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A new lower bound on the number of edges in colour-critical graphs and hypergraphs

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Abstract

A graph G is called k-critical if it has chromatic number k, but every proper subgraph of G is $(k-1)$-colourable. We prove that every k-critical graph $(k \geqslant 6)$ on $n \geqslant k+2$ vertices has at least $\frac{1}{2}(k-1+\frac{k-3}{(k-c)(k-1)+k-3})n$ edges where $c=(k-5)(\frac{1}{2}-\frac{1}{(k-1)(k-2)})$. This improves earlier bounds established by Gallai (Acad. Sci. 8 (1963) 165) and by Krivelevich (Combinatorica 17 (1999) 401).

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1. Introduction

A graph G is k-critical for a positive integer k if G is not $(k-1)$-colourable but every proper subgraph of G is $(k-1)$-colourable. Then every k-critical graph has chromatic number k and every k-chromatic graph contains a k-critical subgraph. The importance of the notion of criticality is that problems for k-chromatic graphs may often be reduced to problems for k-critical graphs, whose structure is more restricted. Critical graphs were first defined and used by Dirac [5] in 1951. In the present paper a new lower bound for the number of edges in a k-critical graph on n vertices is established.

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The complete graph K_k is an example of a k-critical graph and for $k = 1, 2$ it is the only one. The only 3-critical graphs are the odd circuits, so for the remainder of this paper we shall restrict our attention to the case $k \geqslant 4$. Then there are k-critical graphs on n vertices for all $n \geqslant k$ except for $n = k + 1$. For $n \geqslant k + 2$, let $f_k(n)$ denote the minimum number of edges possible in a k-critical graph on n vertices. Since every k-critical graph has minimum degree at least $k - 1$, we have $2f_k(n) \geqslant (k - 1)n$. Brooks' theorem [3] implies

$$2f_k(n) \geqslant (k - 1)n + 1,$$

and Dirac [6] proved

$$2f_k(n) \geqslant (k - 1)n + k - 3.$$

In [7], he also gave a complete description of the extremal cases. Dirac's proof was rather long. Shorter and more elegant proofs were found by Kronk and Mitchem [18], Weinstein [23] and, for the result in [7], by Deuber et al. [4]. In [13], the authors proved

$$2f_k(n) \geqslant (k - 1)n + 2(k - 3)$$

provided that $n \neq 2k - 1$. For a given constant $c \geqslant 0$, let

$$g_k(n, c) = \left(k - 1 + \frac{k - 3}{(k - c)(k - 1) + k - 3}\right) n.$$

In his fundamental paper [9] Gallai characterized the class of graphs that are subgraphs of some k-critical graph G induced by the set of vertices having degree $k - 1$ in G. Based on this result, he proved $2f_k(n) \geqslant g_k(n, 0)$. Recently, this lower bound was improved by Krivelevich [17] to $2f_k(n) \geqslant g_k(n, 2)$. Krivelevich [17] also presents some interesting applications of his lower bound on the number of edges in critical graphs. In what follows, let

$$\alpha_k = \frac{1}{2} - \frac{1}{(k - 1)(k - 2)}.$$

The following theorem is one of the main results of this paper.

Theorem 1.1. *If* $k \geqslant 6$ *and* $n \geqslant k + 2$, *then* $2f_k(n) \geqslant g_k(n, (k - 5)\alpha_k)$.

1.1. Terminology

Concepts and notation not defined in this paper will be used as in standard textbooks. Though the main objects of our study are graphs, it is convenient to define the central concepts for hypergraphs.

A *hypergraph* $G = (V, E)$ consists of a finite set $V = V(G)$ of *vertices* and a set $E = E(G)$ of subsets of V, called *edges*, each having cardinality at least two. An edge e with $|e| \geqslant 3$ is called a *hyperedge* and an edge e with $|e| = 2$ is called an *ordinary edge*. A *graph* is a hypergraph in which each edge is ordinary.

Let G be a hypergraph. The *order* of G is $|V(G)|$. The *degree* $d_G(x)$ of a vertex $x \in V(G)$ is the number of the edges in G containing x. If $d_G(x) = r$ for every vertex

$x \in V(G)$, then G is called *r-regular*. Furthermore, let $d(G) = \sum_{x \in V(G)} d_G(x)$. Clearly, if G is a graph, then $d(G) = 2|E(G)|$.

If H and G are hypergraphs with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, then H is said to be a *subhypergraph* of G.

Let G be a hypergraph and $X \subseteq V(G)$. The subhypergraph $G[X]$ of G induced by X is defined as follows: $V(G[X]) = X$ and $E(G[X]) = \{e \in E(G) \mid e \subseteq X\}$. Furthermore, let $G(X)$ denote the hypergraph with $V(G(X)) = X$ and $E(G(X)) = \{e \cap X \mid e \in E(G)\ \&\ |e \cap X| \geqslant 2\}$. Further, let $G - X = G[V(G) - X]$ and $G \backslash X = G(V(G) - X)$. For $M \subseteq E(G)$, let $G - M = (V(G), E(G) - M)$. Clearly, $G[X]$ is a subhypergraph of $G(X)$ and, if G is a graph, then $G[X] = G(X)$. Note that in general $G(X)$ is not a subhypergraph of G.

A subset X of $V(G)$ will be called a *clique* of G if $G[X]$ is a complete graph. A clique of G with p vertices is also said to be a *p-clique* of G. As usual, K_n denotes the complete graph on n vertices. For an edge e, let $\langle e \rangle$ be the hypergraph $(e, \{e\})$.

For a graph G and a vertex $x \in V(G)$, let $N(x : G)$ be the *neighbourhood* of x in G, that is the set of all vertices $y \in V(G)$ such that $\{x, y\} \in E(G)$. Obviously, $d_G(x) = |N(x : G)|$.

Now consider a hypergraph G and a set $X \subseteq V(G)$. Then the set of all edges $e \in E(G)$ satisfying $|e \cap X| = 1$ is denoted by $E_X(G)$ and in case of $X = \{x\}$ also by $E_x(G)$. By an *X-mapping* of G we mean a mapping v that assigns to every edge $e \in E_X(G)$ a vertex $v(e) \in e - X$. For an X-mapping v and a vertex $x \in X$, let

$$N_X^v(x : G) = \{y \in V(G) \mid y = v(e)\ \&\ e \cap X = \{x\} \text{ for some } e \in E_X(G)\}.$$

Clearly, $N_X^v(x : G) \subseteq V(G) - X$ and $d_G(x) \geqslant d_{G(X)}(x) + |N_X^v(x : G)|$. Furthermore, $d_G(x) = |E_x(G)|$ and, provided that G is a graph, $N_X^v(x : G) = N(x : G) - X$.

1.2. Main results

For the proof of Theorem 1.1 we shall use the concept of list colouring. Consider a hypergraph G and assign to each vertex x of G a set $\Phi(x)$ of colours (positive integers). Such an assignment Φ of sets to vertices in G is referred to as a *list* for G. A *Φ-colouring* of G is a mapping φ of $V(G)$ into the set of colours such that $\varphi(x) \in \Phi(x)$ for all $x \in V(G)$ and $|\{\varphi(x) \mid x \in e\}| \geqslant 2$ for each $e \in E(G)$. If G admits a Φ-colouring, then G is said to be *Φ-colourable*. In the case where $\Phi(x) = \{1, \dots, k\}$ for all $x \in V(G)$, we also use the terms *k-colouring* and *k-colourable*, respectively. The *chromatic number* of G denoted by $\chi(G)$ is the least number k for which G is k-colourable. If $\chi(G) = k$, then G is called *k-chromatic*. The list colouring concept was introduced, independently, by Vizing [22] and by Erdös et al. [8].

Let G be a hypergraph and let Φ be a list for G. We say that G is *Φ-critical* if G is not Φ-colourable but every proper subhypergraph of G is Φ-colourable. In the case where $\Phi(x) = \{1, \dots, k-1\}$ for all $x \in V(G)$, we also use the term *k-critical*. Then G is k-critical if and only if $\chi(G') < \chi(G) = k$ for every proper subhypergraph G' of G.

The following theorem is one of the main results of this paper. In particular, it implies Theorem 1.1.

Theorem 1.2. *Let G be a hypergraph not containing a K_k, and let Φ be a list for G satisfying $|\Phi(x)| = k-1$ for every $x \in V(G)$. If G is Φ-critical, then*

$$d(G) \geqslant g_k(|V(G)|, c) = \left(k - 1 + \frac{k-3}{(k-c)(k-1)+k-3}\right)|V(G)|$$

provided that $k \geqslant 9$ and $c = \frac{1}{3}(k-4)\alpha_k$ or $k \geqslant 6$, $\Phi(x) = \{1, \dots, k-1\}$ for every $x \in V(G)$ and $c = (k-5)\alpha_k$.

Theorem 1.2 is an immediate consequence of Theorem 1.9 in Section 1.4 and this result is proved in Section 4. The proof of the next result is given in Section 2. For k-critical graphs, this result was proved by Gallai [9] in 1963.

Theorem 1.3. *Assume that $k \geqslant 4$ and $G \neq K_k$ is a Φ-critical hypergraph where Φ is a list for G satisfying $|\Phi(x)| = k-1$ for every $x \in V(G)$. Then $d(G) \geqslant g_k(|V(G)|, 0)$.*

Theorem 1.3 is interesting only for Φ-critical hypergraphs containing a K_k. Obviously, if a k-critical hypergraph G contains a K_k, then $G = K_k$. However, the list version of this statement is not true. To see this, let $r \geqslant 2$ be an integer and let G denote the hypergraph whose vertex set is the disjoint union of r sets $A_1, \dots, A_r$ such that $G[A_i] = K_k$ for $i = 1, \dots, r$ and $E(G) = \bigcup_{i=1}^{r} E(G[A_i]) \cup \{e\}$ where $e \cap A_i = \{y_i\}$ for $i = 1, \dots, r$. Furthermore, define the list Φ for the hypergraph G by

$$\Phi(x) = \begin{cases} \{1, \dots, k-1\} & \text{if } x \in V(G) - \{y_1, \dots, y_r\}, \\ \{2, \dots, k\} & \text{if } x \in \{y_1, \dots, y_r\}. \end{cases}$$

Then $|\Phi(x)| = k-1$ for all $x \in V(G)$ and it is easy to check that G is Φ-critical. Clearly, G is a hypergraph of order $n = rk$ containing r copies of a K_k and $d(G) = (k-1)n + r$.

In the next subsection we establish some basic results about list-critical hypergraphs.

1.3. Gallai trees and bad pairs

Let G be a connected hypergraph. A vertex x of G is called a *separating vertex* of G if $G \backslash \{x\}$ is non-empty and disconnected. An edge e of G is called a *bridge* of G if $G - \{e\} = (V(G), E(G) - \{e\})$ has precisely $|e|$ components. By a *block* of G we mean a maximal connected subhypergraph B of G such that no vertex of B is a separating vertex of B. Any two distinct blocks of G have at most one vertex in common and, obviously, a vertex of G is a separating vertex of G iff it is contained in more than one block of G. An *end-block* of G is a block that contains at most one separating vertex of G. Clearly, every non-empty hypergraph has at least one end-block.

The above statements about the block structure are well known for graphs. For hypergraphs, the proof of these statements is left to the reader.

By a *brick* we mean a hypergraph of the form $\langle e \rangle$ for some edge e, or an odd circuit (consisting only of ordinary edges), or a complete graph. A connected hypergraph all of whose blocks are bricks is called a *Gallai tree*; a *Gallai forest* is a hypergraph all of whose components are Gallai trees.

By a *bad pair* we mean a pair (G, Φ) consisting of a non-empty connected hypergraph G and a list Φ of G such that $|\Phi(x)| \geqslant d_G(x)$ for all $x \in V(G)$ and G is not Φ-colourable.

Lemma 1.4 (Kostochka et al. [16]). *If (G, Φ) is a bad pair, then the following statements hold*:

(a) $|\Phi(x)| = d_G(x)$ *for all* $x \in V(G)$.
(b) *Every hyperedge e of G is a bridge of G and, therefore, $\langle e \rangle$ is a block of G.*
(c) *If G has no separating vertex, then $\Phi(x)$ is the same for all $x \in V(G)$.*
(d) *G is a Gallai tree.*

For graphs, Lemma 1.4 was proved, independently, by Borodin [1,2] and by Erdös et al. [8]. Proofs of statements (a) and (c) in the graph version based on a sequential colouring argument were given by Vizing [22] and by Lovász [19]. For a short proof of Lemma 1.4 based on the following simple reduction idea the reader is referred to [16].

