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Balanced edge colorings

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Abstract

This paper contains two principal results. The first proves that any graph G can be given a balanced proper edge coloring by k colors for any $k \geqslant \chi'(G)$. Here balanced means that the number of vertices incident with any set of d colors is essentially fixed for each d, that is, for two different d-sets of colors the number of vertices incident with each of them can differ by at most 2. The second result gives upper bounds for the vertex-distinguishing edge chromatic number of graphs G with few vertices of low degree. In particular, it proves a conjecture of Burris and Schelp in the case when $\Delta(G) \geqslant \sqrt{2|V(G)|} + 4$ and $\delta(G) \geqslant 5$.

1. Introduction

Let G be a simple graph. For $d \geqslant 0$ write V_d for the set of vertices in G of degree d and $n_d = |V_d|$ for the number of these vertices. Let $\chi'(G)$ be the minimum number of colors required in a proper edge-coloring of G. By Vizing's Theorem, $\Delta(G) \leqslant \chi'(G) \leqslant \Delta(G) + 1$. If we have such a proper coloring with colors $\{1, \dots, k\}$ and v is a vertex of G, denote by $S(v)$ the set of colors used to color the edges incident with v. For any subset $S \subseteq \{1, \dots, k\}$, write $V_S = \{v : S(v) = S\}$ for the set of vertices for which $S(v) = S$ and $n_S = |V_S|$ for the number of these vertices.

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A proper edge coloring of a graph is said to be *vertex-distinguishing* if each pair of vertices is incident with a different set of colors. In other words, if $n_S \leqslant 1$ for all $S \subseteq \{1, \ldots, k\}$. A vertex-distinguishing proper edge coloring will also be called a *strong* coloring. A graph has a strong coloring if and only if it has no more than one isolated vertex and no isolated edges. Such a graph will be referred to as a *vdec*-graph. The minimum number of colors required for a strong coloring of a vdec-graph G will be denoted $\chi'_s(G)$.

The concept of vertex-distinguishing colorings has been considered in several papers (see [1–10]). In [6] Burris and Schelp made the following conjectures:

Conjecture 1. *If G is a vdec-graph, then $\chi'_s(G) \leqslant |V(G)| + 1$.*

Conjecture 2. *Let G be a vdec-graph and let k be the minimum integer such that $\binom{k}{d} \geqslant n_d$ for all d with $\delta(G) \leqslant d \leqslant \Delta(G)$. Then $\chi'_s(G) = k$ or $k + 1$.*

Recently, Bazgan et al. [4] have proved Conjecture 1 and Balister et al. [3] have proved Conjecture 2 when G is a union of cycles or a union of paths.

This paper contains two principal results. The first extends the ideas in a coloring lemma that appears in [4] to prove that any graph G can be given a balanced proper edge coloring by k colors for any $k \geqslant \chi'(G)$ (Theorem 1). *Balanced colorings* are those proper edge colorings of a graph by k colors where $|n_S - n_{S'}| \leqslant 2$ for all subsets S, S' of colors for which $|S| = |S'|$. R. Gould, M. Karonski and F. Pfender informed us that they also proved Theorem 1 in the case of d-regular graphs on n vertices when the number k of colors satisfies the inequality $\binom{k}{d} \geqslant n$. In this case the conclusion is equivalent to $n_S \leqslant 2$ for all S.

The second principal result (Theorem 10) gives good upper bounds on $\chi'_s(G)$ when there are not too many vertices of low degree. As a corollary, this proves Conjecture 2 above when $\delta(G) \geqslant 5$ and $\Delta(G) \geqslant \sqrt{2|V(G)|} + 4$ (Corollary 12).

If a strong coloring of G exists with k colors then clearly $\binom{k}{d} \geqslant n_d$. Hence in Conjecture 2 we certainly have $\chi'_s(G) \geqslant k$ for the k defined there. In fact there are many graphs for which $\chi'_s(G) \geqslant k + 1$. As an example, consider a d-regular graph on $n = \binom{k}{d}$ vertices. Such graphs exist provided dn is even. If a strong coloring with k colors exists, then there must be exactly $\binom{k-1}{d-1}$ vertices incident with any given color c. However, each edge colored with c is incident with two vertices, so this number must be even. There are many pairs (d, k) for which $d\binom{k}{d}$ is even but $\binom{k-1}{d-1}$ is odd (for example if k is a power of 2 and d is arbitrary). For these, the graph will need at least $k + 1$ colors. The same examples also show that the statement $|n_S - n_{S'}| \leqslant 2$ is best possible. Indeed, in any k-coloring some d-tuple will appear at least twice, so some other d-tuple will not appear at all.

A similar parity argument shows that there are many graphs G for which $\chi'_s(G) = |V(G)| + 1$ in Conjecture 1. Take $k = n$ to be even and G to be K_n with r edges removed where $r < n/6$. Every vertex in V_{n-1} must miss a distinct color, so there are n_{n-1} colors which are used on at most $n/2 - 1$ edges each. The other colors may be used on at most $n/2$ edges. Hence $|E(G)| \leqslant n^2/2 - n_{n-1}$. However $n - n_{n-1} \leqslant 2r$, so $|E(G)| \leqslant \binom{n}{2} - n/2 + 2r$. But $|E(G)| = \binom{n}{2} - r$, so $r \geqslant n/2 - 2r$ contradicting our assumption that $r < n/6$.

2. Optimal edge colorings

Define an *optimal* k-coloring of G to be a proper edge coloring of G with k colors with minimal value of $\sum_S n_S^2$. Optimal colorings clearly exist whenever $k \geqslant \chi'(G)$.

Since $\sum n_S = |V(G)|$ is fixed, minimizing $\sum n_S^2$ amounts to reducing the "variance" of n_S. In particular, if two values of n_S are brought closer together (keeping their sum fixed), then $\sum n_S^2$ decreases. We will prove that every optimal coloring is balanced.

Theorem 1. *In any optimal k-coloring of G, $|n_S - n_{S'}| \leqslant 2$ for all $S, S' \subseteq \{1, \ldots, k\}$ with $|S| = |S'|$.*

If t_d^+, t_d^- are integers with $(t_d^- + 1)\binom{k}{d} - 1 \leqslant n_d \leqslant (t_d^+ - 1)\binom{k}{d} + 1$ then Theorem 1 implies that in any optimal k-coloring and for any S with $|S| = d$ we have $t_d^- \leqslant n_S \leqslant t_d^+$. To see this, assume some $n_S > t_d^+$, then $n_d = \sum_{|S'|=d} n_{S'} \geqslant n_S + (\binom{k}{d} - 1)(n_S - 2) \geqslant (t_d^+ - 1)\binom{k}{d} + 2$. A similar argument shows $n_S \geqslant t_d^-$. In particular, if $n_d \leqslant \binom{k}{d} + 1$ for all d then $n_S \leqslant 2$ for all S.

For $S_- \subseteq S_+ \subseteq \{1, \ldots, k\}$, define $[S_-, S_+] = \{S : S_- \subseteq S \subseteq S_+\}$. As a special case, if $a, b \in \{1, \ldots, k\}$ define $[a, \bar{b}] = [\{a\}, \{1, \ldots, k\} \backslash \{b\}]$ to be the subsets of $\{1, \ldots, k\}$ containing a but not b. We shall also write $[a \diamond b] = [a, \bar{b}] \cup [b, \bar{a}]$ for the collection of sets containing precisely one of a and b. Define an involution i_{ab} on subsets of $\{1, \ldots, k\}$ by interchanging the colors a and b.

If $\mathscr{T}$ is a collection of subsets of $\{1, \ldots, k\}$, write $\mathscr{T}_d$ for the sets in $\mathscr{T}$ of size d. Also write $V_{\mathscr{T}} = \bigcup_{S \in \mathscr{T}} V_S$ and $n_{\mathscr{T}} = \sum_{S \in \mathscr{T}} n_S$ for the set of and number of vertices with $S(v) \in \mathscr{T}$, respectively.

