A List Version of Dirac's Theorem on the Number of Edges in Colour-Critical Graphs

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Received May 18, 1999

Abstract: One of the basic results in graph colouring is Brooks' theorem [R. L. Brooks, Proc Cambridge Phil Soc 37 (1941) 194–197], which asserts that the chromatic number of every connected graph, that is not a complete graph or an odd cycle, does not exceed its maximum degree. As an extension of this result, Dirac [G. A. Dirac, Proc London Math Soc 7(3) (1957) 161–195] proved that every k-colour-critical graph ($k \geq 4$) on $n \geq k+2$

Contract grant sponsor: Russian Foundation for Fundamental Research; Contract grant number: 99-01-00581; Contract grant sponsor: INTAS; Contract grant number: 97-1001.

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vertices has at least $\frac{1}{2}((k-1)\ n+k-3)$ edges. The aim of this paper is to prove a list version of Dirac's result and to extend it to hypergraphs.
 J Graph Theory 39: 165–177, 2002; DOI 10.1002/jgt.998

Keywords: *list colourings; critical graphs*

1. TERMINOLOGY AND MAIN RESULT

In this paper, we continue studying colour-critical graphs and hypergraphs with few edges (cf. [12–16]).

A *hypergraph* $G = (V, E)$ consists of a finite set $V = V(G)$ of *vertices* and a set $E = E(G)$ of subsets of V, called *edges*, each having cardinality at least two. An edge e with $|e| = 2$ is called an *ordinary edge*. A *graph* is a hypergraph in which each edge is ordinary. The *degree* $d_G(x)$ of a vertex x in G is the number of the edges in G containing x. If $d_G(x) = r$ for all $x \in V(G)$, then G is said to be *r-regular*. Let $d(G) = \sum_{x \in V(G)} d_G(x)$. Clearly, if G is a graph, then $d(G) = 2|E(G)|$.

If H and G are hypergraphs with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, then H is said to be a *subhypergraph* of G. Let G be a hypergraph and $X \subseteq V(G)$. The subhypergraph of G induced by $X \subseteq V(G)$ is denoted by $G[X]$, i.e. $V(G[X]) = X$ and $E(G[X]) = \{e \in E(G) | \, e \subseteq X\}$; further, $G - X = G[V(G) - X]$.

Consider a hypergraph G and assign to each vertex x of G a set $\Phi(x)$ of colours (positive integers). Such an assignment Φ of sets to vertices in G is referred to as a *colour scheme* (or briefly, a *list*) for G. A Φ-*colouring* of G is a mapping φ of $V(G)$ into the set of colours such that $\varphi(x) \in \Phi(x)$ for all $x \in V(G)$ and $|\{\varphi(x) | \, x \in e\}| \geq 2$ for each $e \in E(G)$. If G admits a Φ-colouring, then G is said to be Φ-*colourable*. In case of $\Phi(x) = \{1, \ldots, k\}$ for all $x \in V(G)$, we also use the terms *k-colouring* and *k-colourable*, respectively.

We say that a hypergraph G is Φ-*critical* where Φ is a given list for G if G is not Φ-colourable but every proper subhypergraph of G is Φ-colourable. In case of $\Phi(x) = \{1, \ldots, k-1\}$ for all $x \in V(G)$, we also use the term *k-colour-critical*.

Colour-critical graphs were first defined and used by Dirac [6]. The complete graph K_k on k vertices is an example of a k-colour-critical graph and for $k = 1, 2$ it is the only one. The 3-colour-critical graphs are the odd cycles, so for the remainder of this paper we shall restrict our attention to the case $k \geq 4$. Then there are k-colour-critical graphs on n vertices for every $n \geq k$ except for $n = k + 1$. If G is a k-colour-critical graph on n vertices, then the minimum degree of G is at least $k - 1$ and, therefore, $d(G) \geq (k-1)n$. Furthermore, Brooks' theorem implies $d(G) \geq (k-1)n + 1$ provided that $G \neq K_k$. In 1957 Dirac [7] proved the following extension of Brooks' result.