Remark 1.5. Let G be a hypergraph, Φ be a list for G, $X \subseteq V(G)$, and let v be an X-mapping of G. Furthermore, let $Y = V(G) - X$ and let φ be a Φ-colouring of $G[Y]$. For the hypergraph $G' = G(X) = G \backslash Y$, define the list Φ' by

$$\Phi'(x) = \Phi(x) - \{\varphi(y) \mid y \in N_X^v(x : G)\}$$

for every $x \in V(G')$. In what follows, we denote Φ' by $\Phi(Y, v, \varphi)$ and in case of $Y = \{y\}$ and $\varphi(y) = a$ also by $\Phi(y, a)$. Then it is straightforward to show that the following statements hold.

(a) If G' is Φ'-colourable, then G is Φ-colourable.
(b) If $|\Phi(x)| = d_G(x) + p$ for $x \in V(G')$, then $|\Phi'(x)| \geqslant d_{G'}(x) + p$.
(c) If (G, Φ) is a bad pair, then (G', Φ') is a bad pair provided that G' is connected.

Lemma 1.6. *Let G be a Φ-critical hypergraph where Φ is a given list for G, $H = \{y \in V(G) \mid d_G(y) > |\Phi(y)|\}$ and $L = V(G) - H$. Furthermore, let X be a non-empty subset of L, let v be an X-mapping of G, and let $F = \{e \in E(G) \mid |e \cap X| \geqslant 2 \,\&\, e - X \neq \emptyset\}$. Then the following statements hold*:

(a) $d_G(x) = |\Phi(x)|$ *for every* $x \in L$.
(b) $G(X)$ *is a Gallai forest.*
(c) $d_G(x) = d_{G(X)}(x) + |N_X^v(x : G)|$ *for every* $x \in X$.
(d) *If* $x \in L$, *then* $|e \cap e'| = 1$ *for every two distinct edges* $e, e' \in E_x(G)$.
(e) *If* $e, e' \in F$ *and* $e \neq e'$, *then* $e \cap X \neq e' \cap X$.

(f) *If* $e \in F$, *then* $e \cap X$ *is a bridge of* $G(X)$.

(g) *If* $|\Phi(x)| = k-1$ *for every* $x \in V(G)$ $(k \geqslant 1)$, *then* H *is non-empty or* G *is a* K_k *or* $k=3$ *and* G *is an odd circuit or* $k=2$ *and* $G = \langle e \rangle$. *Furthermore, if* $G(X)$ *contains a* K_k, *then* $G = K_k$.

Proof. In order to prove Lemma 1.6, it is sufficient to consider the case where $G(X)$ is connected. Let $Y = V(G) - X$. Since G is Φ-critical, there is a Φ-colouring φ of $G[Y]$. Now consider the list $\Phi' = \Phi(Y, v, \varphi)$ for the connected hypergraph $G' = G \backslash Y = G(X)$. Then, because $X \subseteq L$, we have

$$|\Phi(x)| \geqslant d_G(x) \geqslant d_{G'}(x) + |N_X^v(x:G)|,$$

and, therefore,

$$|\Phi'(x)| \geqslant |\Phi(x)| - |N_X^v(x:G)| \geqslant d_{G'}(x)$$

for all $x \in X$. Furthermore, G' is not Φ'-colourable. Consequently, (G', Φ') is a bad pair. Then, by Lemma 1.4, G' is a Gallai tree and $|\Phi'(x)| = d_{G'}(x)$ for all $x \in X$ implying that $|\Phi(x)| = d_G(x) = d_{G'}(x) + |N_X^v(x:G)|$ for all $x \in X$. Thus (a)–(c) are proved.

For the proof of (d), suppose that, for some $x \in L$, there are two distinct edges $e, e' \in E_x(G)$ such that $|e \cap e'| \geqslant 2$. Then, for the set $X' = \{x\}$, there exists an X'-mapping v' of G such that $v'(e) = v'(e')$. Consequently, we have $d_G(x) > d_{G(X')}(x) + |N_{X'}^{v'}(x:G)|$, a contradiction to (c).

Clearly, statement (e) is an immediate consequence of (d). For the proof of (f), let $\tilde{e} = e \cap X$ for every $e \in F$. Then $\tilde{e}$ is an edge of $G(X)$ for all $e \in F$.

Now, suppose that $\tilde{e}$ is not a bridge of $G' = G(X)$ for some $e \in F$. Then, because of (b), $\tilde{e}$ is an ordinary edge of G', i.e. $\tilde{e} = \{x_1, x_2\}$ with $x_1, x_2 \in X$ and, therefore, $\tilde{G} = G' - \{\tilde{e}\}$ is a connected hypergraph. Let $\tilde{\Phi}$ be the list for $\tilde{G}$ such that $\tilde{\Phi}(x) = \Phi'(x)$ if $x \neq x_1$ and $\tilde{\Phi}(x_1) = \Phi'(x_1) - \{\varphi(y)\}$ for some $y \in e \cap Y$. Then $|\tilde{\Phi}(x)| \geqslant d_{\tilde{G}}(x)$ for all $x \in X = V(G')$ and $|\tilde{\Phi}(x_2)| > d_{\tilde{G}}(x_2)$. Therefore, by Lemma 1.4, $\tilde{G}$ is $\tilde{\Phi}$-colourable implying that G is Φ-colourable. This contradiction proves (f).

Finally, suppose that $|\Phi(x)| = k-1$ for every $x \in V(G)$. If $H = \emptyset$, then $G = G(L)$ and, since G is Φ-critical, G is connected. Therefore, by (a) and (b), G is a $(k-1)$-regular Gallai tree. Since every block of a Gallai tree is regular, this implies that G consists of one block. Consequently, G is a K_k or $k=3$ and G is an odd circuit or $k=2$ and $G = \langle e \rangle$ for some hyperedge e. If $G(X)$ contains a K_k, then we argue as follows. By (a), the maximum degree of $G(X)$ is at most $k-1$. Consequently, by (b), one block B of $G(X)$ is a K_k. Then, by (d), every edge of B belongs to G and, therefore, B is a subhypergraph of G. Since every vertex of B has degree $k-1$ in G, this implies that B is a component of G. Then, since G is Φ-critical, we infer that $G = B = K_k$. This proves (g). □

For k-critical graphs, statement (b) of Lemma 1.6 is due to Gallai [9] and the first statement of (g) is equivalent to the well-known theorem of Brooks [3].

Following Gallai, a vertex x of a Φ-critical hypergraph G is called a *high vertex* if $d_G(x) > |\Phi(x)|$, otherwise x is called a *low vertex* of G. For this reason, we always write H and L for the corresponding sets of vertices.

Let G be an arbitrary Gallai tree. The set of all blocks of G is denoted by $\mathscr{B}(G)$. If $B \in \mathscr{B}(G)$, then B is regular and we say that B is a *block of type b* if B is $(b-1)$-regular. Clearly, if $B \in \mathscr{B}(G)$ is a block of type b, then $b \geqslant 1$ and $B = K_b$, or $b = 3$ and B is an odd circuit, or $b = 2$ and $B = \langle e \rangle$ for some edge e. Two distinct blocks which have a vertex in common (they cannot have more than one vertex in common) are called *adjacent*.

Let $\mathscr{U}(G)$ denote the set of all mappings u that assign to every block $B \in \mathscr{B}(G)$ of type b a set $u(B)$ of $b-1$ colours such that $u(B) \cap u(B') = \emptyset$ for any two adjacent blocks $B, B' \in \mathscr{B}(G)$. For a given mapping $u \in \mathscr{U}(G)$, define the list $\Phi = \Phi_u$ for the Gallai tree G by $\Phi(x) = \bigcup u(B)$ where B runs through all blocks of G containing the vertex $x \in V(G)$. The graph version of the following result was proved by Borodin [1,2] and by Erdös et al. [8].

Lemma 1.7. *Let (G, Φ) be a bad pair. Then $\Phi = \Phi_u$ for some $u \in \mathscr{U}(G)$. This implies, in particular, that $\Phi(x) = \Phi(y)$ provided that x and y are two non-separating vertices of G contained in the same block of G.*

Proof (By induction on the number m of blocks of G). For $m = 1$, Lemma 1.7 follows from Lemma 1.4.

Now assume $m > 1$. Let G_1 be an end-block of G and let x denote the only separating vertex of G contained in G_1. Let $G_2 = G - (V(G_1) - \{x\})$. Clearly, G_2 is a Gallai tree with $\mathscr{B}(G_2) = \mathscr{B}(G) - \{G_1\}$.

For $i = 1, 2$, let M_i denote the set of all colours $a \in \Phi(x)$ such that there is no Φ-colouring φ of G_i with $\varphi(x) = a$. If there is a colour $a \in \Phi(x) - M_1 - M_2$, then, for $i = 1, 2$, there is a Φ-colouring φ_i of G_i with $\varphi_i(x) = a$. Consequently, $\varphi_1 \cup \varphi_2$ is a Φ-colouring of G. This contradiction shows that $\Phi(x) = M_1 \cup M_2$. For $i = 1, 2$, define a list Φ_i for the hypergraph G_i by

$$\Phi_i(y) = \begin{cases} \Phi(y) & \text{if } y \in V(G_i) - \{x\}, \\ M_i & \text{if } y = x. \end{cases}$$

Clearly, for $i = 1, 2$, the hypergraph G_i is not Φ_i-colourable and, moreover, $|\Phi_i(y)| \geqslant d_{G_i}(y)$ for all $y \in V(G_i) - \{x\}$. Therefore, by Lemma 1.4, $|M_i| = |\Phi_i(x)| \leqslant d_{G_i}(x)$. Since

$$|M_1| + |M_2| \geqslant |M_1 \cup M_2| = |\Phi(x)| = d_G(x) = d_{G_1}(x) + d_{G_2}(x),$$

this implies that $|M_i| = d_{G_i}(x)$ for $i = 1, 2$ and $M_1 \cap M_2 = \emptyset$. Hence (G_i, Φ_i) is a bad pair and, by the induction hypothesis, $\Phi_i = \Phi_{u_i}$ for some $u_i \in \mathscr{U}(G_i)$. Then the mapping u with $u(G_1) = u_1(G_1)$ and $u(B) = u_2(B)$ for all $B \in \mathscr{B}(G_2)$ belongs to $\mathscr{U}(G)$ and $\Phi = \Phi_u$. $\square$

1.4. Basic idea

The next lemma tells us how we can find a lower bound for the degree sum of a list-critical hypergraph.

Lemma 1.8. *Assume that* $k \geqslant 4$ *and* $G \neq K_k$ *is a* Φ*-critical hypergraph where* Φ *is a list for* G *satisfying* $|\Phi(x)| = k-1$ *for every* $x \in V(G)$. *Furthermore, let* $L = \{x \in V(G) \mid d_G(x) = k-1\}$, $H = \{x \in V(G) \mid d_G(x) \geqslant k\}$, $E_1 = \{e \in E(G) \mid |e \cap L| = 1\}$ *and* $E_2 = \{e \in E(G) \mid |e \cap L| \geqslant 2\}$. *Finally, let*

$$\varrho = \sum_{e \in E_1} (|e \cap H| - 1) + \sum_{e \in E_2} |e \cap H|,$$

$$\sigma = \left(k - 2 + \frac{2}{k-1}\right)|L| - d(G(L))$$

and

$$\tau_c = d(G[H]) + \left(k - c - \frac{2}{k-1}\right) \sum_{y \in H} (d_G(y) - k),$$

where $0 \leqslant c \leqslant k - \frac{2}{k-1}$ *is a given constant. If* $\varrho + \sigma + \tau_c \geqslant c|H|$, *then* $d(G) \geqslant g_k(|V(G)|, c)$.

