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Homomorphisms from sparse graphs with large girth

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Abstract

We show that a planar graph with girth at least $\frac{20t-2}{3}$ has circular chromatic number at most $2+\frac{1}{t}$, improving earlier results. This follows from a general result establishing homomorphisms into special targets for graphs with given girth and given maximum average degree. Other applications concern oriented chromatic number and homomorphisms into mixed graphs with colored edges.

Keywords: Graph coloring; Graph homomorphism; Planar graph; Circular coloring; Oriented coloring; Discharging; *t*-Nice graphs

1. Introduction

Circular coloring, introduced by Vince [13], is a model for coloring the vertices of graphs that provides a more refined measure of coloring difficulty than the ordinary

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chromatic number. A (k,d)-*coloring* of a graph G is a map $\phi : V(G) \to \{0, \dots, k-1\}$ such that $d \leqslant |\phi(u) - \phi(v)| \leqslant k - d$ for every edge $uv \in E(G)$; a graph having such a coloring is (k,d)-*colorable*. Note that a $(k,1)$-coloring is an ordinary proper k-coloring.

The *circular chromatic number* $\chi_c(G)$ of a graph G is the infimum of $\frac{k}{d}$ over all pairs (k,d) such that G is (k,d)-colorable. A (k,d)-coloring is "circular" in the sense that we may view the k colors as points on a circle, and the requirement for (k,d)-coloring is that the colors on adjacent vertices must be at least d positions apart on the circle. Zhu [15] provides a thorough survey of results on circular chromatic number.

We call χ_c a refined measure of coloring because $\chi(G) - 1 < \chi_c(G) \leqslant \chi(G)$ for every graph G, as proved in [13] and again in [2]. A 3-chromatic graph is not 2-colorable, but if its circular chromatic number is near 2, then it is somehow "just barely" not 2-colorable. For odd cycles, $\chi_c(C_{2t+1}) = 2 + \frac{1}{t}$. If G contains C_{2t+1}, then $\chi_c(G) \geqslant 2 + \frac{1}{t}$. By the theorem of Grötzsch [7], every triangle-free planar graph is 3-colorable. In generalizing this to circular chromatic number, we ask what threshold on girth is needed to force the circular chromatic number to be at most $2 + \frac{1}{t}$. A relaxation for planar graphs of a conjecture of Jaeger [9] on nowhere-zero flows states the following:

Conjecture 1.1. *For every positive integer t, every planar graph with girth at least $4t$ has circular chromatic number at most $2 + \frac{1}{t}$.*

When $t = 1$, Conjecture 1.1 reduces to Grötzsch's Theorem. The conjecture is sharp if true, as shown by DeVos [5]. The example of DeVos consists of $4t - 1$ paths of length $2t - 1$ with a common endpoint, plus a cycle of length $4t - 1$ through the vertices on the opposite ends of the paths. The graph has girth $4t - 1$, but it has no $(2t + 1, t)$-coloring (the color on the central vertex cannot appear on the peripheral cycle, and with this restriction that cycle cannot be colored). Thus the circular chromatic number exceeds $2 + \frac{1}{t}$.

Nešetřil and Zhu [12] and Galuccio, Goddyn, and Hell [6] proved that every planar graph with girth at least $10t - 4$ has circular chromatic number at most $2 + \frac{1}{t}$. In [6] there are also analogous bounds for all surfaces. The *odd girth* of a graph is the minimum length of its odd cycles; Klostermeyer and Zhang [10] showed that for circular chromatic number at most $2 + \frac{1}{t}$ it suffices to have odd girth at least $10t - 4$, via the "Folding Lemma".

Zhu [14] improved the bound from $10t - 4$ to $8t - 3$. We further lower this threshold to $\frac{20t-2}{3}$. Our result applies in a more general setting for which we interpret (k,d)-colorings as homomorphisms. A *homomorphism* from G into H is a map $\phi : V(G) \to V(H)$ such that adjacent vertices in G are mapped into adjacent vertices in H. Let $H(k,d)$ be the circulant graph with vertex set $\{0, \dots, k-1\}$ in which i and j are adjacent if and only if $d \leqslant |i - j| \leqslant k - d$. Every (k,d)-coloring of a graph G is a homomorphism from G into $H(k,d)$.