Theorem 1. *If $G \neq K_k$ is a k-colour-critical graph on n vertices where $k \geq 4$, then $d(G) \geq (k-1)n + k - 3$.*

In [8], Dirac also gave a complete description of the extremal cases. His proof was rather long. Shorter and more elegant proofs were found by Kronk and Mitchem [18], Weinstein [21] and, for the result in [8], by Deuber, et al. [5]. The aim of this paper is to prove the following list version of Dirac's result.

Theorem 2. *Let G be a hypergraph on n vertices not containing K_k, and let Φ be a list for G with $|\Phi(v)| = k-1$ for every $v \in V(G)$ where $k \geq 4$. If G is Φ-critical, then $d(G) \geq (k-1)n + k - 3$.*

The proof of Theorem 2 is given in Section 3. For further results concerning the number of edges in colour-critical graphs and hypergraphs the reader is referred to [10,12–17] and [11, Chapter 5]. [For an interesting application of Theorem 2 see [1].]

If a k-colour-critical graph G contains a K_k, then $G = K_k$. Hence Theorem 1 is an immediate consequence of Theorem 2. However, Theorem 2 becomes false if we only assume that $G \neq K_k$. To see this, let $k \geq 6$ and let G denote the graph whose vertex set consists of two disjoint sets A, B where $|A| = |B| = k-1$ and two additional vertices a, b such that both $G[A \cup \{a\}]$ and $G[B \cup \{b\}]$ are complete graphs not joined by any edge in G beside the edge $\{a, b\}$. Furthermore, define the list Φ for the graph G by

$$\Phi(x) = \begin{cases} \{1, \ldots, k-1\} & \text{if } x \in A \cup B, \\ \{2, \ldots, k\} & \text{if } x \in \{a, b\}. \end{cases}$$

Then $|\Phi(x)| = k-1$ for all $x \in V(G)$ and it is easy to check that G is Φ-critical. Clearly, G has $n = 2k$ vertices and $d(G) = (k-1)n + 2$.

2. GALLAI TREES AND BAD PAIRS

For a hypergraph G and a vertex x of G, let $G \setminus x$ denote the hypergraph with $V(G \setminus x) = V(G) - \{x\}$ and $E(G \setminus x) = E(G - \{x\}) \cup \{e - \{x\} \mid x \in e \in E(G)$ & $|e| \geq 3\}$. Note that $G \setminus x = G - \{x\}$ provided that G is a graph. For a hyperedge e, let $\langle e \rangle$ denote the hypergraph $(e, \{e\})$.

Let G be a connected hypergraph. A vertex x of G is called a *separating vertex* of G if $G \setminus x$ is disconnected. By a *block* of G we mean a maximal connected subhypergraph B of G such that no vertex of B is a separating vertex of B. Any two distinct blocks of G have at most one vertex in common and, obviously, a vertex of G is a separating vertex of G if it is contained in more than one block of G. An *end-block* of G is a block that contains at most one separating vertex of G.

By a *brick* we mean a hypergraph of the form $\langle e \rangle$ for some hyperedge e, or an odd cycle consisting only of ordinary edges, or a complete graph. A connected hypergraph all of whose blocks are bricks is called a *Gallai tree*; a *Gallai forest* is a hypergraph whose components are Gallai trees.

We call a pair (G, Φ) consisting of a connected hypergraph G and a list Φ for G a *bad pair* if $|\Phi(x)| \geq d_G(x)$ for all $x \in V(G)$ and G is not Φ-colourable. The following result was proved by Kostochka et al. [12].