Proof. Let $n = |V(G)|$ and $\gamma = \sum_{y \in H}(d_G(y) - k)$. Then

$$\sigma = \left(k - 2 + \frac{2}{k-1}\right)|L| - d(G(L)) \quad \text{and}$$
$$\tau_c = d(G[H]) + \left(k - c - \frac{2}{k-1}\right)\gamma.$$

From Lemma 1.4 we conclude that $H \neq \emptyset$ and $n = |L| + |H|$. If $|L| = 0$, then $d(G) \geqslant kn \geqslant g_k(n, c)$. If $|L| \geqslant 1$, then $d(G(L)) = (k-1)|L| - |E_1|$ and we infer that

$$\begin{aligned}\sum_{y \in H} d_G(y) &= d(G[H]) + \sum_{e \in E_1 \cup E_2} |e \cap H| \\ &= d(G[H]) + (k-1)|L| - d(G(L)) + \varrho.\end{aligned}$$

Since $\varrho+\sigma+\tau_c \geqslant c|H|$ and every vertex in L has degree $k-1$ in G, this implies, on the one hand, that

$$\begin{aligned}
d(G) &= (k-1)|L| + \sum_{y\in H} d_G(y)\\
&= d(G[H]) + 2(k-1)|L| - d(G(L)) + \varrho\\
&= d(G[H]) + \sigma + \varrho + |L|\left(k-\frac{2}{k-1}\right)\\
&\geqslant c|H| - \gamma\left(k-c-\frac{2}{k-1}\right) + |L|\left(k-\frac{2}{k-1}\right)\\
&= cn - \gamma\left(k-c-\frac{2}{k-1}\right) + |L|\left(k-c-\frac{2}{k-1}\right).
\end{aligned}$$

On the other hand,

$$d(G) = (k-1)n + |H| + \gamma = kn - |L| + \gamma.$$

Therefore,

$$d(G)\left(1+k-c-\frac{2}{k-1}\right) \geqslant \left(c + k\left(k-c-\frac{2}{k-1}\right)\right)n.$$

Since $k-c-\frac{2}{k-1}\geqslant 0$, this is equivalent to

$$d(G) \geqslant \left(k-1+\frac{k-3}{(k-c)(k-1)+k-3}\right)n = g_k(n,c).$$

Thus Lemma 1.8 is proved. $\square$

Consider a k-critical graph $G\neq K_k$ for some integer $k\geqslant 4$. Furthermore, let L, H, ϱ, σ and τ_c be defined as in Lemma 1.8. By this lemma, $\varrho+\sigma+\tau_c\geqslant c|H|$ implies $d(G)\geqslant g_k(|V(G)|,c)$. For k-critical graphs, this fact was already known to Gallai [9]. Clearly, in the graph case we have $\varrho = 0$, $\tau_c \geqslant 0$ and, moreover, Gallai [9] proved that if c_L is the number of components of $G[L]$, then $\sigma\geqslant 2c_L$. Consequently, $\varrho+\sigma+\tau_c\geqslant 0$ and, therefore, $d(G)\geqslant g_k(|V(G)|,0)$. Krivelevich [17] observed that if c_H is the number of components of $G[H]$, then $\tau_c\geqslant d(G[H]) = 2|E(G[H])|\geqslant 2|H| - 2c_H$ and, therefore, $\varrho+\sigma+\tau_c\geqslant 2c_L+2|H|-2c_H$. Since $c_L-c_H\geqslant 0$ by a result from [21], this implies that $\varrho+\sigma+\tau_c\geqslant 2|H|$ and, therefore, $d(G)\geqslant g_k(|V(G)|,2)$.

The statement $\sigma\geqslant 2c_L$ holds also if $G\neq K_k$ is a Φ-critical graph for some list Φ satisfying $|\Phi(x)| = k-1$ for all $x\in V(G)$ (see Section 2). However, the statement $c_L - c_H\geqslant 0$ is not true in this case.

As an immediate consequence of Lemma 1.8 we obtain that for the proof of Theorem 1.2 it suffices to prove the following result.

Theorem 1.9. *Let G be a hypergraph not containing a K_k, and let Φ be a list for G satisfying $|\Phi(x)| = k-1$ for every $x\in V(G)$. Let L, H, E_1, E_2, ϱ, σ and τ_c be* *defined as in Lemma* 1.8. *If G is Φ-critical, then*

$$\varrho + \sigma + \tau_c \geqslant c|H|$$

provided that $k \geqslant 9$ *and* $c = \frac{1}{3}(k-4)\alpha_k$ *or* $k \geqslant 6$, $\Phi(x) = \{1, \ldots, k-1\}$ *for every* $x \in V(G)$ *and* $c = (k-5)\alpha_k$.

The proof of Theorem 1.9 is given in Section 4. In Section 2 we give a generalization of Gallai's result concerning σ and establish some lower bounds for this parameter. In Section 3 we prove some auxiliary results about bipartite graphs. Section 4 is mainly devoted to the proof of Lemma 4.1 which is the key lemma for the proof of Theorem 1.9.

2. Lower bounds for σ and ε_k-hypergraphs

Let $k \geqslant 4$ be a given integer, and let

$$r_k = k - 2 + \frac{2}{k-1}.$$

For an arbitrary hypergraph F and $x \in V(F)$, define $\sigma(x : F) = r_k - d_F(x)$ and

$$\sigma(F) = \sum_{x \in V(F)} \sigma(x : F) = |V(F)| r_k - d(F).$$

Let $\mathscr{T}_k$ denote the set of all Gallai trees distinct from K_k and with maximum degree at most $k-1$. For $T \in \mathscr{T}_k$ and some end-block B of T, let $T_B = T - (V(B) - \{x\})$ where x is the only separating vertex of T contained in B (if there is no such vertex, then $T = B$ and an arbitrary vertex of B may be taken).

Lemma 2.1. *Let $T \in \mathscr{T}_k$ and $k \geqslant 4$. Then the following statements hold*:

(a) *If $B \in \mathscr{B}(T)$, then $\sigma(B) = 2$ if $B = K_{k-1}$ and $\sigma(B) \geqslant r_k$ otherwise.*
(b) *If B is an end-block of $T \in \mathscr{T}_k$, then $\sigma(T) = \sigma(T_B) + \sigma(B) - r_k$.*

Proof. Let $B \in \mathscr{B}(T)$ be a block of type b, that is B is a brick and B is $(b-1)$-regular for some $b \leqslant k-1$. Then $1 \leqslant b \leqslant k-1$, $B = K_b$ and

$$\sigma(B) = b(r_k - b + 1) \begin{cases} \geqslant r_k & \text{if } 1 \leqslant b \leqslant k-2, \\ = 2 & \text{if } b = k-1 \end{cases}$$

or $b = 3$, B is an odd circuit of order at least five and $\sigma(B) = |V(B)|(r_k - 2) \geqslant 5(r_k - 2) \geqslant r_k$, or $b = 2$, $B = \langle e \rangle$ and $\sigma(B) = |e|(r_k - 1) \geqslant r_k$. This proves (a). Statement (b) follows from the fact that T_B and B have exactly one vertex in common. □

Consider an arbitrary Gallai tree $T \in \mathscr{T}_k$. Let $x \in V(T)$ and let $B_1, \ldots, B_l$ be the blocks of T containing x where B_i is of type b_i $(i = 1, \ldots, l)$. Then x is said to be of *type* $(b_1, \dots, b_l)$ in T. Let $\mathcal{T}'_k$ denote the set of all Gallai trees from $\mathcal{T}_k$ that do not have a block of type 2. For an integer $b \geqslant 1$, let $t(b) = 2 - \frac{2}{b}$.

Let $T \in \mathcal{T}'_k$ and $k \geqslant 6$. Clearly, if T contains a block B of type $k-1$, then $T = B = K_{k-1}$. For a vertex $x \in V(T)$ of type $(b_1, \dots, b_l)$ in T, define

$$\sigma'(x : T) = \begin{cases} \sigma(x : T) + \sum_{i=1}^{l} t(b_i) - 2 & \text{if } T \neq K_{k-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, let

$$\sigma'(T) = \sum_{x \in V(T)} \sigma'(x : T).$$

Lemma 2.2. *If* $T \in \mathcal{T}'_k$ *and* $k \geqslant 6$, *then*

(a) $\sigma(T) \geqslant \sigma'(T) + 2$, *and*
(b) $\sigma'(x : T) \geqslant \alpha_k(k - 1 - d_T(x))$ *for every* $x \in V(T)$ *provided that* $T \neq K_{k-1}$.

Proof. We prove statement (a) by induction on the number m of blocks of T. First, assume $m = 1$. Then T is a complete graph of order b where $1 \leqslant b \leqslant k-1$ and $b \neq 2$ or T is an odd circuit. If $T = K_{k-1}$, then $\sigma'(T) = 0$ and, by Lemma 2.1(a), $\sigma(T) = 2 = \sigma'(T) + 2$. If $T = K_b$ with $1 \leqslant b \leqslant k-2$ and $b \neq 2$, then $\sigma'(T) = \sigma(T) + (t(b) - 2)b = \sigma(T) - 2$. If T is an odd circuit of order $p \geqslant 3$, then $\sigma'(T) = \sigma(T) + (t(3) - 2)p = \sigma(T) - (2/3)p \leqslant \sigma(T) - 2$. This settles the case $m = 1$.

Next, assume $m \geqslant 2$. Let B be some end-block of T and let x be the only separating vertex of T contained in B. Suppose that B is a block of type b and x is of type $(b_1, \dots, b_l)$ in T where $b_l = b$. Since $T \in \mathcal{T}'_k$ has at least two blocks, no block of T is a K_{k-1}. Furthermore, $T' = T_B \in \mathcal{T}'_k$ and x is of type $(b_1, \dots, b_{l-1})$ in T'. Consequently,

$$\sigma'(x : T') = r_k - d_{T'}(x) + \sum_{i=1}^{l-1} t(b_i) - 2,$$

$$\sigma'(x : B) = r_k - d_B(x) + t(b_l) - 2$$

and

$$\sigma'(x : T) = r_k - d_T(x) + \sum_{i=1}^{l} t(b_i) - 2.$$

Since $d_T(x) = d_{T'}(x) + d_B(x)$, this implies that

$$\begin{aligned} \sigma'(T) &= \sigma'(T') + \sigma'(B) + \sigma'(x : T) - \sigma'(x : T') - \sigma'(x : B) \\ &= \sigma'(T') + \sigma'(B) - r_k + 2. \end{aligned}$$

Then, by the induction hypothesis and Lemma 2.1(b), we infer that

$$\begin{aligned}\sigma'(T) &= \sigma'(T') + \sigma'(B) - r_k + 2\\ &\leqslant \sigma(T') - 2 + \sigma(B) - 2 - r_k + 2\\ &= \sigma(T) - 2.\end{aligned}$$

Thus (a) is proved. For the proof of (b), consider an arbitrary vertex $x \in V(T)$. Suppose that x is of type $(b_1, \dots, b_l)$ in T. Then, since $T \in \mathscr{T}'_k$ and $T \neq K_{k-1}$, we have $1 \leqslant b_i \leqslant k-2$ and $b_i \neq 2$ for $i = 1, \dots, m$. Furthermore, $d_T(x) = \sum_{i=1}^{l}(b_i - 1) \leqslant k-1$ and we have to show that

$$\sigma'(x:T) = r_k - d_T(x) + \sum_{i=1}^{l} t(b_i) - 2 \geqslant \alpha_k(k-1-d_T(x)). \tag{1}$$

Let

$$M = (1-\alpha_k)\left(k-1-\sum_{i=1}^{l}(b_i-1)\right) + \sum_{i=1}^{l}\left(2-\frac{2}{b_i}\right).$$

By an easy calculation, it then follows that (1) is equivalent to

$$M \geqslant 3 - \frac{2}{k-1}. \tag{2}$$

First, consider the case $l = 1$. Then

$$M = (1-\alpha_k)(k-b_1) + 2 - 2/b_1.$$

For $b_1 = 1$, this yields $M = (1-\alpha_k)(k-1)$. Then, in case of $k \geqslant 7$ we have $M \geqslant 3$, and in case of $k = 6$ we have $M = (1/2 + 1/20)5 = 55/20$ and $3 - 2/(k-1) = 13/5$. Hence (2) is satisfied for $b_1 = 1$. If $3 \leqslant b_1 \leqslant k-2$, then M is a monotone decreasing function of b_1, and we infer that

$$\begin{aligned}M &\geqslant (1-\alpha_k)(k-(k-2)) + 2 - 2/(k-2)\\ &= 1 + \frac{2}{(k-1)(k-2)} + 2 - \frac{2}{k-2} = 3 - \frac{2}{k-1}.\end{aligned}$$

This settles the case $l = 2$. Next, consider the case $l = 2$. Then $3 \leqslant b_1, b_2$ and $b_1 + b_2 \leqslant k+1$. Hence $-2/b_1 - 2/b_2 \geqslant -4/3$. Therefore, in case of $b_1 + b_2 \leqslant k$ we have

$$M = (1-\alpha_k) + \left(2-\frac{2}{b_1}\right) + \left(2-\frac{2}{b_2}\right) \geqslant \frac{1}{2} + \frac{8}{3} \geqslant 3,$$

and in case of $b_1 + b_2 = k+1$ we have

$$M > \left(2-\frac{2}{b_1}\right) + \left(2-\frac{2}{k+1-b_1}\right) \geqslant \left(2-\frac{2}{k-2}\right) + \left(2-\frac{2}{3}\right) > 3 - \frac{2}{k-1}.$$

Consequently, (2) holds for $l = 2$. Finally, consider the case $l \geqslant 3$. Then $b_i \geqslant 3$ and, therefore,

$$M \geqslant \sum_{i=1}^{3}\left(2-\frac{2}{b_i}\right) \geqslant 3\left(2-\frac{2}{3}\right) = 4.$$

Hence (2) holds for all $l \geqslant 1$ and, therefore, (b) is proved. □

For a hypergraph G and an integer $p \geqslant 2$, let $W^p(G)$ denote the set of all vertices of G that belong to some $(p-1)$-clique of G. If $G \in \mathscr{T}_k$, then $W^{k+1}(G) = \emptyset$ and, for every $(k-1)$-clique X of G, $G(X) = G[X]$ is a block of G.