We actually consider a still more general setting that also applies to oriented homomorphisms and to homomorphisms into mixed graphs with colored edges and arcs. Other applications include a new upper bound on the girth of planar graphs with oriented chromatic number greater than 5. In particular, we show that for every orientation of a planar graph with girth at least 13, there is a homomorphism into some tournament with five vertices. This was known previously for girth at least 14 via a lengthy proof [4], but the improvement to 13 is an easy consequence of our main theorem.

Our original motivation was the result of Zhu on Conjecture 1.1. To see how our theorem improves his threshold from $8t-3$ to $\frac{20t-2}{3}$, we first state the relevant special case of our theorem. For graphs with given girth and given upper bound on the average degree of all subgraphs, we prove the existence of homomorphisms into special targets. For this example, we are interested in the target $H(2t+1,t)$.

Theorem 1.2. *If $g \geqslant 6t-2$ and $d < 2+\frac{3}{5t-2}$, then every graph with girth g whose subgraphs all have average degree at most d is $(2+\frac{1}{t})$-colorable.*

Corollary 1.3. *If G is a planar graph G with girth at least $\frac{20t-2}{3}$, then G is $(2+\frac{1}{t})$-colorable.*

Proof. Since $\frac{20t-2}{3} > 6t-2$, Theorem 1.2 applies if we can show that every subgraph of G has average degree less than $2+\frac{3}{5t-2}$. It suffices to show this bound for G itself, since every subgraph has girth at least as large as the girth of G.

We may assume that G is 2-connected, since we can combine $(2t+1,t)$-colorings of blocks. Let n, m, f be the numbers of vertices, edges, and faces in some planar embedding of G. By summing face-lengths, the bound on girth yields $f \leqslant 6m/(20t-2)$. With d denoting the average degree, we have $m = dn/2$. Using Euler's Formula, we have $2 = n - m + f \leqslant m(\frac{2}{d} - 1 + \frac{3}{10t-1})$. Since the factor in parentheses must be positive, we obtain $\frac{10t-4}{10t-1} < \frac{2}{d}$; that is, $d < 2 + \frac{3}{5t-2}$. □

The argument of Corollary 1.3 holds equally well for graphs embeddable on the projective plane. For graphs embeddable on the torus or Klein bottle, it also works when the girth is strictly greater than $\frac{20t-2}{3}$.

Section 2 introduces definitions and notation and states the main result. In Section 3, we apply the result to graphs embedded in surfaces, to circular chromatic number, to oriented chromatic number of graphs with large girth, and to homomorphisms of planar graphs with colored edges. We prove the main result in Section 4 using a discharging argument.

2. t-Expansive s-graphs

Since we are interested in applications to both graphs and digraphs, we use a model that incorporates both and is still more general. A *mixed graph* is a common generalization of multigraphs and multidigraphs that allows both ordered pairs (arcs) and unordered pairs (edges) of vertices in the edge set. We augment such a structure by allowing a fixed coloring of the edges and arcs.

Definition. Let s denote a pair (s_1, s_2) of nonnegative integers. An *s-graph* is a mixed graph without loops in which the arcs are colored from the set $\{-1, \dots, -s_1\}$ and the edges are colored from the set $\{1, \dots, s_2\}$, in such a way that two edges or two arcs do not have the same color if they have the same two endpoints.

In the applications discussed in Section 3, we will only consider the cases where s is $(0,1)$ or $(1,0)$ or $(0,t)$. A $(0,1)$-graph is a simple undirected graph, and a $(1,0)$-graph is an orientation of such a graph. A $(1,2)$-graph has arcs of color -1 and edges of colors 1 and 2. The notion of homomorphism extends to s-graphs; a homomorphism from an s-graph G to an s-graph H is a map from $V(G)$ into $V(H)$ such that the image of an edge or forward arc with color c in G is an edge or forward arc with color c in H.

A *walk* (of *length* t) in an s-graph is a list $(v_0, e_1, v_1, \dots, e_t, v_t)$ such that each e_i is an edge or an arc, and for each i the endpoints of e_i are v_{i-1} and v_i. The edges or arcs of a walk need not be distinct, and arcs need not be followed from tail to head. A walk with first vertex v and last vertex w is a v,w-walk.