Lemma 1. *If (G, Φ) is a bad pair, then the following statements hold.*

(a) $|\Phi(x)| = d_G(x)$ *for all* $x \in V(G)$.
(b) *If G has no separating vetex, then $\Phi(x)$ is the same for all $x \in V(G)$.*
(c) *G is a Gallai tree.*

For graphs, Lemma 1 was proved independently by Borodin [2,3] and Erdös et al. [9]. The following result is a consequence of Lemma 1 and generalizes a result of Gallai [10] about colour-critical graphs.

Lemma 2. *Assume that $k \geq 4$ and G is a Φ-critical hypergraph where Φ is a list for G satisfying $|\Phi(v)| = k-1$ for every $v \in V(G)$. Let $H = \{y \in V(G) \,|\, d_G(y) \geq k\}$ and $L = V(G) - H$. Then the following statements hold.*

(a) *$G[L]$ is empty or a Gallai forest and $d_G(x) = k-1$ for every $x \in L$.*
(b) *If $H = \emptyset$, then $G = K_k$.*

Proof. For the proof of (a), suppose that $L \neq \emptyset$. Consider the vertex set X of some component of $G[L]$ and let $Y = V(G) - X$. Since G is Φ-critical, there is a Φ-colouring φ of $G[Y] = G - X$. For every edge e of G with $e - X \neq \emptyset$, choose a vertex $v(e) \in e - X$. For the connected hypergraph $G' = G[X]$, define a list Φ' by

$$\Phi'(x) = \Phi(x) - \{\varphi(v(e)) \,|\, x \in e \in E(G) \;\&\; e - X \neq \emptyset\}$$

for every $x \in V(G')$. Since G is not Φ-colourable, G' is not Φ'-colourable. Furthermore, $|\Phi'(x)| \geq d_{G'}(x)$ for all $x \in V(G')$. Consequently, (G', Φ') is a bad pair. Then, by Lemma 1, G' is a Gallai tree and $|\Phi'(x)| = d_{G'}(x)$ for all $x \in X$ implying that $|\Phi(x)| = d_G(x)$ for all $x \in X$. This proves (a).

Now, suppose that $H = \emptyset$. Then $G = G[L]$ and, since G is Φ-critical, G is connected. Therefore, by (a), G is a $(k-1)$-regular Gallai tree. Since every block of a Gallai tree is regular, this implies that G is a block. Since $k \geq 4, G = K_k$. This proves (b). □

Lemma 3. *Let (G, Φ) be a bad pair and let B be an end-block of G. Assume that B is a complete graph. Then $\Phi(x) = \Phi(y)$ for every two non-separating vertices x,y of G contained in B.*

Proof. Suppose that there are two non-separating vertices x, y of G contained in B such that $\Phi(x) \neq \Phi(y)$. W.l.o.g. assume that $\alpha \in \Phi(y) - \Phi(x)$. For the connected hypergraph $G' = G - \{y\}$, define the list Φ' by setting $\Phi'(z) = \Phi(z) - \{\alpha\}$ if $\{y, z\} \in E(G)$ and $\Phi'(z) = \Phi(z)$ otherwise. Since G is not Φ-colourable, G' is not Φ'-colourable. Furthermore, $|\Phi'(z)| \geq d_{G'}(z)$ for all $z \in V(G')$. Consequently, (G', Φ') is a bad pair where $|\Phi'(x)| > d_{G'}(x)$, a contradiction to Lemma 1. This proves Lemma 3. □

Lemma 4. *Let $G = K_t^-$ be a complete graph on $t \geq 3$ vertices with an edge missing. Let x_1, x_2 be the two non-adjacent vertices of G and let $Y = V(G) - \{x_1, x_2\}$. Assume that Φ is a list for G such that $|\Phi(y)| \geq t-1$ for all $y \in Y, |\Phi(x_i)| \geq 1$ for $i = 1, 2$ and $|\Phi(x_1)| + |\Phi(x_2)| \geq t$. Then G is Φ-colourable.*