Assume we have a proper edge-coloring of G and let $a, b \in \{1, \ldots, k\}$. An (a, b)-*Kempe path* is a maximal path in G consisting only of edges colored with either a or b. The endvertices of an (a, b)-Kempe path lie in $V_{[a \diamond b]}$ and we can define a perfect matching K on $V_{[a \diamond b]}$ by setting $uv \in E(K)$ whenever there exists an (a, b)-Kempe path with endvertices u and v.

We now make the following key observation. If $uv \in E(K)$ then we can swap the colors a and b along the Kempe path between u and v. The result will be a new proper coloring with the same color sets $S(w)$ for all $w \neq u, v$. The only color sets to change are $S(u)$ and $S(v)$ which both change by swapping the colors a and b. In other words $S(u)$ and $S(v)$ will be changed to $i_{ab}S(u)$ and $i_{ab}S(v)$, respectively.

Let G_k be the graph with vertex set equal to the set of all subsets S of $\{1, \ldots, k\}$. The edges of G_k consist of all pairs SS' with $|S| = |S'|$ and S and S' differing by the change of one color. In other words, $SS' \in E(G_k)$ iff $S' \neq S$ and $S' = i_{ab}S$ for some a and b. The graph G_k is the vertex-disjoint union of the connected subgraphs $G_{k,d}$

consisting of the subsets of size d. If $|S| = |S'|$, write $d(S, S')$ for the distance between S and S' in G_k. We shall often consider $[S_-, S_+]$ to be an induced subgraph of G_k. It is worth noting that $[S_-, S_+]$ is isomorphic to G_r and $[S_-, S_+]_d$ is isomorphic to $G_{r,s}$ where $r = |S_+| - |S_-|$ and $s = d - |S_-|$. In particular, $[S_-, S_+]_d$ is connected (if non-empty).

Imagine the vertices of G_k as the squares on a board and the vertices v of G as "pieces" placed on the squares $S(v)$ of this board. The edges of G_k are the valid "moves" we can make in a game based on interchanging colors in Kempe paths. If we move a piece u from S to $i_{ab}S$, then our "opponent" moves a piece v from S' to $i_{ab}S'$, where $S = S(u)$, $S' = S(v)$ and u and v are the endpoints of an (a, b)-Kempe path. We will generally regard the pieces on the board as being indistinguishable, and will not concern ourselves if two pieces are swapped.

Lemma 2. *Assume we have an optimal k-coloring of G and $S \in [a \diamond b]$. Then we can change the coloring by interchanging a and b on some edges so that we get a new optimal coloring in which the values of n_S and $n_{i_{ab}S}$ are interchanged. Also every other $n_{S'}$ remains the same, except in the case when n_S and $n_{i_{ab}S}$ differ by one, in which case exactly one other pair $n_{S'}$ and $n_{i_{ab}S'}$ (differing by one) is interchanged. Also, we can make each pair $\{S, i_{ab}S\}$ lead to a different pair $\{S', i_{ab}S'\}$. It is not possible for n_S and $n_{i_{ab}S}$ to differ by more than two.*

Proof. Construct a partial matching J on $V_{[a \diamond b]}$ as follows. For each $S \in [a \diamond b]$ arbitrarily match as many vertices in V_S with those of $V_{i_{ab}S}$ as possible. The partial matching J will just be the union of all these partial matchings. Note that if u is unmatched by J then $n_{S(u)} > n_{i_{ab}S(u)}$. As above we define the matching K so that $uv \in E(K)$ when u and v are endvertices of an (a, b)-Kempe path.

In terms of the board game, J and K determine the strategy used by us and our opponent respectively. If we move a piece u then our opponent moves v with $uv \in E(K)$. If our opponent moves v then we move w where $vw \in E(J)$. We stop when we hit an unmatched vertex. Every move by our opponent corresponds to interchanging the coloring along some (a, b)-Kempe path. We now consider the effect of all these moves.

The union $J \cup K$ is a multigraph on $V(G)$ with maximum degree two. Start with a vertex $v_0 \in V_{[a \diamond b]}$ which is not in the partial matching J (and hence is of degree one in $J \cup K$). For each $i \geqslant 0$, change the coloring of the Kempe path from v_{2i} to v_{2i+1} as above where v_{2i+1} is defined so that $v_{2i}v_{2i+1} \in E(K)$. If v_{2i+1} occurs in the matching J define v_{2i+2} so that $v_{2i+1}v_{2i+2} \in E(J)$ then repeat this process with i replaced with $i + 1$. Since $J \cup K$ is a union of paths and cycles (and double edges) and we started at a degree one vertex, we must terminate at a vertex $v_r \neq v_0$ which is not in the partial matching J.

Now consider the change in n_S for each S. It is clear that whenever we change the coloring along the Kempe path with endvertices v_{2i}, v_{2i+1}, we change $S(v_{2i})$ and $S(v_{2i+1})$ to $i_{ab}S(v_{2i})$ and $i_{ab}S(v_{2i+1})$, respectively. However, $S(v_{2i+1}) = i_{ab}S(v_{2i+2})$, so the only n_S to change are $n_{S(v_0)}$, $n_{S(v_r)}$, $n_{i_{ab}S(v_0)}$ and $n_{i_{ab}S(v_r)}$. Since distinct starting points of paths lead to distinct endpoints of paths in $J \cup K$, each v_0 corresponds to a distinct v_r, (although the correspondence between v_0 and v_r depends on our choice of J). It is worth noting that this process is symmetric in v_0 and v_r; if we had started with v_r we would have ended with v_0.

Now assume the coloring is optimal. If $n_S = n_{i_{ab}S}$ then there is nothing to prove, so assume $n_S > n_{i_{ab}S}$. Since $n_S > n_{i_{ab}S}$, there must be a vertex v_0 with $S(v_0) = S$ and v_0 not in the partial matching J. Applying the algorithm above gives us a change of coloring which changes only n_S, $n_{S'}$, $n_{i_{ab}S}$ and $n_{i_{ab}S'}$ where $S' = S(v_r)$ and $n_{S'} > n_{i_{ab}S'}$ (since v_r is not in the partial matching J). In particular $S \neq i_{ab}S'$.

Assume first that $S \neq S'$. Then n_S and $n_{S'}$ decrease by one and $n_{i_{ab}S}$ and $n_{i_{ab}S'}$ increase by one. If either $n_S \geqslant n_{i_{ab}S} + 2$ or $n_{S'} \geqslant n_{i_{ab}S'} + 2$ then $\sum n_S^2$ is reduced, contradicting optimality. Hence $n_S = n_{i_{ab}S} + 1$ and $n_{S'} = n_{i_{ab}S'} + 1$ and both pairs are interchanged. Different pairs $\{S, i_{ab}S\}$ give different pairs $\{S', i_{ab}S'\}$ since only one vertex in V_S or $V_{S'}$ is unmatched by J. Note also that the new coloring is still optimal since it has the same value of $\sum n_S^2$.

Now assume $S = S'$, so that n_S is decreased by two and $n_{i_{ab}S}$ is increased by two. Hence there are two vertices in V_S unmatched by J and so $n_S \geqslant n_{i_{ab}S} + 2$. If $n_S > n_{i_{ab}S} + 2$ then $\sum n_S^2$ is decreased contradicting optimality, otherwise $n_S = n_{i_{ab}S} + 2$ and the two values n_S and $n_{i_{ab}S}$ are interchanged. All other n_S are unaffected. Once again, the coloring remains optimal. □

Note that the pairing $\{S, i_{ab}S\}$ to $\{S', i_{ab}S'\}$ in Lemma 2 depends on the choice of J. However, for any fixed choice of J we get a matching on the set of all pairs $\{S, i_{ab}S\}$ with $|n_S - n_{i_{ab}S}| = 1$.

From now on we shall simplify the game analogy by assuming all the intermediate moves in Lemma 2 are taken for granted. The moves of the game now correspond to swapping n_S and $n_{i_{ab}S}$ with our opponent only allowed to move if $|n_S - n_{i_{ab}S}| = 1$. In this case he swaps some other pair $n_{S'}$ and $n_{i_{ab}S'}$ with $|n_{S'} - n_{i_{ab}S'}| = 1$.