Following Gallai, G is called an *ε_k-hypergraph* if $G \in \mathscr{T}_k$ and $W^k(G) = V(G)$. For $k \geqslant 5$, a hypergraph G is an ε_k-hypergraph iff $G \in \mathscr{T}_k$ and every separating vertex of G is of type $(k-1, 2)$ and every non-separating vertex of G is of type $k-1$.

If a component G' of $G(W^k(G))$ is an ε_k-hypergraph, then G' is said to be an *ε_k-subcomponent* of G.

Obviously, if $T \in \mathscr{T}_k$, then every vertex of $W^k(T)$ is of type $(k-1, 2)$ or of type $k-1$ and the ε_k-subcomponents of T are precisely the components of $T(W^k(T))$. The number of all ε_k-subcomponents of T is denoted by $s(T)$.

Let $T \in \mathscr{T}_k$. For a vertex $x \in V(T)$, define

$$\sigma^*(x:T) = \begin{cases} \alpha_k(k-1-d_T(x)) & \text{if } x \in V(T) - W^k(T), \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, let

$$\sigma^*(T) = \sum_{x \in V(T)} \sigma^*(x:T).$$

Lemma 2.3. *If $T \in \mathscr{T}_k$ and $k \geqslant 6$, then $\sigma(T) \geqslant \sigma^*(T) + s(T)\alpha_k + 2 - \alpha_k$.*

Proof. We prove Lemma 2.3 by induction on the number m of blocks of type 2 in T. If $m = 0$, then $T \in \mathscr{T}'_k$, $s(T) \leqslant 1$, and, by Lemma 2.2, $\sigma(T) \geqslant \sigma^*(T) + 2 \geqslant \sigma^*(T) + s(T)\alpha_k + 2 - \alpha_k$.

Now assume $m \geqslant 1$. Let B be an arbitrary block of T that is of type 2. Then $B = \langle e \rangle$ where $e \in E(T)$ is a bridge of T. Let $e = \{x_1, \dots, x_p\}$ where $p \geqslant 2$ and, for $i = 1, \dots, p$, let T_i denote the component of $T - \{e\}$ containing the vertex x_i. Assume that $x_i \in W^k(T_i)$ for $i = 1, \dots, l$ and $x_i \in V(T_i) - W^k(T_i)$ for $i = l+1, \dots, p$. Then

$$\sigma^*(T) = \sum_{i=1}^{p} \sigma^*(T_i) - \alpha_k(p-l),$$

and, moreover,

$$s(T) = \begin{cases} \sum_{i=1}^{p} s(T_i) & \text{if } l = 0, \\ \sum_{i=1}^{p} s(T_i) - l + 1 & \text{if } l \geqslant 1. \end{cases}$$

Consequently, using the induction hypothesis, we conclude that

$$\begin{aligned}\sigma(T) &= \sum_{i=1}^{p} \sigma(T_i) - p\\ &\geqslant \sum_{i=1}^{p} \sigma^*(T_i) + \alpha_k \sum_{1=1}^{p} s(T_i) + p(2-\alpha_k) - p\\ &= \sigma^*(T) + \alpha_k(p-l) + \alpha_k \sum_{1=1}^{p} s(T_i) + p(2-\alpha_k) - p\\ &= \sigma^*(T) + \alpha_k\left(\sum_{1=1}^{p} s(T_i) - l\right) + p\\ &\geqslant \sigma^*(T) + \alpha_k s(T) + p - \alpha_k\\ &\geqslant \sigma^*(T) + \alpha_k s(T) + 2 - \alpha_k.\end{aligned}$$

This proves Lemma 2.3. □

Lemma 2.4. *Let* $T \in \mathcal{T}_k$ *and* $k \geqslant 4$. *Then* $\sigma(T) \geqslant 2$ *if* T *is an* ε_k*-hypergraph and* $\sigma(T) \geqslant r_k$ *otherwise.*

Proof (By induction on the number m of blocks of T). For $m = 1$, Lemma 2.4 is an immediate consequence of Lemma 2.1.

Now assume $m > 1$. If T is an ε_k-hypergraph, then T_B is not an ε_k-hypergraph for any end-block B of T and, by the induction hypothesis and Lemma 2.1, $\sigma(T) \geqslant \sigma(T_B) + \sigma(B) - r_k \geqslant \sigma(B) \geqslant 2$.

If T is not an ε_k-hypergraph, then we argue as follows. First, consider the case where T has a block B of type 2. Then $B = \langle e \rangle$ where $e \in E(T)$ is a bridge of T. For $x \in e$, let T_x denote the component of $T - \{e\}$ containing x. Since T is not an ε_k-hypergraph, T_x is not an ε_k-hypergraph for at least one $x \in e$. Furthermore, $r_k \geqslant k - 2 \geqslant 2$. Therefore, by the induction hypothesis,

$$\sigma(T) = \sum_{x \in e} \sigma(T_x) - |e| \geqslant 2(|e| - 1) + r_k - |e| \geqslant r_k.$$

Now, consider the case where T has no block of type 2. Then no block of T is a K_{k-1}. Let B be an end-block of T. Then T_B is not an ε_k-hypergraph and, by the induction hypothesis and Lemma 2.1, $\sigma(T) = \sigma(T_B) + \sigma(B) - r_k \geqslant r_k$. This completes the proof of Lemma 2.4. □

Proof of Theorem 1.3. Assume that $k \geqslant 4$ and $G \neq K_k$ is a Φ-critical hypergraph where Φ is a list for G satisfying $|\Phi(x)| = k - 1$ for every $x \in V(G)$. Let $L = \{x \in V(G) \mid d_G(x) = k - 1\}$. Then, by Lemma 1.6, each component of $G(L)$ belongs to $\mathcal{T}_k$. Therefore, by Lemma 2.4, $\sigma(G(L)) \geqslant 0$. Consequently, by Lemma 1.8, $d(G) \geqslant g_k(|V(G)|, 0)$. □

3. Bipartite graphs

Let G be a graph. An edge $\{x, y\}$ of G is also denoted by xy or yx. We denote by $F = F(A, B)$ a bipartite graph satisfying $V(F) = A \cup B$, $A \cap B = \emptyset$ and $E(F) \subseteq \{xy \mid x \in A \text{ and } y \in B\}$. For an integer x, let $\lceil x \rceil$ denote the upper integer part of x.

Lemma 3.1. *Let $F = F(A, B)$ be a bipartite graph, let $r \geqslant 1$ be an integer, and let B_r be the set of all vertices of B having degree at least r in F. Then there is a subgraph F' of F such that*

(a) $d_{F'}(x) \leqslant \lceil \frac{d_F(x)}{r} \rceil$ *for every* $x \in A$,
(b) $d_{F'}(y) = 1$ *for every* $y \in B_r$ *and* $d_{F'}(y) = 0$ *for every* $y \in B - B_r$.

Proof. For every vertex $x \in A$, there is a partition $\{N_1^x, \ldots, N_{m_x}^x\}$ of $N(x : F)$ into $m_x = \lceil \frac{d_F(x)}{r} \rceil$ subsets satisfying $1 \leqslant |N_i^x| \leqslant r$ for $i = 1, \ldots, m_x$. Now, replace in F every vertex $x \in A$ by $m = m_x$ new vertices $x_{(1)}, \ldots, x_{(m)}$ and join $x_{(i)}$ to every vertex in N_i^x by an edge ($i = 1, \ldots, m_x$). This results in a bipartite graph $H = H(A', B)$ such that $d_H(x') \leqslant r$ for every $x' \in A'$ and $d_H(y) = d_F(y)$ for every $y \in B$.

Consider an arbitrary set $S \subseteq B_r$ and let $N(S) = \bigcup_{x \in S} N(x : H)$. Let m be the number of all edges $x'y \in E(H)$ satisfying $y \in S$ and $x' \in N(S) \subseteq A'$. On the one hand, $m \geqslant r|S|$ and, on the other hand, $m \leqslant r|N(S)|$. Consequently, $|N(S)| \geqslant |S|$. Now, Hall's theorem yields that there is a matching M in H that covers all vertices in B_r, i.e., $M \subseteq E(H)$ and for the graph $H' = (V(H), M)$ we have $d_{H'}(y) = 1$ for every $y \in B_r$ and $d_{H'}(y) = 0$ for every $y \in B - B_r$.

Let F' be the graph with $V(F') = A \cup B$ and $E(F') = \{xy \in E(F) \mid x_{(i)}y \in M \text{ for } 1 \leqslant i \leqslant m_x\}$. Then $d_{F'}(x) \leqslant m_x = \lceil \frac{d_F(x)}{r} \rceil$ for every $x \in A$, $d_{F'}(y) = 1$ for every $y \in B_r$, and $d_{F'}(y) = 0$ for every $y \in B - B_r$. $\square$

Lemma 3.2. *Let $F = F(A, B)$ be a bipartite graph and, for $r \geqslant 1$, let B_r be the set of all vertices of B having degree at least r in F. Assume that $d_F(x) \geqslant 4$ for every $x \in A$. Then there is a subgraph F' of F such that*

(a) $d_{F'}(x) = 2$ *for every* $x \in A$,
(b) $d_{F'}(y) \leqslant d_F(y) - 2$ *for every* $y \in B_4$, *and*
(c) $d_{F'}(y) \leqslant d_F(y) - 1$ *for every* $y \in B_3$.

Proof. Because of Lemma 3.1, there is a subgraph H of F such that $d_H(x) \leqslant \lceil \frac{d_F(x)}{4} \rceil$ for every $x \in A$, $d_H(y) = 1$ for every $y \in B_4$, and $d_H(y) = 0$ for every $y \in B - B_4$. Let $\tilde{F} = F - E(H)$ and let $\tilde{B}_3$ be the set of all vertices of B having degree at least 3 in $\tilde{F}$. Obviously, $\tilde{B}_3 = B_3 \cup B_4$. Then Lemma 3.1 implies that there is a subgraph $\tilde{H}$ of $\tilde{F}$ such that $d_{\tilde{H}}(x) \leqslant \lceil \frac{d_{\tilde{F}}(x)}{3} \rceil$ for every $x \in A$, $d_{\tilde{H}}(y) = 1$ for every $y \in \tilde{B}_3$, and $d_{\tilde{H}}(y) = 0$ for every $y \in B - \tilde{B}_3$.

Let $G = \tilde{F} - E(\tilde{H}) = F - E(H) - E(\tilde{H})$. Then, for $y \in B_4$, we have $d_G(y) = d_F(y) - 2$, and, for $y \in B_3$, we have $d_G(y) = d_F(y) - 1$. Let $x \in A$. Since $d_F(x) \geqslant 4$, we have $d_{\tilde{F}}(x) = d_F(x) - d_H(x) \geqslant d_F(x) - \lceil \frac{d_F(x)}{4} \rceil \geqslant 3$ and, therefore, $d_G(x) = d_{\tilde{F}}(x) - d_{\tilde{H}}(x) \geqslant d_{\tilde{F}}(x) - \lceil \frac{d_{\tilde{F}}(x)}{3} \rceil \geqslant 2$.

Consequently, there is a subgraph F' of G satisfying (a)–(c). Thus Lemma 3.2 is proved. □

Lemma 3.3. *Let $r \geqslant 3$ be an integer. Let $F = F(A, B)$ be a bipartite graph and let $\mathscr{P}$ be a mapping that assigns to every vertex $x \in A$ a partition $\mathscr{P}(x)$ of $N(x : F)$. Assume that $d_F(x) \geqslant |\mathscr{P}(x)| + 2^{r-3}$ for every $x \in A$. Then there is a subgraph F' of F such that the following statements hold*:

(a) *If $x \in A$, then $d_{F'}(x) = 2$ and $N(x : F') \subseteq N$ for some $N \in \mathscr{P}(x)$.*
(b) *If $y \in B$ and $d_F(y) \geqslant s$ where $3 \leqslant s \leqslant r$, then $d_{F'}(y) \leqslant d_F(y) - s + 3$.*

Proof (By induction on r and $|E(F)|$). A subgraph F' of F satisfying the conditions (a) and (b) of Lemma 3.3 is called a *good subgraph* of F with respect to $\mathscr{P}$ and r. Let $F_1 = F_1(A, B)$ be a subgraph of F and define $\mathscr{P}_1$ by

$$\mathscr{P}_1(x) = \{N \cap N(x : F_1) \mid N \in \mathscr{P}(x) \;\&\; N \cap N(x : F_1) \neq \emptyset\}$$

for every $x \in A$. In this case we write $\mathscr{P}_1 = \mathscr{P}|F_1$. It is easy to check that if F' is a good subgraph of F_1 with respect to $\mathscr{P}_1 = \mathscr{P}|F_1$ and r, then F' is a good subgraph of F with respect to $\mathscr{P}$ and r.