The *pattern* of a walk of length t is the list $(C_1, \dots, C_t)$, where C_i is the pair (c_i, σ_i) consisting of the color c_i of e_i and a sign σ_i defined by

$$\sigma_i = \begin{cases} + & \text{if } e_i \text{ is undirected or } v_i \text{ is the head of } e_i, \\ - & \text{if } v_i \text{ is the tail of } e_i. \end{cases}$$

In particular, the element "$(-2,-)$" in the pattern of a walk means that the corresponding step is an arc with color -2 traversed from head to tail. For an s-graph G, the patterns of length t are the t-tuples $(C_1, \dots, C_t)$ such that $C_i = (c_i, \sigma_i)$ with $c_i \in \{-s_1, \dots, -1, 1, \dots, s_2\}$ and $\sigma_i = +$ if $c_i > 0$ and $\sigma_i \in \{+, -\}$ if $c_i < 0$. The set of patterns is determined by s.

Definition. An s-graph G is *t-nice* if for all $v, w \in V(G)$ (not necessarily distinct) and every pattern of length t, there is a v,w-walk with this pattern.

When $s = (0,1)$, an s-graph G is essentially just an undirected graph, and there is only one pattern of length t. Since v,v-walks are needed, no s-graph is 1-nice. The complete graph K_n is 2-nice as a $(0,1)$-graph if $n \geqslant 3$. No bipartite graph is t-nice for any t, since a v,w-walk of length t can exist only when t has the same parity as the distance from v to w. The odd cycle C_{2k+1} is $2k$-nice, since from a fixed vertex v every vertex can be reached by a path of even length at most $2k$, and such a walk can be increased to length $2k$ by repeating edges. On the other hand, C_{2k+1} is not $(2k-1)$-nice, because a path of odd length from a vertex to itself must traverse the full cycle, and then its length is at least $2k+1$.

Remark 2.1. Every t-nice s-graph is also $(t+1)$-nice.

Proof. Let G be a t-nice s-graph; note that $t \geqslant 2$. Given vertices v and w in G and a pattern $(C_1, \ldots, C_{t+1})$, there is a walk of length 1 with pattern C_1 from v to some vertex u, since G is t-nice. Again since G is t-nice, there is a u, w-walk with pattern $(C_2, \ldots, C_{r+1})$. Together, these form a v, w-walk of length $t+1$ with pattern $(C_1, \ldots, C_{r+1})$. □

A graph is *nice* if it is t-nice for some t. Nice graphs were used implicitly in [11]. They were explicitly studied and characterized in [8]. In [8] it was also shown that minimal graphs that are homomorphic images of all planar s-graphs with girth at least g are nice.

For another example, let C_n^2 denote the directed graph whose vertices are the congruence classes modulo n and whose arcs are the ordered pairs of the form $(i, i+1)$ and $(i, i+2)$. This is an s-graph for $s = (1, 0)$. There are 2^t patterns of length t, but to show that C_n^2 is not $(n-2)$-nice we only need to show failure for one pattern and one pair of vertices. A walk whose pattern has only plus-signs follows only forward arcs. We claim that C_n^2 has no such walk of length $n-2$ from vertex 0 to vertex $n-3$. The total upward motion in $n-2$ steps that each add $+1$ or $+2$ is at least $n-2$ and at most $2n-4$. Since no value in this range is congruent to $n-3$ modulo n, the needed walk does not exist. On the other hand, C_n^2 is $(n-1)$-nice. This is easiest to show (see Example 2.3) by using a stronger concept that we introduce next.

Definition. A vertex $v \in V(H)$ is a (c, σ)-*successor* of a set W of vertices in an s-graph H if for some $w \in W$ there is an edge or arc with endpoints w and v that has color c and sign σ when viewed from w to v. An s-graph H with n vertices is t-*expansive* if for all nonempty $W \subset V(H)$ and every pair (c, σ), the number of (c, σ)-successors of W is at least $\min\{n, |W| + \frac{n-1}{t}\}$.

A vertex in W may be a (c, σ)-successor of W. In a $(0, 1)$-graph G, the number of successors of W is the number of vertices of G having neighbors in W. No bipartite graph is t-expansive, because a largest partite set does not have enough successors. We formalize t-expansiveness for the two examples that we discussed earlier and will apply later.