Proof of Theorem 1. Suppose the theorem is false. Among all optimal k-colorings and sets S_1, S_2 with $|S_1| = |S_2|$, $n_{S_1} \geqslant n_{S_2} + 3$, pick an optimal coloring and sets S_1, S_2 with S_1 and S_2 as close as possible in G_k. Let $d = |S_1| = |S_2|$, $S_- = S_1 \cap S_2$ and $S_+ = S_1 \cup S_2$ so that $S_1, S_2 \in [S_-, S_+]_d$ and $S_2 = S_1 \Delta (S_+ \backslash S_-)$. For all $S, S' \in [S_-, S_+]_d$, $d(S, S') = \frac{1}{2}|S \Delta S'|$, so if $S \neq S_1, S_2$ then the distances $d(S, S_1)$ and $d(S, S_2)$ are strictly less than $d(S_1, S_2)$. Hence by minimality of $d(S_1, S_2)$ we have $n_{S_1} > n_S > n_{S_2}$. Assume $a \in S_1$, $a \notin S_2$. Using Lemma 2 we can swap n_{S_1} with any neighboring n_S, that is any n_S with $S_1 S \in E(G_k)$. If $S \in [S_- \cup \{a\}, S_+]_d$ then S is closer to S_2 than S_1, so by minimality of $d(S_1, S_2)$, n_{S_2} must also be swapped with the neighboring $n_{S'}$ with $S' = S \Delta (S_+ \backslash S_-) \in [S, S_+ \backslash \{a\}]_d$. In particular $|n_S - n_{S_1}| = 1$. For any $S \in [S_- \cup \{a\}, S_+]_d$ with $S \neq S_1$, we can move n_{S_1} along a path between them in $[S_- \cup \{a\}, S_+]_d$ (which is a connected subgraph of G_k) and eventually swap n_{S_1} with n_S. Since our opponent is always making moves in $S, S_+ \backslash \{a\}_d$, n_S is unaffected by any of these moves. However swapping n_{S_1} with n_S must also cause our opponent to move S_2, so $|n_S - n_{S_1}| = 1$ by Lemma 2. Since any $S \neq S_2$ is in $[S_- \cup \{a\}, S_+]_d$ for some a, this argument applies for all $S \neq S_1, S_2$. A similar argument shows that $|n_S - n_{S_2}| = 1$ for any $S \neq S_1, S_2$. Hence no such S exists and $[S_-, S_+]_d = \{S_1, S_2\}$. In this case S_1 and S_2 are neighbors in G_k and $|n_{S_1} - n_{S_2}| > 2$, contradicting Lemma 2. □

3. Distinguishing vertices

Our aim is to produce vertex-distinguishing colorings of arbitrary graphs. Our strategy will be to color G so that it is almost vertex-distinguishing, in that each color set that occurs more than once contains certain specified colors. We then recolor the edges of G colored with these, using one or two additional colors so as to ensure all vertices are distinguished. We now prove some technical results needed to do this recoloring. In this section G will correspond to the subgraph of edges colored with the two or three specified colors, and the matching or equivalence relation will define the sets of vertices not distinguished by the remaining colors in our original graph.

Lemma 3. *Let G be a vertex disjoint union of cycles and M a partial matching on the vertices of G (which is otherwise unrelated to G). Assume M does not match at least one vertex on each odd cycle of G. Then there exists a proper edge coloring of G with four colors such that $S(u) \neq S(v)$ whenever $uv \in E(M)$. If G contains no component 6-cycle in which all pairs of opposite vertices are matched by M, then three colors are sufficient.*

Proof. For each component 6-cycle $v_1 \dots v_6$ of G with opposite vertices matched, remove v_1v_4 and v_2v_5 from $E(M)$. If we can color the resulting graph with three colors, then we can color the original by recoloring v_1v_2 with the fourth color. Hence the first part follows from the second.

If there is a cycle $v_1 \dots v_r$, $r \geqslant 4$ with two adjacent vertices matched, say $v_1v_r \in E(M)$, identify the vertices v_1 and v_r and color the remaining graph with $v_1 = v_r$ unmatched. We can then color the original graph by giving edge v_1v_r the color different to those of v_1v_2 and v_rv_{r-1} (which are distinct). If $r = 3$ then we can just remove $v_1v_r = v_1v_3$ from the matching, since v_1v_2 and v_3v_2 automatically have distinct colors in any proper coloring. Hence, we are reduced to proving the result when no adjacent vertices are matched by M.

We color each cycle in turn. Suppose we have already colored some cycles and the next cycle to be colored is $v_1 \dots v_r$. Let u_i be the vertex matched by M to v_i (when it exists). We will call v_i *restricted* if u_i exists and both edges adjacent to u_i have already been colored. Otherwise call v_i *unrestricted*.

Assume first that v_1 is not matched to another vertex in the cycle $v_1 \dots v_r$. If v_1 or v_r is unrestricted or if $S(u_1) \neq S(u_r)$ then we shall color the cycle as follows. Pick some color a in $S(u_r)$ ($\{1, 2, 3\}$ if v_r unrestricted) but not in $S(u_1)$ ($\emptyset$ if v_1 unrestricted). If v_1 and v_r are both restricted then by assumption $S(u_1) \neq S(u_2)$ and $|S(u_1)| = |S(u_2)| = 2$ so such a color exists. Color v_1v_2 with a. This will guarantee $S(v_1) \neq S(u_1)$ in any final coloring if v_1 is restricted. Now color each edge around the cycle in turn so that the coloring is proper and $S(v_i) \neq S(u_i)$ when v_i is restricted. At each stage, two of the three colors will make the coloring proper and we need to avoid at most one of these to ensure $S(v_i) \neq S(u_i)$. It is possible that u_r may be on the current cycle ($u_r = v_i$ for some i). If this happens, make sure that the edge $v_i v_{i+1}$ is colored so that $a \in S(v_i)$. This is possible since $v_i = u_r$ is unrestricted when we color $v_i v_{i+1}$ and so we can color $v_i v_{i+1}$ with a when $v_{i-1} v_i$ is not colored with a. Now assume we have colored all the edges except the last one $v_r v_1$. If v_r is restricted then $a \in S(u_r)$. Color $v_r v_1$ with any color not equal to a, distinct from the color of $v_{r-1} v_r$, and so that $S(v_r) \neq S(u_r)$. Since $a \in S(u_r)$, one of these last two conditions is redundant and so such a color always exists. We now have a suitable coloring of the whole cycle.

If v_r is restricted and $S(u_1) = S(u_r)$ then v_r is also not matched to another vertex in the cycle $v_1 \ldots v_r$. Hence if the above argument does not apply for any vertex of the cycle then either all the vertices are matched to other vertices on the same cycle, or all the v_i are restricted and all the $S(u_i)$ are the same, say $S(u_i) = \{a, b\}$. In the second case there are no unmatched vertices in the cycle, so the cycle length is even and we can color the edges of the cycle alternately a and c where $c \neq a, b$.

Now we may assume all the vertices v_i are matched by M with other vertices on the same cycle. Let $l > 0$ be the minimum value such that $v_i v_{i+l} \in E(M)$ for some i (where we consider $i + l$ as defined mod r). Assume first that $l < r/2$. Not every v_i can be matched with v_{i+l} since then v_l would be matched to both $v_0 = v_r$ and $v_{2l} \neq v_r$. By cyclically permuting the numbering, we can assume $v_1 v_{1+l} \in E(M)$ but $v_r v_l \notin E(M)$. Since v_r must be matched to some v_s we must have $v_r v_s \in E(M)$ and $s > l + 1$ by minimality of l.

Color $v_1 v_2$ with color 1, and proceed to color each edge in turn as before. By minimality of l, all the v_i with $1 \leqslant i \leqslant l$ are matched with v_j with $j > l$ and hence are unrestricted when we color $v_i v_{i+1}$. Since $l > 1$ we can color the edges $v_2 v_3, \ldots, v_l v_{l+1}$ so that $v_l v_{l+1}$ is colored with color 2. Now color $v_{l+1} v_{l+2}$ with color 3 (so $1 \notin S(v_{l+1})$). Now proceed with the remaining edges as before. When we reach $u_r = v_s$, color $v_s v_{s+1}$ so that $1 \in S(u_r)$ (possible since v_s is unrestricted). Continue until we have colored every edge except $v_r v_1$. As before, we can color $v_r v_1$ with a color other than 1 since $1 \in S(u_r)$. Also $S(u_1) \neq S(v_1)$ since $1 \notin S(v_{l+1})$, $1 \in S(v_1)$, so we are done.