We have to show that there is a good subgraph of F with respect to $\mathscr{P}$ and r provided that $d_F(x) \geqslant |\mathscr{P}(x)| + 2^{r-3}$ for every $x \in A$. For $r = 3$ this is evident. Now assume $r \geqslant 4$.

First, assume that, for some $x \in A$, there is a set $N \in \mathscr{P}(x)$ such that $N = \{y\}$. Let $F_1 = F - \{xy\}$ and $\mathscr{P}_1 = P|F_1$. Then $d_{F_1}(x) = d_F(x) - 1 \geqslant |\mathscr{P}(x)| + 2^{r-3} - 1 = |\mathscr{P}_1(x)| + 2^{r-3}$ and, by the induction hypothesis, there is a good subgraph F' of F_1 with respect to $\mathscr{P}_1$ and r. Then F' is a good subgraph of F with respect to $\mathscr{P}$ and r.

Now, assume that $|N| \geqslant 2$ for every $N \in \mathscr{P}(x)$ and every $x \in A$. If $d_F(x) > |\mathscr{P}(x)| + 2^{r-3}$ for some $x \in A$, then let $F_1 = F - \{xy\}$ and $\mathscr{P}_1 = \mathscr{P}|F_1$ where $y \in N_F(x)$. Since $d_{F_1}(x) \geqslant |\mathscr{P}(x)| + 2^{r-3} = |\mathscr{P}_1(x)| + 2^{r-3}$, it then follows from the induction hypothesis that there is a good subgraph F' of F_1 with respect to $\mathscr{P}_1$ and r. Then F' is a good subgraph of F with respect to $\mathscr{P}$ and r.

If $d_F(x) = |\mathscr{P}(x)| + 2^{r-3}$ for every $x \in A$, then we argue as follows. Since every set of $\mathscr{P}(x)$ has at least two elements, $|\mathscr{P}(x)| \leqslant 2^{r-3}$ and, therefore, $d_F(x) \leqslant 2^{r-2}$ for every $x \in A$. By Lemma 3.2, there is a subgraph H of F such that $d_H(x) \leqslant \lceil \frac{d_F(x)}{r} \rceil \leqslant \lceil \frac{2^{r-2}}{4} \rceil = 2^{r-4}$ for every $x \in A$ and $d_H(y) = 1$ for every $y \in B$ with $d_F(y) \geqslant r$. Let $\tilde{F} = F - E(H)$ and $\tilde{\mathscr{P}} = \mathscr{P}|\tilde{F}$. Then, for every $x \in A$, $d_{\tilde{F}}(x) = d_F(x) - d_H(x) \geqslant |\mathscr{P}(x)| + 2^{r-3} - 2^{r-4} = |\mathscr{P}(x)| + 2^{r-4} \geqslant |\tilde{\mathscr{P}}(x)| + 2^{r-4}$. Therefore, by the induction hypothesis, there is a good subgraph F' of $\tilde{F}$ with respect to $\tilde{\mathscr{P}}$ and $r-1$. Then F' is a good subgraph of F with respect to $\mathscr{P}$ and $r-1$. If $y \in B$ and $d_F(y) = r$, then $d_H(y) = 1$ and, therefore, $d_{\tilde{F}}(y) = r - 1$ implying that $d_{F'}(y) \leqslant d_{\tilde{F}}(y) - (r-1) + 3 = d_F(y) - r + 3$. Consequently, F' is a good subgraph of F with respect to $\mathscr{P}$ and r. Thus Lemma 3.3 is proved. □

Remark. Lemma 3.3 remains valid if the condition $d_F(x) \geqslant |\mathscr{P}(x)| + 2^{r-3}$ is replaced by $d_F(x) \geqslant |\mathscr{P}(x)| + m_r$ where $m_3, m_4, \ldots$ is a sequence of integers satisfying $m_3 = 1$ and $m_r - \lceil \frac{2m_r}{r} \rceil \geqslant m_{r-1}$ for $r \geqslant 4$. For $r = 5$, the case we are interested in, this gives $m_5 = 4$.

Lemma 3.4. *Let $F = F(A, B)$ be a bipartite graph, let R, d be integers with $R \geqslant d \geqslant 1$ and, for every $x \in A$, let $a(x) \geqslant 1$ be an integer. Assume that $d_F(y) \geqslant R$ for every $y \in B$. Then*

$$(R - d)|B| \leqslant \sum_{x \in A} a(x)$$

or there are non-empty subsets $A' \subseteq A$ and $B' \subseteq B$ such that for $F' = F[A' \cup B']$ we have $d_{F'}(x) > a(x)$ for every $x \in A'$ and $d_{F'}(y) > d$ for every $y \in B'$.

Proof. For $z \in V(F)$ and $Z \subseteq V(F)$, let $d(z : Z) = |N(z : G) \cap Z|$. Define a sequence $B_0 = \emptyset$, $A_1, B_1, A_2, B_2, \ldots$ of sets as follows. For $i \geqslant 1$, let

$$A_i = \{x \in A \mid d(x : B - B_{i-1}) \leqslant a(x)\}$$

and

$$B_i = \{y \in B \mid d(y : A_i) \geqslant R - d\}.$$

Then, for every $i \geqslant 1$, we have $A_i \subseteq A_{i+1} \subseteq A$ and $B_i \subseteq B_{i+1} \subseteq B$. Let $A' = A - \bigcup A_i$, $B' = B - \bigcup B_i$, and $F' = F[A' \cup B']$.

If A' contains a vertex x, then $d(x : B - B_{i-1}) > a(x)$ for every $i \geqslant 1$ implying that $d_{F'}(x) = d(x : B') > a(x)$ and, hence, $B' \neq \emptyset$. If B' contains a vertex y, then $d(y : A_i) < R - d$ for every $i \geqslant 1$ and, therefore, $d_F(y : \bigcup A_i) < R - d$. This implies that $d_{F'}(y) = d(y : A') = d(y : A) - d(y : \bigcup A_i) > R - (R - d) = d$ and, hence, $A' \neq \emptyset$. Consequently, $A' \neq \emptyset$ iff $B' \neq \emptyset$ and, moreover, Lemma 3.4 is true if A' or B' is non-empty. Otherwise, both sets A' and B' are empty and, therefore, $A = \bigcup A_i$ and $B = \bigcup B_i$. Let $E = \{xy \in E(F) \mid x \in A_i$ and $y \in B - B_{i-1}$ for some $i \geqslant 1\}$. Then

$$(R - d)|B| \leqslant |E| \leqslant \sum_{x \in A} a(x).$$

Thus Lemma 3.4 is proved. □

4. List critical hypergraphs

4.1. The key lemma

The proof of Theorem 1.9 is mainly based on the following technical lemma. Recall that if G is a hypergraph and $p \geqslant 2$ is an integer, then $W^p(G)$ denotes the set of all vertices of G that belong to some $(p-1)$-clique of G.

Lemma 4.1. *Let G be a hypergraph not containing a K_k, and let Φ be a list for G satisfying $|\Phi(x)| = k-1$ for every $x \in V(G)$. Furthermore, let $L = \{x \in V(G) \mid d_G(x) = k-1\}$, $X \subseteq L$, $Y \subseteq \{y \in V(G) \mid d_G(y) = k\}$ and let $W = W^k(G(X))$. Denote by $\mathscr{C}$ the set of all components of $G(X)$ and let v be an X-mapping of G. For $y \in Y$ and $T \in \mathscr{C}$, define*

$$d(y) = |\{T \in \mathscr{C} \mid y \in N_X^v(x : G) \text{ for some } x \in W \cap V(T)\}|,$$

$$d(T) = |\{y \in Y \mid y \in N_X^v(x : G) \text{ for some } x \in W \cap V(T)\}|$$

and

$$d_X^v(y) = |\{x \in W \mid y \in N_X^v(x : G)\}|.$$

If G is Φ-critical, then the following statements hold:

(a) *$d(y) \geqslant d_X^v(y) - 1$ for every $y \in Y$ provided that $k \geqslant 5$.*
(b) *$d(y) \leqslant 4$ for some $y \in Y$ or $d(T) \leqslant s(T) + 3$ for some $T \in \mathscr{C}$ provided that $\Phi(x) = \{1, \ldots, k-1\}$ for every $x \in V(G)$ and $k \geqslant 5$.*
(c) *$d(y) \leqslant 3$ for some $y \in Y$ or $d(T) \leqslant 3$ for some $T \in \mathscr{C}$ provided that every member of $\mathscr{C}$ is an ε_k-hypergraph and $k \geqslant 9$.*

The proof of this result is given in Section 4.2. In Section 4.3 we use Lemma 4.1 to prove Theorem 1.9.

4.2. Proof of Lemma 4.1

Let G be a hypergraph, $z \in V(G)$, and let Φ be a list for G. We call (G, z, Φ, k) a *configuration of type* 1 if the following conditions hold:

(a1) $G \neq K_k$ and every component of $G - \{z\}$ belongs to $\mathscr{T}_k$.
(a2) $d_G(z) \leqslant k$ and z is contained only in ordinary edges of G.
(a3) $|\Phi(z)| \geqslant d_G(z) - 1$ and $|\Phi(x)| \geqslant d_G(x)$ for all $x \in V(G) - \{z\}$.

The proof of Lemma 4.1(a) is based on the following result.

Lemma 4.2. *Let (G, z, Φ, k) be a configuration of type 1 where $k \geqslant 5$, let m be the number of components of $G - \{z\}$ and let $W = W^k(G - \{z\})$. Furthermore, let $N_z = \{x \in V(G) \mid \{z, x\} \in E(G)\}$ and $W_z = N_z \cap W$. Assume that $V(T) \cap W_z \neq \emptyset$ for every component T of $G - \{z\}$. If G is not Φ-colourable, then $m \geqslant |W_z| - 1$.*

Proof. Consider a possible counterexample (G, z, Φ, k) such that $|V(G)|$ is minimum. Let $T_1, \ldots, T_m$ denote the components of $G - \{z\}$. Then, by (a1), $T_i \in \mathscr{T}_k$ for $i = 1, \ldots, m$. Furthermore, for $i = 1, \ldots, m$, let $d_i = |V(T_i) \cap W_z|$ where $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_m$. Then $d_m \geqslant 1$ and $m \leqslant |W_z| - 2 = d_1 + \cdots + d_m - 2$. We claim that $m = 1$ and $d_1 \geqslant 3$ or $m = 2$ and $d_1 = d_2 = 2$.

Obviously, if $m = 1$, then $d_1 \geqslant 3$. Now, assume $m \geqslant 2$. Let $T = T_m$. Then $|\Phi(x)| \geqslant d_G(x) \geqslant d_T(x)$ for all $x \in V(T)$. Since there is a vertex $x \in V(T) \cap W_z$, we have $|\Phi(x)| > d_T(x)$ for this vertex x. Therefore, by Lemma 1.4, there is a Φ-colouring φ of T. Let $G' = G - V(T) = G \backslash V(T)$ and $\Phi' = \Phi(V(T), v, \varphi)$ (see Remark 1.5). Then $\Phi'(x) = \Phi(x)$ for all $x \in V(G') - \{z\}$ and $\Phi'(z) = \Phi(z) - \{\varphi(x) \mid x \in V(T) \cap N_z\}$. Since G is not Φ-colourable, G' is not Φ'-colourable. Moreover, it is easy to check that (G', z, Φ', k) is a configuration of type 1 satisfying the assumption of Lemma 4.2. Therefore, $m - 1 \geqslant |W_z \cap V(G')| - 1 = d_1 + \cdots + d_{m-1} - 1$ implying that $m = 2$ and $d_1 = d_2 = 2$. This proves our claim. Now, we consider two cases.

Case 1: $m = 2$ and $d_1 = d_2 = 2$. Let $i \in \{1, 2\}$ and let $G_i = G[V(T_i) \cup \{z\}]$. For $x \in V(G_i) - \{z\}$, we have $|\Phi(x)| \geqslant d_G(x) = d_{G_i}(x)$. Since z has exactly two neighbours in the Gallai tree $T_i = G_i - \{z\} \in \mathscr{T}_k$ that belong to $(k-1)$-cliques of T_i and every $(k-1)$-clique of T_i is a block of T_i, we conclude that G_i is not a Gallai tree and $|\Phi(z)| \geqslant d_G(z) - 1 > d_{G_i}(z)$.