Proof. First, we prove that there is a *feasible colour pair*, that is a pair (α_1, α_2) such that $\alpha_i \in \Phi(x_i)$ and $|\Phi(y') - \{\alpha_1, \alpha_2\}| \geq t-2$ for some vertex $y' \in Y$. If there is a colour $\alpha \in \Phi(x_1) \cap \Phi(x_2)$, then (α, α) is a feasible colour pair. Otherwise, $\Phi(x_1) \cap \Phi(x_2) = \emptyset$ and, therefore, $|\Phi(x_1) \cup \Phi(x_2)| = |\Phi(x_1)| + |\Phi(x_2)| \geq t$. Because of $|\Phi(y)| \geq t-1$ for all $y \in Y$, this implies that there is a feasible colour pair, too.

Now, let (α_1, α_2) be a feasible colour pair and let Φ' be the list for the complete graph $G' = G[Y]$ with $\Phi'(y) = \Phi(y) - \{\alpha_1, \alpha_2\}$ for all $y \in Y$. Since $|\Phi'(y)| \geq d_{G'}(y)$ for all $y \in Y$ and strong inequality holds for at least one vertex, we conclude from Lemma 1 that there is a Φ'-colouring for G' and, therefore, a Φ-colouring for G. This proves Lemma 4. □

3. PROOF OF THEOREM 2

The proof is by contradiction. Throughout this section, G denotes a possible counterexample, that is,

(a) G is a Φ-critical hypergraph on n vertices where $k \geq 4$ and Φ is a list for G satisfying $|\Phi(v)| = k-1$ for all $v \in V(G)$,
(b) G does not contain a K_k, and
(c) $d(G) \leq (k-1)n + (k-4)$.

To arrive at a contradiction we shall establish that there is a Φ-colouring of G. We start with some preliminaries. For $v \in V(G)$ and $A \subseteq V(G)$, let $d(v) = d_G(v)$ and $d(A) = \sum_{v \in A} d(v)$. Let $L = \{x \in V(G) \mid d(x) = k-1\}$ and $H = \{y \in V(G) \mid d(y) \geq k\}$. Because of (a) and (b), it follows from Lemma 2 that $V(G) = H \cup L, G[L]$ is a Gallai forest, and $H \neq \emptyset$. Consequently, $k \geq 5$ and $d(G) = d(L) + d(H) = (k-1)|L| + d(H) \geq (k-1)n + |H|$. Then we infer from (c) that

$$|H| \leq k-4, \tag{1}$$

and, therefore,

$$d(H) \leq (k-1)|H| + k - 4 \leq k(k-4). \tag{2}$$

3.1. The Auxiliary Hypergraph G^*

We partition the set of edges of G that have a vertex in common with H into two classes, namely

$$E_1 = \{e \in E(G) \,||\, e \cap H| = 1\},$$

and

$$E_2 = \{e \in E(G) || \, e \cap H| \geq 2\}.$$

For every edge $e \in E_1$, choose a subset $e^* = \{x, y\}$ of e with $x \in L$ and $y \in H$. Let $E^* = \{e^* | \, e \in E_1\}$ and let G^* be the hypergraph with $V(G^*) = V(G)$ and $E(G^*) = (E(G) - E_1 - E_2) \cup E^*$. For a vertex $x \in L$, let $d^*(x) = d_{G^*}(x)$ and let $H_x = \{y \in H \,|\, \{x, y\} \in E^*\}$ be the set of all neighbours of x in G^* that belong to H. Note that $d^*(x) = d_{G[L]}(x) + |H_x| \leq d(x) = k - 1$ for all $x \in L$. Clearly, $G^*[L] = G[L]$ and $G^*[H]$ is a hypergraph without any edges.

