored possibility is that z has only one in-neighbour v, only one out-neighbour w, and the distance from v to w on C is 2. But then G is obtained from C by duplicating a vertex, and a mapping corresponding to such a graph is not nilpotent. The first part is proved.

If φ is symmetric then the above argument holds with the values $k = 2$ and $t = 0$, which proves the second claim. ∎

When $|\Lambda| \geqslant 2$, the situation gets more complicated.

A mapping φ is called *increasing* if $X \subseteq \varphi(X)$ for every $X \subseteq V$. Note that if φ is increasing and nilpotent then it is strictly increasing; i.e. X is a proper subset of $\varphi(X)$. Also, if φ^+, φ^- are opposite then $\varphi^+\varphi^-$ is increasing.

PROPOSITION 2. *Let $\Lambda = \{a, b, c_1, \ldots, c_k\}$ be a nilpotent collection such that the mappings ab, ba, and all c_i are increasing. Then Λ is n^3-nilpotent.*

Proof. Take an arbitrary sequence $(f_1, f_2, \ldots, f_m)$ of mappings from Λ, an arbitrary non-empty $X_0 \subseteq V$, and let $X_i = f_i(X_{i-1})$ for $i = 1, 2, \ldots, m$. To each set X_i we assign a level, an integer value l_i, defined as follows:

$l_0 = 0$;

if $f_i = a$ then $l_i = l_{i-1} + 1$;

if $f_i = b$ then $l_i = l_{i-1} - 1$;
if $f_i = c_j$ then $l_i = l_{i-1}$.

Now we shall prove two claims, from which the proposition will immediately follow. Let $0 \leqslant p < q \leqslant m$.

CLAIM 1. *If $l_q = l_p$ and $X_p \neq V$ then X_p is a proper subset of X_q.*

We shall prove this claim by induction on $q - p$. If $q - p = 1$ then $f_q = c_j$ is increasing, as required. If $l_{p+1} \neq l_{q-1}$ then there is an r such that $p < r < q$ and $l_r = l_p$, and we apply the induction hypothesis to (p, r) and (r, q). Finally, let $l_{p+1} = l_{q-1} \neq l_p$. We can assume that $f_{p+1} = a$, $f_q = b$ (the other case is similar). By induction, we have $X_{p+1} \subseteq X_{q-1}$. So,

$$X_p \subset b(a(X_p)) = b(X_{p+1}) \subseteq b(X_{q-1}) = X_q$$

and the claim is proved. (Here the first inclusion is proper since X_p is not a black hole.)

CLAIM 2. *If $|l_q - l_p| \geqslant n(n-1)$ then $X_q = V$.*

Let $l_q = l_p + n(n-1)$; the other case is treated similarly. For $i = 0, \ldots, n(n-1)$ let p_i be the smallest index such that $p \leqslant p_i \leqslant q$ and $l_{p_i} = l_p + i$. In particular, $p_0 = p$. For every $i = 1, \ldots, n(n-1)$ we have $l_{p_i - 1} = l_p + i - 1 = l_{p_{i-1}}$. So, by Claim 1, we have $X_{p_{i-1}} \subseteq X_{p_i - 1}$; and $a(X_{p_{i-1}}) \subseteq a(X_{p_i - 1}) = X_{p_i}$. All these inclusions together imply that $a^{n(n-1)}(X_p) \subseteq X_{p_{n(n-1)}}$ and by Proposition 1, $X_{p_{n(n-1)}} = V$. The claim is proved.

Now, if some value of the level is assigned to n or more sets then Claim 1 implies that the last of these sets is equal to V. On the other hand, if we have more than $n(n-1)$ different values of the level, Claim 2 implies that we have reached V. Therefore we shall reach V after at most $n(n-1)(n-1)$ steps. ∎

THEOREM 3. *Let $\Lambda = \{\varphi_1, \varphi_2\}$. If Λ is nilpotent and either both φ_i are symmetric, or they are opposite to each other, then Λ is $(2n^3)$-nilpotent.*

Proof. When φ_1 and φ_2 are opposite, Proposition 2 applies immediately, and Λ is n^3-nilpotent.