Let M_i be the set of all colours $a \in \Phi(z)$ such that $\varphi(z) \neq a$ for every Φ-colouring φ of G_i. Since G is not Φ-colourable, $M_1 \cup M_2 = \Phi(z)$. From $|\Phi(z)| \geqslant d_G(z) - 1 = d_{G_1}(z) + d_{G_2}(z) - 1$ we conclude that $|M_i| \geqslant d_{G_i}(z)$ for some i, say $i = 1$. Now, let Φ' be the list for G_1 with $\Phi'(x) = \Phi(x)$ for $x \in V(G_1) - \{z\}$ and $\Phi'(z) = M_1$. Since G_1 is a connected hypergraph but not a Gallai tree, we infer from Lemma 1.4 that G_1 is Φ'-colourable. This implies that there is a Φ-colouring φ of G_1 with $\varphi(z) \in M_1$, a contradiction.

Case 2: $m = 1$ and $d_1 \geqslant 3$. Then $T = G - \{z\} \in \mathscr{T}_k$. Since G is not Φ-colourable, we may assume that $|\Phi(x)| = d_G(x)$ for all $x \in V(G) - \{z\}$. Let B be an arbitrary end-block of T and let X be the set of all non-separating vertices of T that belong to B. Consider a vertex $u \in X$. Since $|\Phi(u)| = d_G(u) \geqslant 1$, there is a colour $a \in \Phi(u)$. Let $G' = G \backslash \{u\}$ and $\Phi' = \Phi(u, a)$. Then G' is not Φ'-colourable and (G', z, Φ', k) is a configuration of type 1. If no vertex of B belongs to W_z, then $W_z \cap V(G') = W_z \cap V(G)$ and, therefore, (G', z, Φ', k) is a smaller counterexample, a contradiction. Hence $|V(B) \cap W_z| \geqslant 1$. Since $d_G(x) \leqslant k - 1$ for all vertices x of the Gallai tree $T \in \mathscr{T}_k$, this implies that B is a K_{k-1}.

Let $y \in V(B) \cap W_z$. Since $d_G(y) \leqslant k - 1$, we have $|\Phi(y)| = d_G(y) = k - 1$ and $y \in X$. We claim that $X \subseteq W_z$. Suppose, on the contrary, that there is a vertex $x \in X - W_z$. Then $|\Phi(x)| = d_G(x) = k - 2$ and, therefore, there is a colour $a \in \Phi(y) - \Phi(x)$. Since $|\Phi(z)| \geqslant d_G(z) - 1 \geqslant d_1 - 1 \geqslant 2$, there is a colour $b \in \Phi(z)$ with $b \neq a$. Let $\Phi' = \Phi(z, b)$. Then $T = G - \{z\} = G \backslash \{z\}$ is not Φ'-colourable and $|\Phi'(u)| \geqslant d_T(u)$ for all $u \in V(T)$. Therefore, (T, Φ') is a bad pair and $\Phi'(x) \neq \Phi'(y)$, a contradiction to Lemma 1.7. This proves our claim, i.e., $X \subseteq W_z$.

If B is the only block of T, then $X = V(B) = V(T)$ and, therefore $G = K_k$, a contradiction to (a1). Hence, there is an end-block $B' \neq B$ of T. For the set X' of all vertices of B' that are non-separating vertices of T, we have $X' \subseteq W_z$. Since $k \geqslant 5$, this yields $d_G(z) \geqslant |X| + |X'| \geqslant 2(k-2) \geqslant k+1$, a contradiction to (a2).

Thus in both cases 1 and 2 we arrive at a contradiction. This proves Lemma 4.2. $\square$

4.2.1. Proof of Lemma 4.1(a)

Consider a vertex $y \in Y$. By Lemma 1.6, $G(X)$ is a Gallai forest and so every component of $G(X)$ belongs to $\mathscr{T}_k$. Denote by $\mathscr{C}'$ the set of all components T of $G(X)$ such that $y \in N_X^v(x:G)$ for some vertex $x \in W^k(T) = W \cap V(T)$. Clearly, $d(y) = |\mathscr{C}'|$.

Let $X' = \bigcup V(T)$ where the union is taken over all $T \in \mathscr{C}'$. Then $G(X')$ is a Gallai forest and, for $W' = W^k(G(X'))$, we have $W' = X' \cap W$. Since there is no edge in G having a vertex in common with both X' and $X - X'$, the set $E_X(G)$ is the disjoint union of $E_{X'}(G)$ and $E_{X-X'}(G)$. Therefore, v is an X'-mapping of G and $N_{X'}^v(x:G) = N_X^v(x:G)$ for all $x \in X'$.

Let $N_y = \{x \in X' \mid y \in N_X^v(x:G)\}$ and $W_y = N_y \cap W$. Then $W_y = N_y \cap W'$ and $d_X^v(y) = |W_y|$. Furthermore, let

$$E' = \{e \in E_{X'}(G) \mid y = v(e)\}$$

and

$$E^* = \{e \in E(G) \mid e \cap X' = \emptyset \;\&\; y \in e\}.$$

Then $|N_y| = |E'|$ since otherwise there are two distinct edges $e, e' \in E_x(G)$, for some vertex $x \in X' \subseteq L$, satisfying $|e \cap e'| \geqslant 2$, a contradiction to Lemma 1.6. For all edges $e \in E_{X'}(G) \cup E^*$, choose a vertex $v'(e) \in e$ such that $v'(e) = v(e)$ for all $e \in E'$ and $v'(e) \neq y$ for all $e \in E^*$.

Let G_1 be the hypergraph obtained from the Gallai forest $G' = G(X')$ by adding the vertex y and joining y to every vertex in N_y by an ordinary edge. Since G is Φ-critical, there is a Φ-colouring φ of the subhypergraph $G_2 = G - (X' \cup \{y\})$ of G. Now, define the list Φ_1 of G_1 as follows. For $x \in X'$, let

$$\Phi_1(x) = \Phi(x) - \{\varphi(v'(e)) \mid x \in e \in E_{X'}(G) - E'\}$$

and let

$$\Phi_1(y) = \Phi(y) - \{\varphi(v'(e)) \mid e \in E^*\}.$$

If $e \in E(G)$, then $e \in E(G_2)$, or $e \cap X' \in E(G_1)$, or $e \in E_{X'}(G) \cup E^*$. This implies that $\varphi \cup \varphi_1$ is a Φ-colouring of G for every Φ_1-colouring φ_1 of G_1. Therefore, since G is not Φ-colourable, G_1 is not Φ_1-colourable. Furthermore, for $x \in X' \subseteq L$,

$$|\Phi_1(x)| \geqslant d_G(x) - |\{e \in E(G) \mid x \in e \in E_{X'}(G) - E'\}| \geqslant d_{G_1}(x)$$

and, since $|\Phi(y)| = k - 1 = d_G(y) - 1$,

$$|\Phi_1(y)| \geqslant d_G(y) - 1 - |E^*| \geqslant d_{G_1}(y) - 1.$$

If $G_1 \neq K_k$, then, clearly, (G_1, y, Φ_1, k) is a configuration of type 1 and, by Lemma 4.2, $d(y) = |\mathscr{C}'| \geqslant |W_y| - 1 = d_X^v(y)$.

Now, consider the case $G_1 = K_k$. Then $G_1(X') = G(X')$ is a K_{k-1} and, by Lemma 1.6, X' is a $(k-1)$-clique of G. Since G does not contain a K_k, this implies that there is a vertex $x \in X'$ such that $\{x, y\} \in E(G_1) - E(G)$. Consequently, there is an edge $e \in E_{X'}(G)$ such that $x, y \in e$, $y = v(e)$ and $|e| \geqslant 3$. Let $y' \in e - \{x, y\}$. Remove the edge $\{x, y\}$ from G_1 and, for the resulting graph G_1', define the list Φ_1' by $\Phi_1'(u) = \Phi_1(u)$ for all $u \in V(G_1') - \{x\}$ and $\Phi_1'(x) = \Phi_1(x) - \{\varphi(y')\}$. Then $|\Phi_1'(u)| \geqslant d_{G_1'}(u)$ for all $u \in V(G_1') = V(G_1)$. Since G_1' is connected but not a Gallai tree, we infer from Lemma 1.4 that there is a Φ_1'-colouring φ' of G_1' and, therefore, $\varphi \cup \varphi'$ is a Φ-colouring of G, a contradiction. This completes the proof of Lemma 4.1(a). □

Lemma 4.3. *Assume that* $k \geqslant 4$ *and* (T, Φ) *is a bad pair satisfying* $T \in \mathscr{T}_k$ *and* $\Phi(u) \subseteq \{1, \ldots, k-1\}$ *for every* $u \in V(G)$. *If* x *and* y *are two non-separating vertices of* T *contained in the same* ε_k*-subcomponent of* T, *then* $\Phi(x) = \Phi(y)$.

Proof. By Lemma 1.7, $\Phi = \Phi_u$ for some mapping $u \in \mathscr{U}(G)$. If x, y are contained in the same block, then the statement is evident. Otherwise, there is a sequence $B_1, B_2, \ldots, B_{2l+1}$ of blocks of T such that $x \in V(B_1)$, $y \in V(B_{2l+1})$, B_{2i+1} is a K_{k-1} for $i = 0, \ldots, l$ and B_{2i} is a block of type 2 for $i = 1, \ldots, l$ and $V(B_i) \cap V(B_{i+1}) \neq \emptyset$ for $i = 1, \ldots, 2l$. Then $u(B_i) \cap u(B_{i+1}) = \emptyset$ for $i = 1, \ldots, 2l$. Since $u(B_i) \subseteq \{1, \ldots, k-1\}$, $|u(B_{2i+1})| = k-2$ and $|u(B_{2i})| = 1$, we infer that $u(B_1) = u(B_{2l+1})$ and, therefore, $\Phi(x) = \Phi(y)$. □

4.2.2. *Proof of Lemma 4.1(b)*

Suppose on the contrary that $d(y) \geqslant 5$ for every $y \in Y$ and $d(T) \geqslant s(T) + 4$ for every $T \in \mathscr{C}$. By Lemma 1.6, $G(X)$ is a Gallai forest not containing a K_k and with maximum degree at most $k-1$. Consequently, $\mathscr{C} \subseteq \mathscr{T}_k$.

Let $F = F(A, B)$ be the bipartite graph with $A = \mathscr{C}$ and $B = Y$ where, for every $T \in \mathscr{C}$, the neighbourhood $N(T : F)$ consists of all vertices $y \in Y$ such that $y \in N_X^v(x : G)$ for some $x \in W^k(T) = W \cap V(T)$. Then $d_F(y) = d(y) \geqslant 5$ for every $y \in B = Y$ and $d_F(T) = d(T) \geqslant s(T) + 4$ for every $T \in A = \mathscr{C}$.

For $T \in A$, let $\mathscr{P}(T)$ be a partition of $N(F : T)$ such that for every $N \in \mathscr{P}(T)$ there is an ε_k-subcomponent T' of T with

$$N \subseteq \{y \in Y \mid y \in N_X^v(x : G) \text{ for some } x \in W^k(T') = V(T')\}.$$

Then $d_F(T) \geqslant s(T) + 4 \geqslant |\mathscr{P}(T)| + 4$ for every $T \in A$. Therefore, since $d_F(y) \geqslant 5$ for all $y \in B$, we infer from Lemma 3.3 that there is a subgraph F' of F such that, for every $T \in A$, $d_{F'}(T) = 2$ and $N(T : F') \subseteq N$ for some $N \in \mathscr{P}(T)$ and, for every $y \in B$, $d_{F'}(y) \leqslant d_F(y) - 2$.

Now let G' be the hypergraph obtained from the subhypergraph $G - X$ of G by adding the ordinary edges $N(T : F')$ for all $T \in \mathscr{C}$. If $y \in Y$, then $d_G(y) = k$ and, by the construction of F', $d_{G'}(y) \leqslant k - 2$. Since G is Φ-critical, there is a Φ-colouring φ of $G - X - Y = G' - Y$. For every $y \in Y$, we have $|\Phi(y)| = k - 1 \geqslant d_{G'}(y) + 1$. This implies that φ can be extended to some Φ-colouring φ' of G'.