Example 2.2. As a $(0, 1)$-graph, the undirected cycle C_{2k+1} is $2k$-expansive.

Proof. When $|W| \leqslant 2k$ we need only $|W| + 1$ successors. The elements of W have distinct successors in the clockwise direction. Since the total number of vertices is odd, when $|W| < 2k+1$ there is a vertex v on the cycle such that $v \notin W$ but the vertex w that is two positions later in the clockwise direction belongs to W. Now the common neighbor of v and w is a successor of W that we have not already counted. □

Example 2.3. As a $(1, 0)$-graph, the digraph C_n^2 is $(n-1)$-expansive.

Proof. Recall that the arcs are the pairs of the form $(i, i+1)$ and $(i, i+2)$, modulo n. Here (c, σ) can be $(-1, +)$ or $(-1, -)$, seeking out-neighbors of W or in-neighbors of W in the digraph, respectively. By symmetry, we consider only $(-1, +)$. As in Example 2.2, we need only $|W| + 1$ successors. For each $i \in W$, the vertex $i+1$ is a successor; this yields $|W|$ successors. If $|W| < n$, then there exists i such that $i \in W$ and $i + 1 \notin W$, and now $i+2$ is a successor of W that we have not already counted. □

By starting with $W = \{v\}$ and using t applications of the definition of t-expansive, it follows that from each vertex v in a t-expansive s-graph, we can reach every vertex (including v) via a walk of length t with a specified pattern. Hence every t-expansive s-graph is t-nice. The more detailed statement below is what we need from this concept; again it is immediate from the definition of t-expansive.

Remark 2.4. Let v be a vertex in a t-expansive s-graph H with n vertices. For every pattern $(C_1, \ldots, C_l)$ of length l, the number of vertices w such that H contains a v, w-walk of length l with pattern $(C_1, \ldots, C_l)$ is at least $1 + (n-1)l/t$.

For ordinary graphs, we say that a graph G is *H-colorable* if there is a homomorphism from G into H; this reduces to ordinary k-colorability when $H = K_k$. We use the term H-colorable for s-graphs in the same way. When H is an s-graph, an s-graph G is *critically non-H-colorable* if G is not H-colorable but every proper subgraph of G is H-colorable.

The *skeleton* of an s-graph G is the multigraph obtained from G by ignoring the orientations of its arcs and erasing all colors. Our main result is the following.

Theorem 2.5. *Let H be a t-expansive s-graph. Let G be an s-graph whose skeleton has girth g and has no subgraph with average degree more than d. If* $g \geq 3t-2$ *and* $d < 2 + \frac{6}{5t-4}$, *then G has a homomorphism into H.*

Since every subgraph of an H-colorable s-graph is also H-colorable, an equivalent phrasing of Theorem 2.5 is that if H is a t-expansive s-graph, and G is a critically non-H-colorable s-graph whose skeleton has girth at least $3t-2$, then the average vertex degree of the skeleton is at least $2 + \frac{6}{5t-4}$.

3. Applications

In this section we present several applications of Theorem 2.5. We begin by considering graphs embedded on surfaces. Note that all graphs embeddable in the plane also embed on higher surfaces.

Corollary 3.1. *Let H be a t-expansive s-graph, and let G be a critically non-H-colorable s-graph of order n. If the skeleton of G embeds on the projective plane, then* *its girth is less than* $\frac{10t-2}{3}$. *If it embeds on the torus or Klein bottle, then its girth is at most* $\frac{10t-2}{3}$.

Proof. Let m be the number of edges in the skeleton of G. If the claimed conclusion fails, then the girth g of the skeleton is at least $3t-2$, and Theorem 2.5 applies. With average degree at least $2+\frac{6}{5t-4}$, we have $m \geqslant \frac{5t-1}{5t-4}n$. Let f be the number of faces in an embedding of the skeleton on a surface of Euler characteristic N. Since $fg \leqslant 2m$, Euler's Formula yields

$$2-N = n-m+f \leqslant m\left(\frac{5t-4}{5t-1}-1+\frac{2}{g}\right) = m\left(\frac{2}{g}-\frac{3}{5t-1}\right).$$

For the surfaces mentioned, $N \leqslant 2$. Hence $\frac{2}{g} \geqslant \frac{3}{5t-1}$, with equality possible only when $N=2$. □

We review the application to circular coloring described in Corollary 1.3, using the more general language.