When both mappings are symmetric, define four new mappings: $a = \varphi_1\varphi_2$, $b = \varphi_2\varphi_1$, $c_1 = \varphi_1\varphi_1$, and $c_2 = \varphi_2\varphi_2$. It is easy to check that these mappings satisfy the conditions of Proposition 2. Now, every sequence of $2N$ mappings φ_i can be considered as a sequence of N mappings a, b, c_i. Therefore Λ is $2n^3$-nilpotent. ∎

The case of φ_1 and φ_2 being opposite corresponds precisely to the original notion of directed nice graphs [3]. Thus, Theorem 3 in this case asserts that "every nice graph on n vertices is n^3-nice."

Recall that every nilpotent Λ on $V = \{0, 1, \ldots, n-1\}$ is $(2^n - 2)$-nilpotent. Our next example shows that if $|\Lambda| = n$, then Λ may be nilpotent but not $(2^n - 3)$-nilpotent.

EXAMPLE 3. For each $v \in V$ define the mapping a_v as follows: $a_v(\{v\}) = V$, $a_v(\{x\}) = \{v\}$ if $x < v$, $a_v(\{x\}) = \{x, v\}$ if $x > v$. All these mappings are symmetric. Let $X_1, \ldots, X_{2^n-1}$ be the lexicographic ordering of $\mathscr{P}(V)^+$; each X_i is the set of positions at which the binary expansion of i has ones.

To make sure that the collection $\{a_0, \ldots, a_{n-1}\}$ is nilpotent, it is enough to check that in this ordering $a_i(X) > X$ for every i and every $X \subset V$. This is straightforward.

On the other hand, for $1 \leqslant i < 2^n - 1$, let $z(i)$ be the position of the first zero in the binary expansion of i; or, equivalently, $z(i) = \min\{j \mid j \in V \backslash X_i\}$. Now it is straightforward to check that $a_{z(i)}(X_i) = X_{i+1}$.

Thus, the collection $\{a_0, \ldots, a_{n-1}\}$ is nilpotent of class $2^n - 2$.

Proposition 1 and Theorem 3 together with Example 3 lead to the following natural conjecture.

CONJECTURE. For every natural k there exists an exponent $f(k)$ and a constant $c(k)$ such that every nilpotent family of k endomorphisms of the semilattice $(\mathscr{P}(n)^+, \cup)$ of all non-empty subsets of an n-element set has nilpotency class at most $c(k)n^{f(k)}$.

This conjecture is open even for $k = 2$.

4. BLACK HOLES

Here we consider families of endomorphisms which are not nilpotent. This means that some element of the semigroup generated by the family has a black hole. We say that the family Λ has a black hole X with a pattern w if w is a word over the alphabet Λ, and $X \subset V$ is a black hole of the endomorphism corresponding, in the obvious way, to this word.

A non-nilpotent family can have many different patterns of black holes; in particular, every power of a pattern is again a pattern. So, the question arises: what can be said about the shortest pattern of a black hole of a non-nilpotent family? It was proved in [3] that every non-nilpotent family corresponding to an uncoloured directed graph has a black hole with the pattern of length at most two. Here we show that general situation is more complicated.

THEOREM 4. *Let Λ be a finite alphabet, and w be an arbitrary word over it. The following statements are equivalent*:

(1) *There is no word v over Λ so that $w = v^{pq}$ for a composite number pq.*

(2) *There exists a set V and a collection of mappings denoted by elements of Λ such that w is a shortest pattern of any black hole of this collection.*

Proof. (1) $\rightarrow$ (2). Throughout this part of the proof, all arithmetic is modulo n; this allows a word $x_0x_1 \ldots x_{n-1}$ to be thought of as a cycle. Let $w = x_0x_1 \ldots x_{n-1}$ and let $V = \{0, 1, \ldots, n-1\}$. For $X \subseteq V$ and an integer i, let $X + i = \{x + i \mid x \in X\}$, a cyclic shift of X.