Let $G^* = G \backslash V(G') = G(X)$ and let $\Phi^* = \Phi(V(G'), v, \varphi')$ (see Remark 1.5). Then G^* is not Φ^*-colourable and, since $|\Phi(x)| = d_G(x)$, we have $|\Phi^*(x)| \geqslant d_{G^*}(x)$ for every $x \in X$. Consequently, there is a component T of $G(X)$, such that (T, Φ_1) is a bad pair where Φ_1 is the restriction of Φ^* to T. Consider the two vertices y_1, y_2 of $N(T : F')$. Then there is a ε_k-subcomponent T' of T and two vertices x_1, x_2 in $V(T')$ such that $y_i \in N_X^v(x_i : G)$ for $i = 1, 2$. Since every vertex of $T \in \mathscr{T}_k$ has degree $k-1$ in G and T' is an ε_k-subcomponent of T, it follows that $d_T(x_1) = d_T(x_2) = k-2$. Consequently, x_1, x_2 are two distinct non-separating vertices of T. Moreover, $\Phi_1(x_i) = \Phi(x_i) - \{\varphi'(y_1)\} = \{1, \dots, k-1\} - \{\varphi'(y_1)\}$ for $i = 1, 2$. Since $\varphi'(y_1) \neq \varphi'(y_2)$, this implies $\Phi_1(x_1) \neq \Phi_1(x_2)$, a contradiction to Lemma 4.3. This completes the proof of Lemma 4.1(b). □

Let G be a hypergraph, let F be a subhypergraph of G, $Y \subseteq V(G)$, and let Φ be a list for G. Then we call (G, F, Y, Φ) a *configuration of type* 2 if the following conditions hold:

(b1) $G - Y$ is a Gallai forest and $|\Phi(x)| \geqslant d_G(x)$ for every $x \in V(G) - Y$.
(b2) $|\Phi(y)| \geqslant d_{G[Y]}(y) + d_F(y) + 1$ for every $y \in Y$.
(b3) Every edge of G intersecting both Y and $V(G) - Y$ is an ordinary edge. For $x \in V(G) - Y$, let $N_x = \{y \in Y \mid \{x, y\} \in E(G)\}$.
(b4) F is a graph and, for every component T of $G - Y$, there are two edges $\{x_1, y_1\}$, $\{x_2, y_2\} \in E(F)$ such that x_1, x_2 are two distinct non-separating vertices of T, y_1, y_2 are two distinct vertices of Y and, for $i = 1, 2$, $N_{x_i} = \{y_i\}$. Furthermore, if B_i $(i = 1, 2)$ is the only block of T containing x_i, then $B_1 = B_2$ or, for some $i \in \{1, 2\}$, there is a non-separating vertex x of T such that $x \in V(B_i)$ and $N_x = \emptyset$.

Lemma 4.4. *If (G, F, Y, Φ) is a configuration of type 2, then G is Φ-colourable*

Proof (By induction on $m = |V(G) - Y|$). If $m = 0$, then $G = G[Y]$ and $|\Phi(y)| \geqslant d_G(y) + 1$ for every $y \in V(G)$ implying that G is Φ-colourable.

Now assume $m \geqslant 1$. Then let T be a component of $G - Y$ and let $\{x_1, y_1\}$, $\{x_2, y_2\}$ be the two edges of F given by condition (b4). For $i = 1, 2$, let B_i be the only block of T containing x_i. Let $G' = G - V(T) = G \backslash V(T)$ and $F' = F - V(T)$. We consider two cases.

Case 1: $B_1 = B_2$. First, assume $\Phi(x_1) = \Phi(x_2)$. Let G^* be the hypergraph obtained from G' by adding the edge $\{y_1, y_2\}$. Then (G^*, F', Y, Φ) is a configuration of type 2 and, by the induction hypothesis, there is a Φ-colouring φ of G'. Consider the list $\Phi' = \Phi(V(G'), \varphi)$ for $T = G - V(G')$, that is

$$\Phi'(x) = \Phi(x) - \{\varphi(y) \mid y \in N_x\}$$

for all $x \in V(T)$. Note that, by (b3), every edge of G containing $x \in V(T)$ belongs to T or is an ordinary edge. By (b2), $|\Phi'(x)| \geqslant d_T(x)$ for all $x \in V(T)$. Consequently, if T is not Φ'-colourable, then (T, Φ') is a bad pair and, since $N_{x_i} = \{y_i\}$ and $\varphi(y_1) \neq \varphi(y_2)$, we have $\Phi'(x_1) \neq \Phi'(x_2)$, a contradiction to Lemma 1.7. Therefore, T is Φ'-colourable implying that G is Φ-colourable.

Now, assume $\Phi(x_1) \neq \Phi(x_2)$, say $a \in \Phi(x_1) - \Phi(x_2)$. Let Φ' be the list obtained from Φ by removing colour a from $\Phi(y_1)$. Then (G', F', Y, Φ') is a configuration of type 2 and, by the induction hypothesis, there is a Φ'-colouring φ of G'. Consider the list $\Phi_1 = \Phi(V(G'), \varphi)$ for $T = G - V(G')$. Then $|\Phi'(x)| \geqslant d_T(x)$ for all $x \in V(T)$ and $\Phi_1(x_1) \neq \Phi_1(x_2)$. Consequently, by Lemma 1.7, T is Φ_1-colourable and, therefore, G is Φ-colourable.

Case 2: $B_1 \neq B_2$. Then, by (b4), one of these two blocks, say B_1, contains a non-separating vertex x of T such that $N_x = \emptyset$. We may assume that $|\Phi(x)| = d_G(x)$. Then $|\Phi(x_1)| \geqslant d_G(x_1) > d_G(x)$ and, therefore, there is a colour $a \in \Phi(x_1) - \Phi(x)$. Let Φ' be the list obtained from Φ by removing colour a from $\Phi(y_1)$. Then (G', F', Y, Φ') is a configuration of type 2 and, by the induction hypothesis, there is a Φ'-colouring φ of G'. Consider the list $\Phi_1 = \Phi(V(G'), \varphi)$ for $T = G - V(G')$. Then $\Phi_1(x_1) \neq \Phi_1(x)$ and we infer from Lemma 1.7 that T is Φ_1-colourable. Hence G is Φ-colourable.

Therefore, in both cases we have established that G is Φ-colourable. Thus Lemma 4.4 is proved. □

4.2.3. Proof of Lemma 4.1(c)

Suppose on the contrary that $d(y) \geqslant 4$ for every $y \in Y$ and $d(T) \geqslant 4$ for every $T \in \mathscr{C}$. To arrive at a contradiction, we show that G is Φ-colourable.

Since G is Φ-critical, we infer from Lemma 1.6 that $\mathscr{C} \subseteq \mathscr{T}_k$. Furthermore, by the assumption of Lemma 4.1(c), every component T of $G(X)$ is an ε_k-hypergraph and, therefore, $V(T) \subseteq W = W^k(G(X)) = X$.

Let X_n denote the set of all non-separating vertices of $G(X)$. Then $d_{G(X)}(x) = d_G(x) = k-1$ for all $x \in X - X_n$, and $d_{G(X)}(x) = d_G(x) - 1 = k-2$ for all $x \in X_n$. Consequently, for every $x \in X_n$, there is exactly one edge $e_x \in E(G) - E(G(X))$ containing x. Clearly, if $x \in X_n$, then $e_x \cap X = \{x\}$ and, moreover, $y \in N_X^v(x : G)$ iff $y = v(e_x)$.

Let E' be the set of all edges $e \in E(G)$ satisfying $e \cap X = \emptyset$ and $e \cap Y \neq \emptyset$. For every edge $e \in E'$, choose a vertex $v'(e) \in e - Y$ provided that $e - Y \neq \emptyset$.

Next, we construct the hypergraph G_1 as follows. Let $V(G_1) = X \cup Y$ and let $E(G_1) = E(G(X)) \cup E^1 \cup E^2$ where

$$E^1 = \{e \cap Y \mid e \in E' \;\&\; |e \cap Y| \geqslant 2\}$$

and

$$E^2 = \{\{x, y\} \mid x \in X_n \;\&\; y = v(e_x) \in Y\}.$$

For $x \in X$, let

$$N_x = N_X^v(x : G) = \{y \in Y \mid \{x, y\} \in E(G_1)\}.$$

Then $|N_x| = 1$ if $x \in X_n$ and $|N_x| = 0$ if $x \in X - X_n$. Since G is Φ-critical, there is a Φ-colouring φ of $G' = G - X - Y$. Now, we define a list Φ_1 for the hypergraph G_1 as follows. For a vertex $y \in Y$, let

$$\Phi_1(y) = \Phi(y) - \{\varphi(v'(e)) \mid e \in E' \;\&\; e \cap Y = \{y\}\}.$$

For a vertex $x \in X$, let

$$\Phi_1(x) = \Phi(x) - \{\varphi(v(e_x))\}$$

if $x \in X_n$ and $v(e_x) \notin Y$, and let $\Phi_1(x) = \Phi(x)$ otherwise, that is if $x \in X - X_n$ or $x \in X_n$ and $v(e_x) \in Y$.

Our aim is to show that G_1 is Φ_1-colourable. If this is true, then there is a Φ_1-colouring φ_1 of G_1 and $\varphi \cup \varphi_1$ is a Φ-colouring of G, a contradiction. Note that if e is an edge of G, then e is an edge of $G' = G - X - Y$, or $e \in E'$, or $e \cap X \in E(G(X)) \subseteq E(G_1)$, or $e = e_x$ for some vertex $x \in X_n$.

To prove that G_1 is Φ_1-colourable, we use Lemma 4.4. First, we need some notation. For $Z \subseteq X$, let $N(Z) = \bigcup_{x \in Z} N_x$, and, for a set of blocks $\mathscr{B}$ of $G(X)$, let $X(\mathscr{B})$ be the set of all vertices contained in some block of $\mathscr{B}$.

Consider an arbitrary component $T \in \mathscr{C}$. Since T is an ε_k-hypergraph, $V(T) \subseteq W$ and, therefore, $|N(V(T))| = d(T) \geqslant 4$. Let S denote the set of all vertices x of T such that $N_x \neq \emptyset$ and let R denote the set of all non-separating vertices of T. Then $S \subseteq R$. From Lemma 4.1(a) it follows that, for every vertex $y \in Y$, there are at most two vertices $x, x' \in V(T)$ such that $N_x = N_{x'} = \{y\}$. This implies, in particular, that $|N(Z)| \geqslant 4$ provided that $|Z \cap S| \geqslant 7$.

Let $\mathscr{B}_1$ denote the set of all blocks B of T such that $V(B) \cap (R - S) \neq \emptyset$, i.e., B contains a non-separating vertex x of T such that $N_x = \emptyset$.

We claim that there is a set $\mathscr{B} = \mathscr{B}_T$ of blocks of T such that all but at most one block of $\mathscr{B}$ belong to $\mathscr{B}_1$ and $|N(X(\mathscr{B}))| \geqslant 4$. If some end-block B of T is not in $\mathscr{B}_1$, then $V(B) \cap R \subseteq S$ and, since B is a K_{k-1} and $k \geqslant 9$, $|V(B) \cap R| = k - 2 \geqslant 7$. This implies that the claim is true for $\mathscr{B} = \{B\}$.

Now, assume that every end-block of T belong to $\mathscr{B}_1$ and $|N(\mathscr{B}_1)| \leqslant 3$. Since $|N(V(T))| \geqslant 4$, there is a block B of T not contained in $\mathscr{B}_1$. Let $\mathscr{B} = \mathscr{B}_1 \cup \{B\}$. Since $\emptyset \neq V(B) \cap R \subseteq S$ and T has at least $|V(B) - R|$ end-blocks, we conclude that B is a K_{k-1} and $|X(\mathscr{B}) \cap S| \geqslant |V(B)| = k - 1 \geqslant 8$ and, therefore, $|N(X(\mathscr{B}))| \geqslant 4$. This proves our claim.

Next, let $F = F(\mathscr{C}, Y)$ be the bipartite graph such that $N(T : F) = N(X(\mathscr{B}_T))$ for every $T \in \mathscr{C}$. Then $d_F(T) \geqslant 4$ for every $T \in \mathscr{C}$ and, by Lemma 3.2, there is a subgraph F' of F such that $d_{F'}(T) = 2$ for every $T \in \mathscr{C}$ and $d_{F'}(y) \leqslant d_F(y) - 2$ for every $y \in Y$ with $d_F(y) \geqslant 4$ and $d_{F'}(y) \leqslant 2$ for every $y \in Y$ with $d_F(y) \geqslant 3$.

For every component $T \in \mathscr{C}$, the set $N(T : F')$ consists of two distinct vertices $y_1(T), y_2(T)$ and, moreover, there are two distinct vertices $x_1(T), x_2(T) \in X(\mathscr{B}_T)$ such that $N_{x_i(T)} = \{y_i(T)\}$ for $i = 1, 2$. Let F_1 be the subgraph of G_1 with the same vertex set as G_1 and with $E(F_1) = \{\{x_i(T), y_i(T)\} \mid T \in \mathscr{C} \,\&\, i = 1, 2\}$. Then it is easy to check that (G_1, F_1, Y, Φ_1) is a configuration of type 2. Therefore, by Lemma 4.4, G_1 is Φ_1-colourable. Hence, Lemma 4.1(c) is proved. □

4.3. Proof of Theorem 1.9

In this subsection, let G be a hypergraph not containing a K_k, and let Φ be a list for G satisfying $|\Phi(x)| = k - 1$ for every $x \in V(G)$. Suppose that G is Φ-critical.