Let $\mathcal{F}$ denote the set of all Φ-colourings φ of $G^*[H]$ such that $\varphi(y) \neq \varphi(y')$ for every two distinct vertices $y, y' \in H$. Because of $|H| \leq k - 4$, $\mathcal{F} \neq \emptyset$ and our aim is to show that a certain colouring $\varphi \in \mathcal{F}$ can be extended to a Φ-colouring φ^* of G^*. Then we have $|\{\varphi^*(x) \,|\, x \in e\}| \geq |\{\varphi^*(x) \,|\, x \in e^*\}| \geq 2$ for all $e \in E_1$ and, because of $\varphi \in \mathcal{F}$, we also have $|\{\varphi^*(x) \,|\, x \in e\}| \geq |e \cap H| \geq 2$ for all $e \in E_2$. Consequently, φ^* is a Φ-colouring of G, a contradiction to (a).

For a colouring $\varphi \in \mathcal{F}$, let Φ_φ denote the list for the Gallai tree $G[L] = G^*[L]$ with

$$\Phi_\varphi(x) = \Phi(x) - \{\varphi(y) \,|\, y \in H_x\}$$

for all $x \in L$. Obviously, $\varphi \in \mathcal{F}$ can be extended to a Φ-colouring of G^* if and only if there is a Φ_φ-colouring of $G[L]$. Therefore, we need only to show that $G[L]$ is Φ_φ-colourable for some $\varphi \in \mathcal{F}$.

For this purpose we partition the set $\mathcal{C}$ of all components of the Gallai forest $G[L]$ into two classes. The first class $\mathcal{C}_1$ consists of all components $T \in \mathcal{C}$ such that $d^*(x) \leq k - 2$ for some vertex x of T and the second class $\mathcal{C}_2$ consists of the remaining components of $\mathcal{C}$. Clearly, for every component $T \in \mathcal{C}_2$ we have $d^*(x) = d(x) = k - 1$ for all $x \in V(T)$.

Proposition 1. *Let $T \in \mathcal{C}$ and $\varphi \in \mathcal{F}$. Then $|\Phi_\varphi(x)| \geq d_T(x)$ for all $x \in V(T)$. This implies that (T, Φ_φ) is a bad pair if T is not Φ_φ-colourable.*

Proof. For every $x \in V(T)$, we have $d^*(x) = d_T(x) + |H_x| \leq d(x) = k - 1$. Consequently, $|\Phi_\varphi(x)| \geq |\Phi(x)| - |H_x| \geq d_T(x)$. This proves Proposition 1. □

Proposition 2. *If $T \in \mathcal{C}_1$, then T is Φ_φ-colourable for all $\varphi \in \mathcal{F}$.*

Proof. Since $T \in \mathcal{C}_1$, there is a vertex $x \in V(T)$ such that $d^*(x) \leq k-2$. Then $|\Phi_\varphi(x)| > d_T(x)$ and Proposition 2 is a consequence of Proposition 1 and Lemma 1. □

Therefore, in order to prove Theorem 2 it is sufficient to show that there is a colouring $\varphi \in \mathcal{F}$ such that T is Φ_φ-colourable for all components $T \in \mathcal{C}_2$.

3.2. The Components in $\mathcal{C}_2$

In this subsection we shall establish some results about the components in $\mathcal{C}_2$.

Proposition 3. *Let $T \in \mathcal{C}_2$ and let B be an end-block of T. Then B is a K_t where $t \geq 4$ and there are two non-separating vertices x, x' of T satisfying $x, x' \in V(B)$ and $H_x \neq H_{x'}$.*

Proof. Since T is a Gallai tree, B is regular of some degree $r \geq 0$ and, because of $T \in \mathcal{C}_2$, we have $d^*(x) = d_T(x) + |H_x| = d(x) = k-1$. By (1), $|H_x| \leq k-4$ for all $x \in V(T)$ and, therefore, $d_T(x) \geq 3$. Consequently, $r \geq 3$ and B is a K_{r+1}.

Now suppose that there is a subset N of H such that $H_x = N$ for all non-separating vertices x of T contained in B. To arrive at a contradiction we shall establish that there is a Φ-colouring of G.