Fix a natural number k. We define the mappings so that every set of k consecutive elements of V—interval of length k—forms a black hole whose pattern is some cyclic shift of w.

For each $v \in V$, set $w_v = x_{v-k+1} \cdots x_{v-1}x_v$. If $x \in V$ does not occur in w_v, set $x(\{v\}) = V$. If x does occur in w_v, let i_x and j_x be the first and the last, respectively, of the indices within w_v such that $x_{i_x} = x_{j_x} = x$ and set $x(\{v\}) = \{j_x + 1, j_x + 2, \ldots, i_x + k\}$.

Note that $v + 1 \in x(\{v\})$ for every x and v. Therefore, for every $A \subseteq V$ we have $|x(A)| \geqslant |A|$, and if $|x(A)| = |A|$ then $x(A) = A + 1$.

Let $I = \{v, v+1, \ldots, v+l-1\}$ be an arbitrary (cyclic) interval of length $l < n$. Its image $x(I)$ under any mapping $x \in \Lambda$ is again an interval, because the image of each element is an interval and these intervals, respectively, contain consecutive elements $v+1, \ldots, v+l$. It follows that $x(I) = I + 1$ if and only if $v \notin x(I)$ and $v + l + 1 \notin x(I)$. From the definition of x, we have:

$v \notin x(v)$ if and only if $x_v = x$,

$v + l + 1 \notin x(v + l - 1)$ if and only if $x_{v+l-k} = x$.

Conversely, if these two conditions are satisfied then $x(I) = I + 1$; unless there is a subinterval of I of length k containing no letter x—then $x(I) = V$.

Let A be an arbitrary subset of V. A is the union of disjoint intervals; let their initial vertices be $v_1, \ldots, v_m$, and their lengths $l_1, \ldots, l_m$. From the above remarks we have that $|x(A)| = |A|$ for at most one mapping x, and if x is such then $x_{v_i} = x_{v_i + l_i - k} = x$ for all $i = 1, \ldots, m$.

Thus every interval of length k is a black hole whose pattern is uniquely determined, and this pattern is a cyclic shift of w. On the other hand, if a black hole contains an interval of length $l \neq k$, or two disjoint intervals with initial vertices v_1, v_2, then the word w is periodic with period $|l - k|$, resp. $|v_1 - v_2|$.

Thus, if the word w is not periodic, we can choose $k = 1$, so the only black holes of the resulting collection of mappings will be one-vertex subsets, and their shortest patterns will be cyclic shifts of w. If w is periodic, $w = w_0^p$ for a prime $p > 1$ and a non-periodic word w_0, then we can take $k = |w_0|$, and the resulting collection of mappings will satisfy the required property.

(2) $\to$ (1) Suppose the contrary. Let $w = u^{pq}$ for some word u and $p, q > 1$; suppose that w is a shortest pattern of a black hole, and let A be a black hole of minimum size with pattern w. Consider the sets $A_0 = A,\ A_1 = u(A), \ldots, A_{pq-1} = u^{pq-1}(A)$. These sets are all distinct, and all non-empty. We claim that they are also pairwise disjoint. Indeed, if $B = A_i \cap A_j \neq \emptyset$ then $u(B) \subseteq A_{i+1} \cap A_{j+1}$, etc. (indices taken modulo pq), and $w(B) \subseteq B$—therefore, some non-empty subset of B is a black hole with pattern w, contrary to our choice of A.

But now we see that the sets

$$X_i = \bigcup_{j=0}^{q} A_{i+pj}$$

for $i = 0, \ldots, p-1$, form an orbit of length p of the mapping u, and so X_0 is a black hole with pattern u^q, contrary to our assumption about w. ∎

Thus, exception of composite powers is the only restriction on shortest patterns of non-nilpotent families.

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