On Graphs with Small Ramsey Numbers

A. V. Kostochka[1,2] and V. Rödl[3]

[1]INSTITUTE OF MATHEMATICS
NOVOSIBIRSK-90, 630090, RUSSIA

[2]UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN
URBANA, ILLINOIS 61801

[3]DEPT. OF MATHEMATICS AND COMPUTER SCIENCE
EMORY UNIVERSITY, ATLANTA
GEORGIA 30322

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Abstract: Let $R(G)$ denote the minimum integer N such that for every bicoloring of the edges of K_N, at least one of the monochromatic subgraphs contains G as a subgraph. We show that for every positive integer d and each $\gamma, 0 < \gamma < 1$, there exists $k = k(d, \gamma)$ such that for every bipartite graph $G = (W, U; E)$ with the maximum degree of vertices in W at most d and $|U| \leq |W|^{\gamma}$, $R(G) \leq k|W|$. This answers a question of Trotter. We give also a weaker bound on the Ramsey numbers of graphs whose set of vertices of degree at least $d+1$ is independent. J Graph Theory 37: 198–204, 2001

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Correspondence to: V. Rödl.

1. INTRODUCTION

The classical Ramsey number $R(k,l)$ is the minimum positive integer N such that for every graph H on n vertices, H contains either a complete subgraph on k vertices or an independent set on l vertices. More generally, for arbitrary graphs G and H, define $R(G,H)$ to be the minimum positive integer N such that in every bicoloring of edges of K_N with, say red and blue colors, there is either a red copy of G or a blue copy of H. Burr and Erdős [2] conjectured that for every d,

$$\text{there exists } k = k(d) \text{ such that } R(G,G) \leq k|V(G)| \tag{1}$$

(a) for every graph G with maximum degree at most d;
(b) for every d-degenerate graph G.

The first conjecture was proved by Chvatal, Rödl, Szemerédi, and Trotter [4], and the second (which is much stronger) is still wide open. Chen and Schelp [3] proved the conjecture for planar graphs and, more generally, for so called *k-arrangeable graphs*. Rödl and Thomas [5] proved that graphs with no K_p-subdivisions are p^8-arrangeable, which implies that for every p, the graphs with no K_p-subdivisions have linearly bounded Ramsey number. Also, Alon [1] proved that (1) holds if $G = (W, U; E)$ is a bipartite graph and the degree of every vertex in W is at most two. We present here a simple lemma which implies the following two results.

Theorem 1. *Let a real $0 < \gamma < 1$ and a positive integer $d \geq 3$ be fixed and let $k = k(d,\gamma) = \lceil 2\exp\{\frac{d}{1-\gamma}\}\rceil$. Let n be sufficiently large and let $G = (W, U; E)$ be a bipartite graph with the bipartition (W, U) and such that $|W| \leq n$, $|U| \leq n^\gamma$, and*

$$\deg(w) \leq d \qquad \textit{for every} \quad w \in W. \tag{2}$$

If n is sufficiently large, then for any bicoloring of the edges of $K_{kn,kn}$, there exists a monochromatic copy of G.

Theorem 2. *Let a positive integer d be fixed and n be sufficiently large. Let $G = (V, E)$ be a graph with $|V| \leq n$, and such that the set U of vertices of degree at least $d+1$ in G is an independent set. Let $k = k(d,n) = \exp\{6d + 6d^2\sqrt{\ln n}\}$. Then $R(G,G) \leq kn$. In particular, for every $\varepsilon > 0$, there exists $C = C(d,\varepsilon)$ such that for every graph $G = (V,E)$ with the independent set of vertices of degree at least $d+1$,*

$$R(G,G) \leq C|V(G)|^{1+\varepsilon}.$$

Theorem 1 confirms the conjecture by Trotter that (1) holds if G is a *crown*. Theorem 2 shows that in a wider class, the Ramsey number is not far from linear.