Finally, if each v_i is matched to v_{i+l} and $l = r/2$, then we assume $r \neq 6$. If $r \equiv 0 \bmod 4$, color the edges $1, 2, 1, 2, 1, \ldots, 2$ for the first half of the cycle from v_1 to v_{l+1} and then $1, 3, 1, 3, 1, \ldots, 3$ for the other half going from v_{l+1} to v_1. If $r \equiv 2 \bmod 4$ and $r \geqslant 10$, color them $1, 2, 3, 1, 2, 1, 2, 1, \ldots, 2$ for the first half of the cycle and then $1, 3, 1, 2, 3, 1, 3, 1, \ldots, 3$ for the other half. □

Corollary 4. *Assume G has a proper edge coloring with two colors. Assume also that $\sim$ is an equivalence relation on $V(G)$ such that $u \sim v$ and $S(u) = S(v)$ implies $|S(u)| = 2$ and no equivalence class of $\sim$ has more than two elements of degree 2. Then there exists a proper 4-coloring with $S(u) \neq S(v)$ for all $u \sim v$. Also, if G does not contain a 6-cycle in which $u \sim v$ for all diagonal pairs, then three colors are sufficient.*

Proof. Since G has a proper 2-coloring, $\Delta(G) \leqslant 2$ and G is a union of paths and even cycles. Without loss of generality, we can assume $u \sim v$ implies that u and v have the same degree and that if this degree is one then u and v are not endvertices of the same path of length 2 (otherwise $S(u) \neq S(v)$ is automatic in any proper coloring). Note that any $\sim$-equivalence class contains at most two vertices, so $\sim$ is a partial matching on $V(G)$. If $u \sim v$ and u and v are of degree 1, identify u and v. Any proper coloring of the resultant graph will give a coloring of G in which $S(u) \neq S(v)$. The original 2-coloring is still proper on the graph with u and v identified since $S(u) \neq S(v)$ in the original coloring with two colors. Finally, if P is a path with endpoints not related to anything by $\sim$, we can enlarge P to an even cycle by adding new vertices and edges (with any new vertices unrelated to any other vertex by $\sim$). Applying Lemma 3 to the resulting union of even cycles and removing any added vertices and edges gives the result. $\square$

Lemma 5. *Let H be a 6-cycle $v_1 \dots v_6$ with six extra vertices $u_1, \dots, u_6$ and edges $v_i u_i$ (see Fig. 1). Assume $v_i u_i$ are colored with colors in $\{1,2,3,4\}$ and the color of $v_i u_i$ is distinct from the color of $v_{i+3}u_{i+3}$ for $i = 1,2,3$. Then we can extend this coloring to a proper edge coloring of H so that $S(v_i) \neq S(v_{i+3})$ for $i = 1,2,3$. In addition, we can remove some or all of the pairs u_i, u_{i+3} and replace them with edges $v_i v_{i+3}$ (initially uncolored) and the result still holds.*

Proof. The proof is a case by case check (and was verified by computer). Indeed, the only coloring sequences on the edges $v_i u_i$ which do not lead to a suitable coloring of H are those isomorphic under cyclic permutations, reflections and interchange of colors to one of the sequences 111111, 112112, 112113 and 112122. Each of these has a pair of opposite edges colored with the same color. The following table gives suitable colorings of the hexagon in the other cases. The subscripts denote the colors of the edges $v_i v_{i+1}$ between the corresponding edges $v_i u_i$ and $v_{i+1} u_{i+1}$. For the second part of the lemma, a diagonal $u_i u_{i+3}$ is equivalent to $v_i u_i$ and $v_{i+3} u_{i+3}$ colored with the same color. Since we

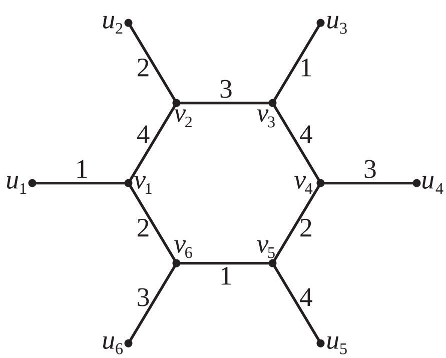

Fig. 1. Example coloring ${}_2 1_4 2_3 1_4 3_2 4_1 3_2$.

can choose this color arbitrarily, we can avoid the four special cases above.

$$\begin{array}{cccccc}
{}_3 1_2 1_3 1_4 1_2 1_4 2_3 & {}_3 1_2 1_3 1_4 1_3 2_4 2_3 & {}_2 1_3 1_2 1_4 1_3 2_4 3_2 & {}_3 1_2 1_3 1_4 2_3 1_4 2_3 & {}_2 1_3 1_2 1_4 2_3 1_4 3_2 & {}_3 1_2 1_3 1_4 2_1 2_4 2_3 \\
{}_2 1_3 1_2 1_4 2_1 2_4 3_2 & {}_3 1_2 1_3 1_4 2_1 3_4 2_3 & {}_2 1_3 1_2 1_3 2_4 3_1 4_2 & {}_2 1_3 1_4 2_3 1_4 2_1 3_2 & {}_3 1_2 1_4 2_3 1_4 3_1 2_3 & {}_4 1_2 1_4 2_3 1_2 3_1 3_4 \\
{}_3 1_2 1_4 2_3 1_4 3_1 4_3 & {}_4 1_2 1_4 2_3 2_4 3_1 3_4 & {}_3 1_2 1_3 2_1 2_4 3_1 4_3 & {}_2 1_3 1_4 2_1 3_4 2_1 3_2 & {}_2 1_3 1_4 2_1 3_4 2_3 4_2 & {}_3 1_2 1_4 2_1 3_4 3_1 2_3 \\
{}_2 1_4 1_3 2_1 3_2 3_1 4_2 & {}_3 1_2 1_3 2_4 3_2 4_1 2_3 & {}_3 1_4 2_3 1_4 2_3 1_4 2_3 & {}_2 1_4 2_3 1_4 2_3 1_4 3_2 & {}_2 1_3 2_4 1_3 2_4 3_1 4_2 & {}_2 1_3 2_4 1_2 3_4 1_3 4_2 \\
{}_2 1_3 2_4 1_2 3_4 2_1 3_2 & {}_2 1_4 2_3 1_2 3_1 2_3 4_2 & {}_2 1_4 2_3 1_4 3_2 4_1 3_2 & {}_2 1_3 2_1 3_4 1_3 2_4 3_2 & {}_2 1_3 2_1 3_2 1_4 2_1 4_2 &
\end{array} \qquad \square$$

Theorem 6. *Assume G has a proper edge coloring with three colors. Assume also that $\sim$ is an equivalence relation on $V(G)$ such that $u \sim v$ and $S(u) = S(v)$ implies $|S(u)| = 3$ and no equivalence class of $\sim$ contains more than two vertices of degree 3. Then there exists a proper 4-coloring with $S(u) \neq S(v)$ for all $u \sim v$.*

Proof. Assume G is colored with the three colors $\{1, 2, 4\}$. We may assume, without loss of generality, that $u \sim v$ implies $|S(u)| = |S(v)|$. As a consequence, $\sim$-equivalence classes of vertices of degrees 0, 1, 2 and 3 can contain at most 1, 3, 3 and 2 vertices respectively. We define a *bad hexagon* to be a 6-cycle $v_1 \dots v_6$ in G which is colored alternately 1 and 2 and for which opposite vertices are equivalent under $\sim$. In this case, each v_i is of degree 3 and is adjacent to some vertex u_i with $v_i u_i$ colored with 4. If h is a bad hexagon, write the vertices of h as $V(h) = \{v_1^h, \dots, v_6^h\}$ and the vertices adjacent to h as $U(h) = \{u_1^h, \dots, u_6^h\}$. Note that any two bad hexagons are vertex-disjoint.