Let $L = \{x \in V(G) \mid d_G(x) = k-1\}$, $H = \{x \in V(G) \mid d_G(x) \geqslant k\}$, $W = W^k(G(L))$ and $L' = L - W$. Furthermore, let

$$E_1 = \{e \in E(G) \mid |e \cap L| = 1\} \quad \text{and} \quad E_2 = \{e \in E(G) \mid |e \cap L| \geqslant 2\}.$$

Let $\mathscr{C}$ be the set of all components of $G(L)$ and let $\mathscr{D}$ be the set of all components of $G(W)$. By Lemma 1.6, $H \neq \emptyset$ and $\mathscr{C}, \mathscr{D} \subseteq \mathscr{T}_k$. Obviously, $W = W^k(G(W))$ and, therefore, every member of $\mathscr{D}$ is an ε_k-hypergraph. This implies, in particular, that every member of $\mathscr{D}$ is an ε_k-subcomponent of some member in $\mathscr{C}$.

Denote by v an arbitrary L-mapping of G and let v' be a W-mapping of G such that $v'(e) = v(e)$ for all $e \in E_W \cap E_L$. Then $N_L^v(x : G) \subseteq N_W^{v'}(x : G)$ for every $x \in W$ and, therefore, $d_W^{v'}(y) \geqslant d_L^v(y)$ for every $y \in H$.

Let ϱ, σ and τ_c be defined as in Lemma 1.8 and, for $y \in H$, let

$$\tau_c(y) = d_{G[H]}(y) + \left(k - c - \frac{2}{k-1}\right)(d_G(y) - k).$$

Then, we have

$$\sigma = \sigma(G(L)) = \sum_{T \in \mathscr{C}} \sigma(T) \quad \text{and} \quad \tau_c = \sum_{y \in H} \tau_c(y).$$

Since $\mathscr{C} \subseteq \mathscr{T}_k$, it follows from Lemma 2.3 that

$$\sigma \geqslant \sum_{T \in \mathscr{C}} (\sigma^*(T) + s(T)\alpha_k + 2 - \alpha_k)$$

provided that $k \geqslant 6$. For $x \in L$, let $d_1(x) = |\{e \in E_1 \mid x \in e\}|$. If the vertex $x \in L$ belongs to a component $T \in \mathscr{C}$, then, by Lemma 1.6,

$$d_1(x) = k - 1 - d_T(x) = d_G(x) - d_{G(L)}(x) = |N_L^v(x : G)|. \tag{3}$$

Consequently,

$$\sigma \geqslant \sum_{x \in L'} \alpha_k d_1(x) + \sum_{T \in \mathscr{C}} (s(T)\alpha_k + 2 - \alpha_k) \tag{4}$$

provided that $k \geqslant 6$. For an edge $e \in E_1 = E_L(G)$, let x_e denote the vertex satisfying $e \cap L = \{x_e\}$. For a vertex $y \in H$, define

$$d^1(y) = |\{e \in E_1 \mid y \in e \text{ and } (x_e \in L - W \text{ or } y \neq v(e))\}|$$

and

$$d^2(y) = |\{e \in E_2 \mid y \in e\}|.$$

It follows from (3) that

$$\sum_{y \in H} (d^1(y) + d^2(y)) + \sum_{x \in W} d_1(x) = \sum_{e \in E_1 \cup E_2} |e \cap H|.$$

Since $|E_1| = \sum_{x \in L} d_1(x)$, this implies that

$$\sum_{y \in H} (d^1(y) + d^2(y)) = \sum_{e \in E_1 \cup E_2} |e \cap H| + \sum_{x \in L'} d_1(x) - |E_1|. \tag{5}$$

Furthermore, for $y \in H$, we have $d_L^v(y) = |\{x \in W \mid y \in N_L^v(x : G)\}|$ and, therefore,

$$d_G(y) = d^1(y) + d^2(y) + d_L^v(y) + d_{G[H]}(y). \tag{6}$$

Let $H' = \{y \in H \mid d_G(y) = k\}$ and let

$$S = \varrho + \sigma + \sum_{y \in H'} \tau_c(y) = \sum_{e \in E_1 \cup E_2} |e \cap H| - |E_1| + \sigma + \sum_{y \in H'} d_{G[H]}(y). \tag{7}$$

Next, define the bipartite graph $F = F(A, B)$ as follows:

(a) $B = H'$ and A is the disjoint union of the sets A_1, A_2 and A_3.
(b) $A_1 = \mathscr{C}$ and a component $T \in A_1$ is joined to a vertex $y \in B$ in F if and only if $y \in N_L^v(x : G)$ for some $x \in W^k(T) = W \cap V(T)$.
(c) For each vertex $y \in B$, let $A_2(y)$ be a set of $d^1(y) + d^2(y)$ vertices which are all joined to y in F. Let A_2 be the disjoint union of all these sets $A_2(y)$, $y \in B$.
(d) For each vertex $y \in B$, let $A_3(y)$ be a set of $d_{G[H]}(y)$ vertices which are all joined to y in F. Let A_3 be the disjoint union of all these sets $A_3(y)$, $y \in B$.

Now, we prove the two parts (Cases 1 and 2) of Theorem 1.9.

Case 1: $k \geqslant 6$, $\Phi(x) = \{1, \dots, k-1\}$ for every $x \in V(G)$ and $c = (k-5)\alpha_k$. We have to show that $\varrho + \sigma + \tau_c \geqslant c|H|$. Since for $y \in H - H'$ we have $\tau_c(y) \geqslant k - c + \frac{2}{k-1} \geqslant c$, it is sufficient to show that $S \geqslant c|H'|$. The proof of this statement is based on Lemma 4.1 where $X = L$, $Y = H'$ and v is the given L-mapping of G.

Consider the bipartite graph $F = F(A, B)$. If $T \in A_1$, then $d_F(T) = d(T)$. If $y \in B = H'$, then Lemma 4.1(a) implies that $|N(y : F) \cap A_1| = d(y) \geqslant d_L^v(y) - 1$ and, by (6), we conclude that

$$d_F(y) \geqslant d_L^v(y) - 1 + d^1(y) + d^2(y) + d_{G[H]}(y) = d_G(y) - 1 = k - 1.$$

Furthermore, we infer from (4), (5) and (7) that

$$\begin{aligned} S &\geqslant \sum_{e \in E_1 \cup E_2} |e \cap H| - |E_1| + \sum_{x \in L'} \alpha_k d_1(x) + \sum_{T \in \mathscr{C}} (s(T)\alpha_k + 2 - \alpha_k) \\ &\quad + \sum_{y \in H'} d_{G[H]}(y) \\ &\geqslant \alpha_k |A_2| + \sum_{T \in \mathscr{C}} (s(T) + 3)\alpha_k + |A_3| \\ &\geqslant \alpha_k \left(|A_2| + \sum_{T \in \mathscr{C}} (s(T) + 3) + |A_3| \right). \end{aligned}$$

Now, we apply Lemma 3.4 to $F = F(A, B)$ where $R = k - 1$, $d = 4$ and $a(x) = 1$ if $x \in A_2 \cup A_3$ and $a(x) = s(T) + 3$ if $x = T \in A_1$. If $(R - d)|B| \leqslant \sum_{x \in A} a(x)$, then the above inequality for S implies

$$S \geqslant \alpha_k \sum_{x \in A} a(x) \geqslant \alpha_k (k-5)|B| = c|H'|.$$

Otherwise, by Lemma 3.4, there are non-empty subsets $A' \subseteq A$ and $B' \subseteq B = H'$ such that for $F' = F[A' \cup B']$ we have $d_{F'}(x) > a(x)$ for every $x \in A'$ and $d_{F'}(y) > d = 4$ for every $y \in B'$. Since every vertex of $A_2 \cup A_3$ has degree 1 in F, we have $A' \subseteq A_1 = \mathscr{C}$. This gives a contradiction to Lemma 4.1(b).

Case 2: $k \geqslant 9$, $|\Phi(x)| = k-1$ for every $x \in V(G)$ and $c = \frac{1}{3}(k-4)\alpha_k$. We have to show that $\varrho + \sigma + \tau_c \geqslant c|H|$. Since $\tau_c(y) \geqslant c$ for every $y \in H - H'$, it is sufficient to show that $S \geqslant c|H'|$. The proof of this statement is based on Lemma 4.1 where $X = W$, $Y = H'$ and v' is the W-mapping of G obtained from the given L-mapping v.

Let $F^* = F^*(A^*, B)$ be the bipartite graph obtained from $F - A_1$ by adding the set $A_1^* = \mathscr{D}$ where $T \in \mathscr{D}$ and $y \in B$ are joined by an edge in F^* iff $y \in N_W^{v'}(x : G)$ for some vertex $x \in W^k(T) = W \cap V(T)$. Since every ε_k-hypergraph $T \in \mathscr{D}$ is an ε_k-subcomponent of some member in $\mathscr{C}$, we infer from (4), (5) and (7) that

$$\begin{aligned} S &\geqslant \sum_{e \in E_1} (|e \cap H| - 1) + \sum_{e \in E_2} |e \cap H| + \sum_{x \in L'} \alpha_k d_1(x) + \sum_{T \in \mathscr{D}} \alpha_k + \sum_{y \in H'} d_{G[H]}(y) \\ &\geqslant \alpha_k |A_2| + \alpha_k |A_1^*| + |A_3| \geqslant \alpha_k |A^*|. \end{aligned}$$

Since $d_W^{v'}(y) \geqslant d_L^v(y)$ for every $y \in H$, we conclude from (6) and Lemma 4.1(a), similarly to Case 1, that $d_F(y) \geqslant k-1$ for every $y \in B$. Now, we apply Lemma 3.4 to F^* where $R = k-1$, $d = 3$ and $a(x) = 3$ for every $x \in A^*$. If $(R-d)|B| \leqslant 3|A^*|$, then we obtain

$$S \geqslant \alpha_k |A^*| \geqslant \tfrac{1}{3}(k-4)\alpha_k |B| = c|H'|.$$

Otherwise, by Lemma 3.4, there are non-empty subsets $A' \subseteq A$ and $B' \subseteq B = H'$ such that for $F' = F^*[A' \cup B']$ we have $d_{F'}(x) > 3$ for every $x \in A'$ and $d_{F'}(y) > 3$ for every $y \in B'$. Since every vertex of $A_2 \cup A_3$ has degree 1 in F^*, we have $A' \subseteq A_1^* = \mathscr{D}$. This gives a contradiction to Lemma 4.1(c). Therefore, Theorem 1.9 is proved. □

5. Concluding remarks

The main result of this paper is that $2f_k(n) \geqslant g_k(n,c)$ where $c = (k-5)\alpha_k$ and $k \geqslant 6$. Our method of proof yields two restrictions for the possible values of the constant c, namely $c \leqslant k - 2/(k-1)$ (see Lemma 1.8) and $c \leqslant \frac{1}{2}(k - 2/(k-1))$ (see the proof of Theorem 1.9, the part where we show that $\tau_c(y) \geqslant c$ provided that $d_G(y) > k$). For integers p, k satisfying $k \geqslant 4$ and $2 \leqslant p \leqslant k$, let

$$c_{k,p} = f_k(k+p) - \frac{1}{2} g_k\left(k+p, k - \frac{2}{k-1}\right)$$

and

$$h_{k,p}(n) = \frac{1}{2} g_k\left(n, k - \frac{2}{k-1}\right) + c_{k,p} = \frac{1}{2}\left(k - 1 + \frac{k-3}{k-1}\right) n + c_{k,p}.$$

We claim that if $n \geqslant k+2$ and $n \equiv p-1 \bmod (k-1)$ where $2 \leqslant p \leqslant k$, then there is a k-critical graph with n vertices and $h_{k,p}(n)$ edges implying that

$$2f_k(n) \leqslant 2h_{k,p}(n) = g_k\left(n, k - \frac{2}{k-1}\right) + 2c_{k,p}. \tag{8}$$

For $n = k+p$, we have $h_{k,p}(n) = f_k(n)$ and the claim is evidently true. Now, assume $n \equiv p-1 \bmod (k-1)$. If G is a k-critical graph with n vertices and $h_{k,p}(n)$ edges, then we apply the Hajós construction (see [11] or [12]) to G and K_k. This results in a k-critical graph with $n+k-1$ vertices and

$$m = |E(G)| + \binom{k}{2} - 1$$

edges. By an easy calculation, we then obtain

$$m = h_{k,p}(n) + \binom{k}{2} - 1 = h_{k,p}(n+k-1).$$

This proves our claim.

Ore [20] (see also [12, Problem 5.3]) conjectured that equality holds in (8). In [10] Gallai proved that

$$2f_k(k+p) = (k-1)(k+p) + p(k-p)$$

provided that $2 \leqslant p \leqslant k-1$ and in [13] it was proved that $f_k(2k) = k^2 - 3$. Ore's conjecture implies, in particular, that

$$\lim_{n \to \infty} \frac{2f_k(n)}{n} = k - \frac{2}{k-1}.$$

Some further results concerning list critical graphs and hypergraphs with few edges can be found in [14,15].

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