2. MAIN LEMMA

Lemma 1. *Let a positive integer d be fixed and n be sufficiently large. Suppose that positive integers $m \geq 2$, $l \geq 4$, and kn satisfy the inequalities*

$$\left(\frac{k}{2}\right)^l \geq 6(nk)^d \tag{3}$$

and

$$\frac{1}{6}\,kn2^{-l} > m - 1. \tag{4}$$

Then for every subgraph $H = (V_1, V_2; E)$ of the complete bipartite graph $K_{kn,kn}$ with the bipartition (V_1, V_2) and such that $|E| \geq (kn)^2/2$, there exists $M \subset V_1$ with $|M| \geq m$ with the property that for every d-element subset D of M, the number of vertices of H adjacent to all vertices in D is at least n.

Proof. Call a d-tuple $\{x_1, \ldots, x_d\}$ of vertices in V_1 *poor* if $|N(x_1) \cap \cdots \cap N(x_d)| < n$. An l-tuple $\{y_1, \ldots, y_l\}$ of vertices in V_2 will be called *bad* if it is contained in the set $N(x_1) \cap \cdots \cap N(x_d)$ for some poor d-tuple $\{x_1, \ldots, x_d\}$ of vertices in V_1. Other l-tuples of vertices in V_2 will be called *good*. By the definition, the number of bad l-tuples in V_2 is at most

$$\binom{|V_1|}{d}\binom{n-1}{l} < k^d n^{d+l}/l\,!.$$

It follows that the number b of pairs (x, L) such that $x \in V_1$ and $L \subset N(x)$ is a bad l-tuple is estimated as follows

$$b < kn\; k^d n^{d+l}/l! = k^{d+1} n^{d+l+1}/l!. \tag{5}$$

On the other hand, the total number of pairs (x, L) such that $x \in V_1$ and $L \subset N(x)$ is an l-tuple, is at least

$$\sum_{x \in V_1} \binom{\deg x}{l}.$$

Under condition that $\sum_{x \in V_1} \deg x \geq (kn)^2/2$, the last sum is at least $|V_1|\binom{\lfloor 0.5kn \rfloor}{l}$. Hence

$$\sum_{x \in V_1} \binom{\deg x}{l} \geq |V_1| \frac{(0.5kn - l)^l}{l!} = 2 \cdot \frac{(0.5kn)^{l+1}}{l!}\left(1 - \frac{2l}{kn}\right)^l$$
$$\geq 2 \cdot \frac{(0.5kn)^{l+1}}{l!}\left(1 - \frac{2l^2}{kn}\right).$$

Since due to (4), $2^l < \frac{kn}{6(m-1)} \le \frac{kn}{6}$, we have

$$1 - \frac{2l^2}{kn} \ge 1 - \frac{2^{l+1}}{kn} > 1 - \frac{2kn}{6kn} = \frac{2}{3}.$$

Therefore,

$$\sum_{x \in V_1} \binom{\deg x}{l} > \frac{(0.5kn)^{l+1}}{l!}.$$

It follows that the number g of pairs (x, L) such that $x \in V_1$ and $L \subset N(x)$ is a good l-tuple, is at least

$$\frac{(0.5kn)^{l+1}}{l!} - \frac{k^{d+1}n^{d+l+1}}{l!} > \frac{n^{l+1}k}{l!}\left(0.5(k/2)^l - (nk)^d\right).$$

By (3), $g \ge \frac{1}{3}(0.5kn)^{l+1}/l!$.

There exists a good l-tuple $L_0 \subset N(x)$ participating in at least $g \cdot \binom{kn}{l}^{-1}$ such pairs. We have

$$\frac{g}{\binom{kn}{l}} \ge \frac{(0.5kn)^{l+1}}{3l!} \frac{l!}{(kn)^l} = \frac{1}{6}kn2^{-l}.$$

By (4), the last expression is greater than $m - 1$. It follows that there is a subset M of V_1 with $|M| = m$ such that $L_0 \subset N(x)$ for every $x \in M$. Since L_0 is good, none of the d-tuples of elements of M is poor. This proves the lemma.