Corollary 3.2. *Let t be a positive integer. If G is a projective planar graph with girth at least* $\frac{20t-2}{3}$, *or a graph embedding on the torus or Klein bottle with girth greater than* $\frac{20t-3}{3}$, *then* $\chi_c(G) \leqslant 2+\frac{1}{t}$.

Proof. The conclusion is equivalent to the existence of a C_{2t+1}-coloring of G. By Remark 2.2, C_{2t+1} is $2t$-expansive as a $(0,1)$-graph. Since taking subgraphs cannot reduce girth, Corollary 3.1 implies that G has no subgraph that fails to be C_{2t+1}-colorable. □

The bound of Corollary 3.2 generalizes easily for other surfaces. Next we apply Corollary 3.1 to oriented coloring.

Definition. The *oriented chromatic number* of a simple graph G is the minimum k such that every orientation of G admits a homomorphism into some simple digraph with k vertices.

The target digraph can be different for different orientations of G, and we may assume that in each case the target is an orientation of K_k.

In [3,4,11], bounds on the oriented chromatic number of planar graphs with given girth were considered. It was proved in [11] that there are planar graphs with arbitrarily large girth having oriented chromatic number 5, and that every planar graph with girth at least 16 has oriented chromatic number at most 5. It was also mentioned that oriented chromatic number exceeds 5 for some planar graph with girth 7. The question then is what is the minimum threshold g such that all planar graphs with girth at least g have oriented chromatic number at most 5. In [4], the threshold 16 was reduced to 14 by a somewhat lengthy argument. Corollary 3.1

yields an immediate strengthening to 13 and applies also for three additional surfaces.

Corollary 3.3. *Every graph with girth at least* 13 *that embeds on the torus or Klein bottle has oriented chromatic number at most* 5. *In fact, every orientation of such a graph has a homomorphism into the same* 5*-vertex regular tournament* C_5^2.

Proof. The orientations of a simple graph G are precisely the $(1,0)$-graphs with skeleton G. If some orientation of G is not C_5^2-colorable, then there is a critical such digraph D, and its girth is as large as that of G. By Remark 2.3, C_5^2 is 4-expansive. By Corollary 3.1, the girth of D is at most $\frac{40-2}{3}$, which is less than 13. □

Let $\mathcal{M}_\alpha$ denote the family of all graphs such that the average degree of every subgraph is strictly less than α. Theorem 2 in [3] says that every graph in $\mathcal{M}_{16/7}$ with girth at least 11 admits a homomorphism into the octahedron $K_{2,2,2}$. Since a planar n-vertex graph with girth g has at most $\frac{g}{g-2}(n-2)$ edges, it follows that every planar graph with girth at least 16 has average degree less than $16/7$ and hence is $K_{2,2,2}$-colorable. The proof of Theorem 2 in [3] takes more than 3 pages. However, a stronger version of its corollary for planar graphs follows directly from Theorem 2.5.

Corollary 3.4. *If G is a planar graph with girth at least* 13, *then G is $K_{2,2,2}$-colorable.*

Proof. Theorem 2.5 applies, since $K_{2,2,2}$ is 5-expansive, $13 = 3 \cdot 5 - 2$, and $2 + \frac{6}{5t-4} = \frac{16}{7}$ when $t = 5$. □

Finally, we apply the edge-partitioning aspect of Theorem 2.5. In connection with questions in group theory, Alon and Marshall [1] studied homomorphisms of graphs with colored edges. Given a t-edge-coloring of a simple graph G, let $\lambda(G,c)$ be the least number of vertices in an edge-colored graph H such that there is a coloring-preserving homomorphism from G into H. In our terminology, this is a homomorphism of s-graphs, where $s = (0,t)$. Alon and Marshall discussed the maximum of $\lambda(G,c)$ when G is planar and c uses at most t colors. We obtain an upper bound for planar graphs with sufficiently large girth.