Let $U = (\bigcup_h U(h)) \setminus (\bigcup_h V(h))$ be the set of all vertices adjacent to bad hexagons but which are not in any bad hexagon themselves. Since $4 \in S(u)$ for all $u \in U$, the restriction of $\sim$ to U is a partial matching (no three subsets of $\{1, 2, 4\}$ containing 4 have the same size).

Define the subgraph G' of G by removing the edges $E(h)$ of all the bad hexagons from G and define the relation $\sim'$ by removing all $\sim$-equivalences on the vertices $V(h)$ of all the bad hexagons. Now the $v \in V(h)$ are all unmatched vertices of degree 1. If we also remove all the edges colored 4, the resulting graph G'' and relation $\sim'$ satisfy the conditions of Corollary 4 with no bad hexagons. Hence G'' can be recolored with $\{1, 2, 3\}$ so that $u \sim' v$ implies $S(u) \neq S(v)$ in G''. If we add back the edges colored with 4 we get a proper 4-coloring of G' in which $u \sim' v$ implies $S(u) \setminus \{4\} \neq S(v) \setminus \{4\}$ and hence $S(u) \neq S(v)$. This however is not the only such 4-coloring on G'.

For each vertex $u \in U$, let $e_u = vu$ be the (unique) edge joining u to a bad hexagon. This edge is colored 4 in the above coloring of G'. We shall show that we can recolor it with some other color so that G' is still properly colored and $u \sim' u'$ still implies $S(u) \neq S(u')$.

If u has degree 1, then it is $\sim'$-related to at most two other vertices in G'. Hence there is a color $k_u \neq 4$ which can be used to color e_u so that the $S(u) \neq S(u')$ when $u \sim' u'$. The other endvertex of e_u is an unmatched vertex of degree 1, so imposes no restriction on the coloring of e_u. Similarly, if the degree of u in G' is 2 then it is related to at most one vertex v with $4 \notin S(v)$. This is because in the original coloring of G

there is only one subset of $\{1,2,4\}$ of size two not containing 4. We can therefore recolor e_u with some color $k_u \in \{1,2,3\}$ so that $S(u) \neq S(u')$ when $u \sim' u'$ and the coloring is still proper at u. Finally, if u is of degree 3, then it is $\sim'$-related to at most one other vertex u'. In the original coloring $S(u') = \{1,2,4\}$ so $4 \in S(u')$ in the G' coloring. Hence we can recolor e_u with $k_u \in \{1,2,3\} \backslash S(u)$ so that the coloring is still proper and $S(u) \neq S(u')$.

Given any subset $U' \subseteq U$ of vertices of U which does not contain any pair of $\sim'$-related elements, we can recolor all the edges e_u, $u \in U'$ with k_u simultaneously. As long as we do not recolor both e_u and $e_{u'}$ when $u \sim' u'$ then the argument above still holds and we have a proper 4-coloring of G' in which $u \sim' v$ implies $S(u) \neq S(v)$ for all u and v.

Now consider the bad hexagons. Construct a multigraph H with loops as follows. The vertices of H will be obtained from $(\bigcup_h V(h)) \cup U$ by identifying pairs of opposite vertices in the bad hexagons. The edges of H will be the edges $v_i^h u_i^h$ originally colored 4 in G which meet the bad hexagons. The graph H may contain loops and multiple edges. The vertices $u \in U$ will be of degree 1 in H and the pairs of opposite vertices of the bad hexagons will be of degree 2. Also the vertices of U are adjacent only to vertices of the bad hexagons. Hence H decomposes as a vertex disjoint union of paths, cycles, double edges and single vertex loops. All the paths are of length at least two and have endpoints in U. Proper color the edges of H with $\{1,2,3,4\}$ as follows. Starting at some $u \in U$ color the first edge in the path starting at u with 4, then color subsequent edges so as to make the coloring proper. When we get to the other endvertex u' of this path, make sure the last edge is colored either 4 or $k_{u'}$. Repeat with each path in turn, starting with u'' where $u'' \sim' u'$ if possible. Eventually, all the paths will be colored in such a way that if $u, u' \in U$ and $u \sim' u'$ then at least one of $S(u)$ or $S(u')$ will be $\{4\}$. Now give every cycle and double edge an arbitrary proper coloring. The loops will remain uncolored for now.

Putting the colorings of G' and H together, we get a coloring on all the edges of G except the edges of the bad hexagons and any diagonal edges of these hexagons (the loops in H). We can match up the colorings on the edges e_u since they are colored either 4 or k_u in H and either color is acceptable in G'. If $e_{u'}$ is another such edge and $u \sim u'$ then $u \sim' u'$ and so at least one of these edges is colored 4 in H. We therefore have a partial coloring of G with $S(u) \neq S(v)$ whenever $u \sim v$ and u and v are not in bad hexagons.

It remains to color the bad hexagons themselves. If h is a bad hexagon then h corresponds to three vertices of H. If h has r diagonal edges in G, $0 \leqslant r \leqslant 3$, then r of the vertices in h are isolated loops and the remaining $3-r$ have two edges colored with distinct colors incident with them. In terms of G, this means that the pairs of opposite vertices in the bad hexagon that are not joined by a single edge have one edge incident with each vertex of the pair which is already colored, and these colors are distinct. By Lemma 5 we can complete the coloring of h and any diagonal edges so that opposite vertices see distinct color sets. We have now colored the whole of G as required. $\square$

4. Vertex-distinguishing colorings

Assume $k \geqslant \chi'(G)$ and $n_d \leqslant \binom{k}{d} + 1$ for all d. We can apply Theorem 1 to get a coloring in which each color set is used at most twice ($n_S \leqslant 2$ for all $S \subseteq \{1, \dots, k\}$). Such a coloring will be called *semi-vertex-distinguishing*. We aim to use each color set at most once. We say a vertex v of G is *bad* if $n_{S(v)} = 2$. An optimal k-coloring is now just a semi-vertex-distinguishing coloring of G with k colors and minimal number of bad vertices.

Considering G_k as a board and the vertices $v \in V(G)$ as pieces, the bad vertices are just the pieces that occur in piles of height two on some square (2-piles). The good vertices are those pieces that occur in 1-piles.

Our strategy is to move the 2-piles around so that they all end up on squares S that contain some specified colors. Write $[a, -] = [\{a\}, \{1, \dots, k\}]$ and $[-, \bar{a}] = [\emptyset, \{1, \dots, k\} \backslash \{a\}]$ for the sets that contain a and do not contain a, respectively. By Lemma 2 we can swap any pair of neighboring piles. If we push a 2-pile onto an empty square (0-pile), no other piles will move. However, if we swap a 2-pile with a 1-pile or move a 1-pile onto an empty square, then our opponent will swap some other piles as well. It is therefore important that when we move a 2-pile from $[-, \bar{a}]$ to $S \in [a, -]$ that this square S was previously empty, otherwise some other 2-pile may move from $[a, -]$ back to $[-, \bar{a}]$. The following lemma shows that if we have enough 0-piles in $[a, -]$ then we can move 0-piles in $[a, -]$ so that some are adjacent to any given 2-pile in $[-, \bar{a}]$. We can then move this 2-pile into $[a, -]$. Repeating this process for each 2-pile in turn allows us to move all the 2-piles into $[a, -]$. We actually prove a "relative" version that works in $[S_-, S_+]$ rather than just G_k since we shall need this to move the 2-piles onto squares that contain several specified colors.

Let $o_{\mathscr{T}}$, $g_{\mathscr{T}}$ and $b_{\mathscr{T}}$ be the number of sets $S \in \mathscr{T}$ with $n_S = 0$, 1 and 2, respectively. Note that $|\mathscr{T}| = o_{\mathscr{T}} + g_{\mathscr{T}} + b_{\mathscr{T}}$ and $n_{\mathscr{T}} = g_{\mathscr{T}} + 2b_{\mathscr{T}}$. Define an *S-recoloring* of an optimal coloring of G to be any optimal coloring of G obtained by a sequence of applications of Lemma 2 with $a, b \in S$ (so that only colors in S are changed). If $\mathscr{T}$ is i_{ab}-invariant for all $a, b \in S$ (for example if $\mathscr{T} = [S_-, S_+]_d$ with $S \subseteq S_+ \backslash S_-$) then the quantities $o_{\mathscr{T}}$, $g_{\mathscr{T}}$ and $b_{\mathscr{T}}$ are the same in any S-recoloring.