3. APPLICATIONS OF THE LEMMA

Proof of Theorem 1. Let $k = \lceil 2\exp\{\frac{d}{1-\gamma}\}\rceil$. Let H_1 and H_2 be two subgraphs of the complete bipartite graph $K_{kn,kn}$ with bipartition (V_1, V_2) whose union is $K_{kn,kn}$. We may assume that $|E(H_1)| \ge |E(H_2)|$ and hence $|E(H_1)|| \ge (kn)^2/2$. Set $l = \lfloor (1-\gamma)\log_2 n \rfloor$ and $m = \lceil n^\gamma \rceil$. Then conditions (3) and (4) are satisfied. Thus, by Lemma 1, there exists $M \subset V_1$ with $|M| \ge m$ with the property that for every d-element subset D of M, the number of vertices of H adjacent to all vertices in D is at least n. Now we construct embedding $f : W \cup U \to M \cup V_2$ of $G = (W, U; E)$ into the subgraph of H_1 induced by $M \cup V_2$ in a greedy manner. Let f be an arbitrary 1–1 mapping of U to M. We extend this mapping to $w_1, w_2, \ldots w_n$-elements of W and define f-images as follows: For $i = 1, 2, \ldots n$ consider $D(w_i) = f(N_G(w_i))$. Since $|D(w_i)| \le d$ there are at least n vertices in V_2 adjacent to each vertex in $D(w_i)$. We choose for $f(w_i)$ any of them not used as $f(w_j)$ for $j < i$. *Theorem* 1 *is proved.*

Instead of proving Theorem 2 directly, we first derive a more general statement. Let $\mathcal{H}(s, n, d)$ denote the family of graphs $G = (V, E)$ on at most n vertices such that there exists a partition $V = V_1 \cup \cdots \cup V_{s+1}$ with the properties:

(a) every V_i is an independent set;
(b) for every $i = 1, \ldots, s$ and every $v \in V_i$, the degree of v in $G - V_1 - \cdots - V_{i-1}$ is at most d.

For example, $\mathcal{H}(0, n, d)$ is the family of graphs without edges on at most n vertices and $\mathcal{H}(1, n, d)$ is the family of bipartite graphs on at most n vertices in which all the vertices of one of the parts have degrees at most d.

Let $F(s, t, n, d)$ be the smallest positive integer such that for every $G_1 \in \mathcal{H}(s, n, d)$ and every $G_2 \in \mathcal{H}(t, n, d)$,

$$R(G_1, G_2) \leq nF(s, t, n, d).$$

Theorem 3. *Let a positive integer $d \geq 3$ be fixed. For every non-negative integers s and t with $s + t \geq 1$, and for sufficiently large n,*

$$F(s, t, n, d) \leq 10 \exp\{2(s + t - 1)(1 + d\sqrt{\ln n})\}.$$

In particular, for every $\varepsilon > 0$, there exists $n_0 = n_0(s, t, d, \varepsilon)$ such that for every $n > n_0$,

$$F(s, t, n, d) \leq n^{\varepsilon}.$$

Proof of Theorem 3. We prove the theorem for a fixed d by induction on $s + t$. Clearly, for any $s \geq 1$ and $t \geq 1$,

$$F(s, 0, n, d) = F(0, t, n, d) = 1.$$

Suppose that the theorem is proved for all pairs (s', t') with $s' + t' < s + t$ and assume that $s \geq 1$ and $t \geq 1$. Consider arbitrary graphs $G_1 \in \mathcal{H}(s, n, d)$ and $G_2 \in \mathcal{H}(t, n, d)$.

Set

$$k = \lfloor 5\exp\{2(s + t - 1)(1 + d\sqrt{\ln n})\}\rfloor \quad \text{and} \quad N = kn. \tag{6}$$

Consider red-blue coloring of edges of K_{2N} and let H_1, H_2 with $V(H_1) = V(H_2) = V(K_{2N})$ be the subgraphs consisting of red and blue edges respectively. Let $V(K_{2N}) = U \cup W$ be an arbitrary partition with $|U| = |W| = kn$. We may assume that at least half of edges connecting U with W belongs to $E(H_1)$. Set

$$l = \sqrt{\ln n}, \quad \text{and} \quad m = 10n \exp\{2(s + t - 2)(1 + d\sqrt{\ln n})\}.$$