Corollary 3.5. *Let G be a planar graph with girth at least $\frac{20t-2}{3}$. For every coloring of G with at most t colors, there is a color-preserving homomorphism from G into an edge-colored graph with $2t+1$ vertices. The same image always suffices: a coloring of K_{2t+1} in which each color class is a spanning cycle.*

Proof. It is well known that the complete graph K_{2t+1} decomposes into t Hamiltonian cycles. Let H denote the $(0,t)$-graph obtained by coloring the ith cycle in the decomposition of K_{2t+1} with color i. We obtain that H is $2t$-expansive, since for each item in a pattern, we can apply Remark 2.2 to the cycle in the corresponding color. By Corollary 3.1, every t-edge-colored planar graph with girth at least $\frac{20t-2}{3}$ is H-colorable. $\square$

4. The main result

We will break the proof of Theorem 2.5 into several lemmas. Throughout this section, fix H as a t-expansive s-graph with n vertices, and let G be a critically non-H-colorable s-graph whose skeleton has girth at least $3t-2$. We develop various properties of the skeleton to show that its average vertex degree is large. We begin by introducing convenient notation.

Let $\tilde{G}$ denote the skeleton of G. We use $\delta(\tilde{G})$ for the minimum degree of $\tilde{G}$ and $d(v)$ for the degree in $\tilde{G}$ of a vertex v.

If $d(v)=1$, then $\{v\}$ can have only one (c,σ)-successor. Since H is t-expansive, we conclude that $\delta(\tilde{G})\geqslant 2$.

Definition. A *thread* in a graph $\tilde{G}$ is a path whose internal vertices have degree 2 in $\tilde{G}$. Two vertices are *weak neighbors* or *weakly adjacent* if they are the endpoints of a thread (this includes adjacent vertices, since threads may have no internal vertices).

Our approach in proving the lemmas is as follows. If the desired conclusion fails, then from G we delete some vertices to obtain a subgraph G'. By the criticality of G, G' is H-colorable. Using the t-expansiveness of H, we extend the resulting homomorphism from G' into H to obtain an H-coloring of G, thus producing a contradiction.

Lemma 4.1. *Every thread in $\tilde{G}$ has length at most $t-1$.*

Proof. Otherwise, let P be a u,v-thread of length t in $\tilde{G}$, and let G' be the s-graph obtained from G by deleting the internal vertices of P. Since $t\geqslant 2$, G' is a proper subgraph of G. Criticality of G yields an H-coloring ϕ of G'. Let $(C_1,\dots,C_t)$ be the pattern of P. By the definition of t-nice, H contains a $\phi(u),\phi(v)$-walk with pattern $(C_1,\dots,C_t)$. By defining ϕ on the internal vertices of P using the vertices of this walk in succession, we extend ϕ to an H-coloring of G, which is impossible. $\square$

Corollary 4.2. *No three vertices of $\tilde{G}$ with degree at least* 3 *are pairwise weakly adjacent, and no two threads have the same set of endpoints.*

Proof. Otherwise, by Lemma 4.1, $\tilde{G}$ has a cycle of length at most $3t-3$. $\square$

Definition. When u and v are weakly adjacent, let l_{uv} denote the length of a shortest u,v-thread. Let $Y=\{v\in V(\tilde{G})\colon d(v)\geqslant 3\}$. A weak neighbor u of v is a *weak Y-neighbor* if $u\in Y$; otherwise it is a *weak 2-neighbor*.

For $v \in V(\tilde{G})$, let $N_Y(v)$ denote the set of weak Y-neighbors of v in $\tilde{G}$. For $v \in Y$, let $f(v) = -t + \sum_{u \in N_Y(v)} (t - l_{vu})$.

The next two lemmas place lower bounds on $f(v)$ and on $\sum_{u \in N_Y(v)} f(u)$. The motivation for doing this is as follows. If $f(v)$ is small, then v has few weak Y-neighbors or has long threads to them. Both conditions tend to reduce the average vertex degree. Since we want a lower bound on the average vertex degree in G, it helps to place lower bounds on values of f.

Lemma 4.3. *If $v \in Y$, then $f(v) \geqslant 1$.*

Proof. Let G' be the s-graph obtained from G by deleting v and all its weak 2-neighbors. By the minimality of G, there is an H-coloring ϕ of G'. Suppose that the desired inequality fails; we extend ϕ to an H-coloring of G.