Lemma 7. *Assume $S_- \subseteq S_+$, $a \in S_+ \backslash S_-$ and $S_0 \in [S_-, S_+ \backslash \{a\}]_d$. Given an optimal k-coloring of G such that $n_{S_0} = 2$ and $o_{[S_- \cup \{a\}, S_+]_d} \geqslant \frac{1}{d - |S_-| - 1} \binom{|S_+| - |S_-| - 1}{d - |S_-| - 1} - 1$ then we can find an $(S_+ \backslash (S_- \cup \{a\}))$-recoloring such that $n_{S_0} = 2$ and $n_S = 0$ for some $S \in [S_- \cup \{a\}, S_+]$ with $d(S, S_0) = 1$.*

Proof. For $i > 0$ let $\mathscr{T}_i = \{S \in [S_- \cup \{a\}, S_+]_d \colon d(S, S_0) = i\}$ be the number of sets containing the color a at distance i from S_0 in $[S_-, S_+]$. Write $O_i = \{S \in \mathscr{T}_i \colon n_S = 0\}$ for the set of 0-piles in $\mathscr{T}_i$ and $o_i = o_{\mathscr{T}_i} = |O_i|$ for the number of such 0-piles. As a special case write $O_0 = \mathscr{T}_0 = \{S_0\}$. Write $r = |S_+| - |S_-|$ and $s = d - |S_-|$. Then by assumption

$$\sum_{i=1}^{s} o_i \geqslant \frac{1}{s-1}\binom{r-1}{s-1} - 1. \tag{1}$$

Among all $(S_+\backslash S_-\backslash\{a\})$-recolorings with $n_{S_0} = 2$, choose those with maximal o_1; and among these, choose the ones with maximal o_2, and so on. If $o_1 > 0$ then we are done, so assume $o_1 = 0$.

If $S' = i_{bc}S$ for some $b, c \in S_+\backslash S_-\backslash\{a\}$, call S' an *improvement* of S if $d(S', S_0) < d(S, S_0)$ and call S' a *worsening* of S if $d(S', S_0) > d(S, S_0)$.

Now assume $S \in O_j$, $j \geqslant 2$ and $S' = i_{bc}S$ is an improvement of S. If $n_{S'} = 2$ we could move the 2-pile on S' to S by Lemma 2 and get a better coloring, hence $n_{S'} < 2$. If $n_{S'} = 1$ we can move the 1-pile on S' to S by Lemma 2. By the choice of our coloring, we must get a coloring not better than the original one. Hence there exists an $i < j$ and some $S'' \in O_i$ such that the recoloring moves a 0-pile (2-pile if $i = 0$) from S'' to $i_{bc}S''$, which is a worsening of S''. In this case we say that the improvement $S' = i_{bc}S$ of S *matches* the worsening $i_{bc}S''$ of S''. If $n_{S'} = 0$ we say the improvement S' of S matches the worsening S of S'. Hence every improvement of S is matched with some worsening of some $S'' \in O_i$ with $0 \leqslant i < j$. Moreover, distinct improvements are matched with distinct worsenings.

For every $0 \leqslant i < j \leqslant d$, let x_{ij} denote the number of worsenings in O_i matched with improvements in O_j. For $i > 0$, every $S \in O_i$ has $(i-1)i$ improvements and $(s-i)(r-s-i)$ worsenings. Moreover, S_0 has $s(r-s-1)$ worsenings. Hence we have:

$$\sum_{i=0}^{j-1} x_{ij} = o_j j\,(j-1), \quad j = 2, \ldots, s, \tag{2}$$

$$\sum_{j=i+1}^{s} x_{ij} \leqslant o_i(s-i)(r-s-i), \quad i = 2, \ldots, s-1, \tag{3}$$

$$\sum_{j=2}^{s} x_{0j} \leqslant s(r-s-1) \quad \text{and} \quad o_1 = x_{1i} = 0. \tag{4}$$

We shall now find the maximum of $\sum_{i=1}^{s} o_i$ under the linear restrictions (2)–(4) assuming only that o_j and x_{ij} are non-negative real numbers. Assume we have a solution of (2)–(4) with $\sum o_j$ maximal and assume that $x_{i,j+1} > 0$ for some i, j with $j > i$ (or $j > 1$ if $i = 0$).

Case 1: For all $h > j+1$, $x_{j+1,h} = 0$. Choose $\varepsilon > 0$, increase x_{ij} by $\varepsilon j(j-1)$, and reduce $x_{i,j+1}$ by $\varepsilon j(j-1)$ keeping $x_{i,j+1}$ non-negative.

Case 2: There exists $h > j+1$ with $x_{j+1,h} > 0$. Choose $\varepsilon > 0$, increase x_{ij} by $\varepsilon j(j-1)$, increase x_{jh} by $\varepsilon(s-j)(r-s-j)$, reduce $x_{i,j+1}$ by $\varepsilon j(j-1)$, and reduce $x_{j+1,h}$ by $\varepsilon(s-j)(r-s-j)$ keeping all of these non-negative.

To make (2) hold, o_j must increase by ε and o_{j+1} must reduce by $\frac{j-1}{j+1}\varepsilon$. Eq. (4) is unaffected, so it remains to check (3). These are unaltered except when i is replaced by j and $j+1$. For j both sides are increased by $\varepsilon(s-j)(r-s-j)$ in case 2 and the right-hand side only is increased in case 1. For $j+1$ in case 2 the left-hand side is reduced by $\varepsilon(s-j)(r-s-j)$ and the right-hand side reduces by $\varepsilon\frac{j-1}{j+1}(s-j-1)(r-s-j-1)$ which is less. In case 1, the original inequality is strict since the left side is zero and $o_{j+1}\geqslant x_{i,j+1}/j(j+1)>0$. Hence for sufficiently small ε it remains true. Now (2)–(4) all hold with $\sum o_j$ increased by $\frac{2\varepsilon}{j+1}$, contradicting the maximality of $\sum o_j$.

The only non-zero x_{ij} are therefore x_{02} and $x_{i,i+1}$ for $i\geqslant 2$. This implies that the maximum $\sum o_j$ is attained for

$$o_2 = \frac{s(r-s-1)}{2} \quad \text{and} \quad o_{i+1} = o_i\frac{(s-i)(r-s-i)}{i(i+1)} \quad \text{for } i=2,\dots,s-1.$$

By induction we conclude that in this case,

$$o_i = \frac{1}{s-1}\binom{s}{s-i}\binom{r-s-1}{i-1} \quad \text{for } i=2,\dots,s-1.$$

It follows that in any case,

$$\sum_{i=2}^{s} o_i \leqslant \frac{1}{s-1}\sum_{i=2}^{s}\binom{s}{s-i}\binom{r-s-1}{i-1} = \frac{1}{s-1}\left\{\binom{r-1}{s-1}-s\right\}$$
$$< \frac{1}{s-1}\binom{r-1}{s-1}-1,$$

a contradiction to (1). □

Lemma 8. *Given an optimal k-coloring of G, sets $S_-\subseteq S_+$ and $c\in S_+\backslash S_-$, we can find an $(S_+\backslash S_-)$-recoloring such that for all d,*

$$g_{[S_-\cup\{c\},S_+]_d} \leqslant \left\lceil \frac{d-|S_-|}{|S_+|-|S_-|}\, g_{[S_-,S_+]_d}\right\rceil.$$

Also, for any single value d_0 we can also ensure that

$$g_{[S_-\cup\{c\},S_+]_{d_0}} \leqslant \frac{d_0-|S_-|}{|S_+|-|S_-|}\, g_{[S_-,S_+]_{d_0}}.$$

Proof. To simplify the notation, write $g_{[S_-\cup\{a\},S_+]_d}$. Consider the set of recolorings for which the double sum $\sum_d\sum_{a\in S_+\backslash S_-} g_{a,d}^2$ is minimal. Among these consider those with $\sum_d g_{c,d}$ minimal and among these pick one with g_{c,d_0} minimal. We shall show that this coloring satisfies the conditions of the lemma.