We will prove that these parameters satisfy conditions of Lemma 1. Indeed, we can assume that for $n \geq n_0(d)$,

$$5 \exp\{2(1 + d\sqrt{\ln n})\} < n/6.$$

Then for k defined by (6), we have $k < (n/6)^{s+t-1}$ and hence

$$\left(\frac{k}{2}\right)^{l} \geq (\exp\{2(s+t-1)(1+d\sqrt{\ln n})\})^{\sqrt{\ln n}} \geq n^{2(s+t-1)d} \geq n^{(s+t-1)d} n^{d} > 6^{d} k^{d} n^{d}.$$

Similarly, for $n \geq n_0(d)$,

$$\frac{1}{6} kn2^{-l} > \frac{n}{6} \exp\{2(s+t-1)(1+d\sqrt{\ln n}) - \sqrt{\ln n}\} >$$
$$> 10n \exp\{2(s+t-2)(1+d\sqrt{\ln n})\} > m-1.$$

Applying Lemma 1, we get that there exists $M \subset U$ with $|M| \geq m$ with the property that for every d-element subset D of M, the number of vertices in W adjacent in H_1 to all vertices in D is at least n.

Let a partition of $V(G_1)$ verifying that $G_1 \in \mathcal{H}(s, n, d)$ be the partition $V(G_1) = X_1 \cup \ldots \cup X_{s+1}$. Let $G_1' = G_1 - X_1$. Then $G_1' \in \mathcal{H}(s-1, n, d)$. By the induction assumption, $R(G_1', G_2) \leq m$. It follows that either $H_1(M)$ contains a copy of G_1' or $H_2(M)$ contains a copy of G_2. In the latter case, we are done, so we assume that there exists an embedding f of G_1' into $H_1(M)$. Now (similarly to the proof of Theorem 1) we embed vertices of X_1 into W in a greedy manner. Let $x_1, \ldots, x_p$ be the vertices in X_1; note that $p < n$. Consecutively, for $i = 1, \ldots, p$, we do the following. Consider $D(x_i) = f(N_{G_1}(x_i))$. Since $|D(x_i)| \leq d$, there are at least n vertices in W adjacent in H_1 to each vertex in $D(x_i)$. We choose for $f(x_i)$ any of them not used as $f(x_j)$ for $j < i$. Theorem is proved.

Proof of Theorem 2. Let X_{d+1} be any maximal (by inclusion) independent set in G containing all vertices of degree at least $d+1$. By the maximality of X_{d+1}, the maximum degree of $G - X_{d+1}$ is at most $d-1$, and hence $V(G) \setminus X_{d+1}$ can be partitioned into d independent sets $X_1, \ldots, X_d$. It follows that $G \in \mathcal{H}(d, n, d)$. Thus, by Theorem 3,

$$R(G, G) \leq 10\, n \exp\{2(2d-1)[(1+d\sqrt{\ln n})]\}.$$

This proves the theorem.

Remark. Note that replacing 2-coloring by r-coloring one can verify straightforward extensions of Theorems 1, 2, and 3.

Note added in proof: It was recently observed by Shi Lingsheng in [6] that as a consequence of Theorem 1 and Lemma 1, one can derive that the Ramsey number of a t-dimensional cube $R(Q_t) \leq m^c$, where $m = 2^t$ and $c > 0$.

Indeed, with the choice of d, l and m as in Theorem 1, (3) is satisfied if

$$\frac{1}{\gamma}\left(\frac{1}{\ln 2} - 1\right)\ln(m-1) \geq \frac{\ln 6}{d} + \ln 2 + \frac{1+d}{1-\gamma}, \tag{1}$$

while (4) holds true.

Thus for example, setting $\gamma = \frac{1}{5}$, $d = t$, and $m = 2^t$, one can see that $R(Q_t) \leq m^c$ holds, where $c \leq 7$. By a careful optimization of constants, $c = \frac{\sqrt{5}+3}{2}$ is proved in [6].

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