Consider $u \in N_Y(v)$. Let P_u be the u,v-thread in G. Let $W_0 = \{u\}$, and for $i > 0$ let W_i be the set of vertices at which the ith vertex of P could be embedded in extending ϕ along P_u. Since H is t-expansive, $|W_i| \geqslant 1 + \frac{n-1}{t} i$. Letting $i = l_{vu}$, we conclude that at most $\frac{n-1}{t}(t - l_{vu})$ vertices of H are excluded from serving as the image of v in an extension of ϕ to P_u.

If some vertex of H is allowed by P_u for all $u \in N_Y(v)$, then ϕ extends to G. Hence we conclude that

$$\frac{n-1}{t} \sum_{u \in N_Y(v)} (t - l_{vu}) \geqslant n > n - 1,$$

which yields $\sum_{u \in N_Y(v)} (t - l_{vu}) > t$ and hence $f(v) > 0$. Now $f(v) \geqslant 1$ follows from the integrality of $f(v)$. □

Lemma 4.4. *If $v \in Y$, then $\sum_{u \in N_Y(v)} f(u) \geqslant t + 1$.*

Proof. Say that a vertex $u \in N_Y(v)$ is *v-free* if $f(u) \leqslant t - l_{uv}$. Let G' be obtained from G by deleting the vertex v, the v-free vertices, and all their weak 2-neighbors. By the criticality of G, there is an H-coloring ϕ of G'.

By Corollary 4.2, ϕ is defined on all of $N_Y(u) - \{v\}$ for each $u \in N_Y(v)$. If u is v-free, then as in Lemma 4.3 the vertices of $N_Y(u) - \{v\}$ exclude at most $\frac{n-1}{t} \sum_{y \in N_Y(u) - \{v\}} (t - l_{yu})$ vertices of H from serving as the image of u in an extension of ϕ along the threads from $N_Y(u) - \{v\}$ to u. By the definition of f, this quantity equals $\frac{n-1}{t}(f(u) + l_{vu})$. Since u is v-free, we have $f(u) + l_{vu} \leqslant t$, and the number of vertices excluded from serving as the image of u is at most $n - 1$. In particular, at least one is available. Accounting for the possibility that $f(u) + l_{vu}$ is strictly less than t, we compute that at least $1 + \frac{n-1}{t}(t - l_{vu} - f(u))$ vertices of H are available for the image of u.

In extending the homomorphism along the thread from u to v, the number of possible images for the current vertex increases by at least $\frac{n-1}{t}$ with each of the l_{vu}

steps, because H is t-expansive. By the time we reach v, there are thus at least $1+\frac{n-1}{t}(t-f(u))$ vertices available for the image of v in an extension of ϕ to all the weak neighbors of u. The number of vertices forbidden from being the image is at most $n-1-\frac{n-1}{t}(t-f(u))$, which equals $\frac{n-1}{t}f(u)$.

When $u\in N_Y(v)$ is not v-free, $\phi(u)$ is fixed. In this case, as in Lemma 4.3, at least $1+\frac{n-1}{t}l_{vu}$ vertices of H can serve as the image of v in an extension of ϕ to the u,v-thread, and at most $\frac{n-1}{t}(t-l_{vu})$ vertices are excluded. Since $t-l_{vu}<f(u)$, fewer than $\frac{n-1}{t}f(u)$ vertices are excluded.

Since G has no H-coloring, every vertex of H must be excluded from serving as the image of v in at least one of these extensions. This requires $\frac{n-1}{t}\sum_{u\in N_Y(v)}f(u)\geqslant n$. Since every $f(u)$ is an integer, we have $\sum_{u\in N_Y(v)}f(u)\geqslant t+1$. $\square$

We complete the proof using a discharging argument. We treat $d(v)$ as an initial "charge" on the vertex $v\in V(\tilde{G})$. We will move charge from vertex to vertex, without changing the total, to obtain a new charge $d^*(v)$ such that

$$d^*(v)\geqslant 2+\frac{4d(v)-2}{5t} \quad \text{for all } v\in V(\tilde{G}). \tag{1}$$

Let $p=|V(\tilde{G})|$ and $m=|E(\tilde{G})|$. If (1) holds, then

$$2m=\sum_{v\in V(\tilde{G})}d^*(v)\geqslant\sum_{v\in V(\tilde{G})}\left(2+\frac{4d(v)-2}{5t}\right)=\left(1-\frac{1}{5t}\right)2p+\frac{4}{5t}2m,$$

and hence $\frac{5t-1}{5t}p\leqslant\frac{5t-4}{5t}m$. This makes the average degree of $\tilde{G}$ at least $\frac{2(5t-1)}{5t-4}$, which equals $2+\frac{6}{5t-4}$. Hence it suffices to obtain d^* so that (1) holds.