We follow the proof of Lemma 2. Pick $a,b\in S_+\backslash S_-$ and let K be the matching on $V_{[a\diamond b]}$ which matches the endvertices of (a,b)-Kempe paths. Let J be a union of maximal partial matchings between V_S and $V_{i_{ab}S}$ for $S\in[a\diamond b]$. If $S\in[a\diamond b]$ and $|n_S-n_{i_{ab}S}|=1$ then precisely one of n_S or $n_{i_{ab}S}$ is equal to 1. Assume $n_S=1$. There is also precisely one vertex in $V_{\{S,i_{ab}S\}}$ which is unmatched by J. We shall call this vertex v_S. Note that if $n_{i_{ab}S}=0$ then $S(v_S)=S$, however if $n_{i_{ab}S}=2$ then $S(v_S)=i_{ab}S$. For each d, construct maximal partial matchings J'_d between

$\{v_S\colon S\in[S_-\cup\{a\},S_+\backslash\{b\}]_d\}$ and $\{v_S\colon S\in[S_-\cup\{b\},S_+\backslash\{a\}]_d\}$. Let J' be the union of all these J'_d. Note that if $g_{a,d}>g_{b,d}$ then there are more v_S in the first set, so there exist some v_S unmatched by J' with $S\in[S_-\cup\{a\},S_+\backslash\{b\}]_d$. The partial matchings J and J' are disjoint by construction.

Now follow the proof of Lemma 2 with $J\cup J'$ in place of J. If $g_{a,d}>g_{b,d}$ we can pick some $v_0=v_S$ which is unmatched by $J\cup J'$ and change colors as in Lemma 2. The result of the color changes is to swap pairs $n_{S'}$ and $n_{i_{ab}S'}$ whenever $S'=S(v_i)$ and v_i is not in the partial matching J. However, the effect of such swaps on $g_{a,d'}$ and $g_{b,d'}$ are reversed on the next step every time we use the matching J' to obtain the next vertex. Hence only the values of $g_{a,d}$, $g_{b,d}$, $g_{a,d'}$ and $g_{b,d'}$ can change where d' is the degree of the final vertex v_r. As in Lemma 2, the minimality of $\sum_d\sum_{a\in S_+\backslash S_-} g_{a,d}^2$ implies that $|g_{a,d}-g_{b,d}|\leqslant 2$ and if it equals two then the values of $g_{a,d}$ and $g_{b,d}$ can be swapped with no other $g_{a',d'}$ changed.

Applying this to $g_{c,d}$ and using the fact that $\sum_d g_{c,d}$ is minimal we deduce that for all a and d, $g_{c,d}\leqslant g_{a,d}+1$. Since $\sum_{a\in S_+\backslash S_-} g_{a,d}=(d-|S_-|)g_{[S_-,S_+]_d}$ the first part of the lemma follows.

For the second part, if $g_{c,d_0}>\frac{d_0-|S_-|}{|S_+|-|S_-|}g_{[S_-,S_+]_{d_0}}$ then there exists some b with $g_{c,d_0}=g_{b,d_0}+1$. We can then swap the values of g_{c,d_0} and g_{b,d_0} as above. Some other pair $g_{c,d'}$ and $g_{b,d'}$ differing by one will also swap, however this does not increase either $\sum_d g_{c,d}$ or $\sum_d\sum_{a\in S_+\backslash S_-} g_{a,d}^2$ and gives a lower value of g_{c,d_0} contradicting the choice of coloring. □

Lemma 9. *Assume $k\geqslant\chi'(G)$ and $a,b,c\in\{1,\dots,k\}$,*

1. *If $n_0,n_1\leqslant 1$, $n_2,n_3,n_k\leqslant 2$, $n_{k-1}\leqslant k+1$ and for $4\leqslant d\leqslant k-2$*
$$n_d\leqslant\frac{d-3}{d-2}\min\left\{2\binom{k-2}{d-2},\binom{k}{d}\right\}, \tag{5}$$
then we can find an optimal k-coloring with $a,b\in S(v)$ for every bad vertex v.
2. *If $n_0,n_1,n_2\leqslant 1$, $n_3,n_4,n_k\leqslant 2$, $n_{k-1}\leqslant k+1$ and for $5\leqslant d\leqslant k-2$*
$$n_d\leqslant\frac{d-4}{d-3}\min\left\{2\binom{k-3}{d-3},\binom{k}{d}\right\}-2, \tag{6}$$
then we can find an optimal k-coloring with $a,b,c\in S(v)$ for every bad vertex v.

Proof. Let $U_0=[\emptyset,\{1,\dots,k\}]$, $U_1=[\{a\},\{1,\dots,k\}]$, $U_2=[\{a,b\},\{1,\dots,k\}]$, and $U_3=[\{a,b,c\},\{1,\dots,k\}]$ be the collection of all subsets containing $\emptyset$, $\{a\}$, $\{a,b\}$ and $\{a,b,c\}$ respectively. Since $n_d\leqslant\binom{k}{d}+1$ in both parts 1 and 2, a semi-vertex-distinguishing k-coloring exists. Use Lemma 8 with $S_-=\emptyset$, $S_+=\{1,\dots,k\}$ to obtain some optimal k-coloring with few 1-piles in U_1. We now move all the 2-piles into U_1. We can do this safely only when we move the pile onto an empty square, otherwise some other 2-pile may move. We therefore need our 2-piles to be adjacent to empty squares in U_1. Use Lemma 7 with $S_-=\emptyset$, $S_+=\{1,\dots,k\}$ to move a 0-pile next a 2-pile outside U_1. Now move this 2-pile inside U_1 using Lemma 2. Repeat with each 2-pile outside U_1 in turn until they all lie in U_1. We shall be able to move them all inside U_1 provided for all d,

$$|(U_1)_d| - (b_{(U_0)_d} - 1) - g_{(U_1)_d} \geqslant \frac{1}{d-1}\binom{k-1}{d-1} - 1. \tag{7}$$

The left-hand side is the number of empty squares in $(U_1)_d$ just before we move the final 2-pile into $(U_1)_d$. We now repeat the process, moving the 2-piles into U_2. Using Lemmas 7 and 8 with $S_- = \{a\}$ and $S_+ = \{1, \dots, k\}$ ensures that no piles move in or out of U_1 when we move the 2-piles into U_2. Hence the value of $g_{(U_1)_d}$ and $b_{(U_0)_d}$ are unaffected by these recolorings. Using Lemma 7 with color b allows us to move all the 2-piles into U_2 provided that for all d,

$$|(U_2)_d| - (b_{(U_0)_d} - 1) - g_{(U_2)_d} \geqslant \frac{1}{d-2}\binom{k-2}{d-2} - 1. \tag{8}$$

Similarly, we shall be able to move all the 2-piles into U_3 provided

$$|(U_3)_d| - (b_{(U_0)_d} - 1) - g_{(U_3)_d} \geqslant \frac{1}{d-3}\binom{k-3}{d-3} - 1. \tag{9}$$

Using Lemma 8 and the fact that $d \leqslant k$ gives

$$g_{(U_1)_d} \leqslant \frac{xd}{k} + 1, \quad g_{(U_2)_d} \leqslant \frac{xd(d-1)}{k(k-1)} + 2 \quad \text{and}$$
$$g_{(U_3)_d} \leqslant \frac{xd(d-1)(d-2)}{k(k-1)(k-2)} + 3, \tag{10}$$

where $x = g_{(U_0)_d}$. Since $|(U_i)_d| = \binom{k-i}{d-i}$, we now get the following sufficient conditions.

$$y + \frac{xd}{k} \leqslant \frac{d-2}{d-1}\binom{k-1}{d-1} + 1, \tag{11}$$

$$y + \frac{xd(d-1)}{k(k-1)} \leqslant \frac{d-3}{d-2}\binom{k-2}{d-2}, \tag{12}$$

$$y + \frac{xd(d-1)(d-2)}{k(k-1)(k-2)} \leqslant \frac{d-4}{d-3}\binom{k-3}{d-3} - 1, \tag{13}$$

where $y = b_{(U_0)_d}$. By multiplying the first by $\frac{d-1}{k-1} \leqslant 1$, it can be shown that the first inequality always follows from the second. Similarly the second follows from the third. The second and third in turn follow from the conditions in parts 1 and 2, respectively of the lemma when $4 \leqslant d \leqslant k-2$ or $5 \leqslant d \leqslant k-2$ using the fact that $2y + x = n_d$. For small d, $n_d \leqslant 1$ ensures that there are no bad vertices of degree d and $n_d \leqslant 2$ ensures that if a bad vertex of degree d exists, then there is precisely one 2-pile and no 1-pile in $G_{k,d}$. Hence it can easily be moved so that $\{a, b\} \in S(v)$ or $\{a, b, c\} \in S(v)$ without affecting any of the other piles. This leaves only the case when

$d = k - 1$. Choose the d_0 of Lemma 8 to be $d = k - 1$. The graph $G_{k,k-1}$ is isomorphic to K_k and there is only one vertex in each of $(U_2 \backslash U_3)_d$, $(U_1 \backslash U_2)_d$ and $(U_0 \backslash U_1)_d$. Lemma 8 then implies that if any 1-piles are in $(U_i)_d$ then every set in $(U_0 \backslash U_i)_d$ has a 1-pile. Since every set in $G_{k,k-1}$ is adjacent to every other, we can move all the 2-piles into $(U_2)_d$ or $(U_3)_d$ by swapping them with empty squares. This will work provided $y + \max(x-2, 0) \leqslant k-2$ or $y + \max(x-3, 0) \leqslant k-3$. Since $2y + x = n_{k-1}$ and we can assume $y \geqslant 1$ this will hold if $n_{k-1} \leqslant \min(k+1, 2k-3)$ or $\min(k+1, 2k-5)$, respectively. These follow from $n_{k-1} \leqslant k+1$ when $k \geqslant 4$ or $k \geqslant 6$, respectively. For smaller k we have already assumed $n_{k-1} \leqslant 2$ and used a different argument. □

Theorem 10. *Assume* $k \geqslant \Delta(G)$.

1. *If* $n_0, n_1 \leqslant 1$, $n_2, n_3, n_k \leqslant 2$, $n_{k-1} \leqslant k+1$ *and for* $4 \leqslant d \leqslant k-2$
$$n_d \leqslant \frac{d-3}{d-2} \min\left\{ 2\binom{k-2}{d-2}, \binom{k}{d} \right\}, \tag{14}$$
then we can find a strong coloring of G with at most $k+2$ *colors.*
2. *If* $n_0, n_1, n_2 \leqslant 1$, $n_3, n_4, n_k \leqslant 2$, $n_{k-1} \leqslant k+1$ *and for* $5 \leqslant d \leqslant k-2$
$$n_d \leqslant \frac{d-4}{d-3} \min\left\{ 2\binom{k-3}{d-3}, \binom{k}{d} \right\} - 2, \tag{15}$$
then we can find a strong coloring of G with at most $k+1$ *colors.*

Proof. By Vizing's Theorem $\chi'(G) \leqslant \Delta(G) + 1$, and by a strengthened version of Vizing's Theorem [11] it is known that if $n_\Delta \leqslant 2$ then $\chi'(G) = \Delta(G)$. Hence $k \geqslant \chi'(G)$ and a proper k-coloring exists. Fix an optimal k-colorings as in Lemma 9. Each bad vertex is incident with edges of both colors in $S_0 = \{a, b\}$ (or all three colors in $S_0 = \{a, b, c\}$). Let G' be the subgraph of G containing all edges colored with colors in S_0. Let $\sim$ be the partial matching defined by $u \sim v$ if $S(u) \backslash S_0 = S(v) \backslash S_0$ and $|S(u)| = |S(v)|$. Recolor G' with four colors as in Corollary 4 or Theorem 6. This uses a total of $k - 2 + 4 = k + 2$ (or $k - 3 + 4 = k + 1$) colors and gives a strong coloring of G. It is strong since if $S(u) = S(v)$ in the new coloring then $u \sim v$ and $S_{G'}(u) = S_{G'}(v)$ so $u = v$. □

Corollary 11. *Assume that for all d,* $n_d \leqslant \binom{2d-3}{d}$ *and let k be the smallest integer such that for all d,* $n_d \leqslant \binom{k}{d}$. *Then* $k \leqslant \chi'_s(G) \leqslant k+3$.

Proof. Clearly $k \geqslant \Delta(G)$. Apply part 1 of Theorem 10 with k replaced by $k+1$. We need to show that
$$\binom{k}{d} \leqslant \frac{d-3}{d-2} \min\left\{ 2\binom{k-1}{d-2}, \binom{k+1}{d} \right\}, \tag{16}$$

which reduces to

$$(d-2)k(k-d+1) \leqslant 2(d-3)d(d-1) \quad \text{and}$$
$$(d-2)(k-d+1) \leqslant (d-3)(k+1). \tag{17}$$

These both hold when $k \leqslant 2d-3$ and $d \geqslant 4$. For $d < 4$ the conditions imply $n_d \leqslant 1$. For $d = (k+1)-1$, $n_d \leqslant 1 < k+2$. Finally for $d = (k+1)$, $n_d = 0 < 2$. □

Note that this implies a slightly weaker version of Conjecture 2 when $\delta(G) \geqslant O(\log|V(G)|)$ since then $\binom{2d-3}{d}$ will be greater than $|V(G)|$ for all $d \geqslant \delta(G)$.

Corollary 12. *If G is a graph with n vertices,* $\Delta(G) \geqslant \sqrt{2n}+4$, $\delta(G) \geqslant 5$ and k *is the smallest integer such that* $n_d \leqslant \binom{k}{d}$ *for all* d, *then* $k \leqslant \chi'_s(G) \leqslant k+1$.

Proof. The conditions of part 2 of Theorem 10 follow (after some calculation) when $5 \leqslant d \leqslant k-2$ since $n_d \leqslant n$. (The worst case is when $d = 5$ which gives the condition $(k-3)(k-4) \geqslant 2(n+2)$. This holds since $k \geqslant \Delta(G) \geqslant \sqrt{2n}+4$.) If $d = k-1$ then $n_d \leqslant \binom{k}{k-1} < k+1$, if $d = k$ then $n_d \leqslant \binom{k}{k} = 1$, and if $d < 5$ then $n_d = 0$. Hence Theorem 10 applies and the result follows. □

5. Conclusion

Theorem 1 is likely to hold as well when the given color set S is replaced by a collection $\mathscr{T}$ of sets, i.e., $|n_{\mathscr{T}} - n_{\mathscr{T}'}| \leqslant 2$ at least when $\mathscr{T}$ and $\mathscr{T}'$ are sets of the form $[S_-, S_+]_d$ of the same cardinality and same degree d. Hopefully, the balanced coloring described in Theorem 1 will have other applications.

It is worth noting that in Theorem 10, the bound on n_d is close to the best possible obtainable with our method. Indeed, we require $n_d \leqslant \binom{k}{d}$ to have any chance of getting a strong coloring and we require $n_d \leqslant 2\binom{k-2}{d-2}+1$ or $n_d \leqslant 2\binom{k-3}{d-3}+1$ to move all the 2-piles so that they contain the specified colors $\{a,b\}$ or $\{a,b,c\}$. To get closer to Conjecture 2 than this would therefore need new ideas. The extra factors $\frac{d-3}{d-2}$ and $\frac{d-4}{d-3}$ can however be improved by a stronger version of Lemma 7. The best result that we have achieved so far gives factors of the form $1 - O(\frac{1}{d(k-d)})$. However, these improvements are quite complicated, so we did not include them here.

It has also been noted by Bruce Reed that one can use the Lovasz Local Lemma to improve the bounds on n_d for small d. Using this, one can remove the restriction $\delta(G) \geqslant 5$ in Corollary 12 although the bound on $\Delta(G)$ becomes $\Delta(G) \geqslant C\sqrt{n}$ for some larger constant C. It remains to be seen if Conjecture 2 in [6] holds for $\Delta(G) < \sqrt{2|V(G)|}+4$. In particular, it would be most interesting to know if Conjecture 2 holds for regular graphs of low degree, or even a weakening of this conjecture with $(k + \text{constant})$ in place of $(k+1)$.

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