Discharging Rules. Given a multigraph $\tilde{G}$ with $d(u)$ denoting the degree of vertex u as an initial charge, define an adjusted charge $d^*(u)$ for each $u\in V(\tilde{G})$ by the following operations:

R1. Every $v\in Y$ gives each weak 2-neighbor the amount $\frac{3}{5t}$.

R2. Every $v\in Y$ gives each weak Y-neighbor the amount $\frac{3f(v)+(t+1)(d(v)-3)}{5td(v)}$.

Lemma 4.5. *Every $v\in Y$ receives from its weak Y-neighbors at least $\frac{t+1}{5t}$.*

Proof. If every $u\in N_Y(v)$ sends v at least $\frac{f(u)}{5t}$, then v receives from $N_Y(v)$ at least $\frac{1}{5t}\sum_{u\in N_Y(v)}f(u)$. By Lemma 4.4, $\sum_{u\in N_Y(v)}f(u)\geqslant t+1$.

Hence we may assume that $\frac{3f(u)+(t+1)(d(u)-3)}{5td(u)}<\frac{f(u)}{5t}$ for some $u\in N_Y(v)$. This requires $d(u)\geqslant 4$. Hence we can cancel $d(u)-3$ after clearing fractions to obtain $f(u)>t+1$. When $f(u)\geqslant t+1$, the formula in R2 yields that u by itself gives v at least $\frac{t+1}{5t}$. For $y\in N_Y(v)$, we have $d(y)\geqslant 3$ and $f(y)\geqslant 1$ (by Lemma 4.3), so all other amounts sent to v are nonnegative. $\square$

Lemma 4.6. *Inequality* (1) *holds for the charge* d^* *obtained for* $\tilde{G}$ *via the Discharging Rules.*

Proof. If $d(v) = 2$, then v sends out nothing and receives $\frac{3}{5t}$ from each of its two weak Y-neighbors. Thus $d^*(v) = 2 + \frac{6}{5t} = 2 + \frac{4d(v)-2}{5t}$.

Now consider $v \in Y$. Vertex v sends out $\frac{3}{5t}\sum_{w \in N_Y(v)} (l_{vw} - 1)$ to its weak 2-neighbors and $\frac{3f(v)+(t+1)(d(v)-3)}{5t}$ to its weak Y-neighbors. By Lemma 4.5, v also receives at least $\frac{t+1}{5t}$ from its weak Y-neighbors. Since also $f(v) = -t + \sum(t - l_{vw})$ by definition, we obtain (with each sum running over $w \in N_Y(v)$)

$$\begin{aligned} d^*(v) &\geqslant d(v) - \frac{3}{5t}\sum(l_{vw} - 1) - \frac{3f(v) + (t+1)(d(v) - 4)}{5t} \\ &= d(v) - \frac{3}{5t}\Big[-t + \sum(t - l_{wv} + l_{vw} - 1)\Big] - \frac{(t+1)(d(v) - 4))}{5t} \\ &= \frac{d(v)}{5t}[5t - 3(t-1) - (t+1)] + \frac{1}{5t}[3t + 4(t+1)] = \frac{(t+2)d(v) + 7t + 4}{5t}. \end{aligned}$$

Since $d(v) \geqslant 3$ and $t \geqslant 2$, we have

$$\begin{aligned} (t+2)d(v) + 7t + 4 &= (t+2)d(v) - 3(t-2) + (10t - 2) \\ &= t(d(v) - 3) + 2d(v) + 6 + (10t - 2) \geqslant 4d(v) + 10t - 2. \end{aligned}$$

Therefore, $\frac{(t+2)d(v)+7t+4}{5t} \geqslant 2 + \frac{4d(v)-2}{5t}$, which completes the proof of the lemma and also the theorem. □

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