Let $Z = V(B) \cup N$. Then $|Z| = k$ and, by (b), the subgraph $G[Z]$ is not a complete graph. Therefore, there are two distinct vertices $z_1, z_2 \in Z$ such that $f = \{z_1, z_2\}$ does not belong to *E(G)*. If B contains a separating vertex b of T, then b is adjacent to a vertex of $T - V(B)$ and, therefore, $N - H_b \neq \emptyset$. In this case we choose $z_1 = b$ and $z_2 \in N - H_b$. Otherwise we choose z_1, z_2 arbitrarily. Since $f \notin E(G)$, at most one of the two vertices of f belongs to B.

Let $X = V(B) - f, Y = N - f, p = |X|$ and $q = |Y|$. Then $p + q = k-2, G[X]$ is a K_p and, for every $x \in X$ and $i = 1, 2$, either $\{x, z_i\}$ is an edge of G or it is contained in an edge $e \in E_1$.

Denote by E' the set of all edges e of G satisfying $|e| \geq 3, e \cap f \neq \emptyset$ and $e \cap X \neq \emptyset$. Since every vertex of X is a non-separating vertex of T contained in the end-block B, we have $e \cap H \neq \emptyset$ for all $e \in E'$. Because of $T \in \mathcal{C}_2$, no vertex of T is contained in an edge of E_2. Consequently, $E' \subseteq E_1$.

Let $U = X \cup \{z_1, z_2\}$ and let K' denote the graph obtained from the complete graph with vertex set U by deleting the edge $\{z_1, z_2\}$. For every edge e of G with $e - U \neq \emptyset$, choose a vertex $v(e) \in e - U$. Since G is Φ-critical, there is a Φ-colouring φ of $G - U$. Now, define for the graph K' a list Φ' by

$$\Phi'(x) = \Phi(x) - \{\varphi(v(e)) \mid x \in e \in E(G) - E' \,\&\, e - U \neq \emptyset\}$$

for all $x \in U$. We claim that K' is Φ'-colourable. The proof of this claim is based on Lemma 4.

For $x \in X = U - \{z_1, z_2\}$ we have $d^*(x) = d(x) = k - 1$ and, therefore, $|\Phi'(x)| \geq p + 1$ where $p = |X|$. Since $Y = N - \{z_1, z_2\} \subseteq H$ and $q = |Y| = k - 2 - p$, we infer from (c) that $d(z_1) + d(z_2) + q \leq 2(k-1) + (k-4) = 3k - 6$ and, therefore, $d(z_1) + d(z_2) \leq 2k - 4 + p$ and, for $i = 1, 2, d(z_i) \leq k - 3 + p$. Consequently, we have $|\Phi'(z_i)| \geq k - 1 - d(z_i) + p \geq 2$ for $i = 1, 2$ and $|\Phi'(z_1)| + |\Phi'(z_2)| \geq 2(k-1) - (d(z_1) + d(z_2)) + 2p \geq p + 2$. Since K' has $p + 2$ vertices, Lemma 4 implies that there is a Φ'-colouring φ' of K'. Then $\varphi \cup \varphi'$ is a Φ-colouring of G. This contradiction proves Proposition 3. □

Let T be a component in $\mathcal{C}_2$. By $g(T)$ we denote the number of edges of G^* that have a vertex in common with both T and H, i.e., $g(T) = \sum_{x \in V(T)} |H_x|$. Furthermore, we say that $\{x, x'\}$ is a *light pair* of T if x, x' are non-separating vertices of T contained in an end-block B of T of maximum size, and $H_x \neq H_{x'}$. For a light pair $\{x, x'\}$ of T, let $s(x, x') = |H_x - H_{x'}|$. Since $k - 1 = d^*(v) = d_T(v) + |H_v|$ for all $v \in V(T)$ and $d_T(x) = d_T(x')$, we have $|H_x - H_{x'}| = |H_{x'} - H_x|$.

Proposition 4. *Let $T \in \mathcal{C}_2$ and let $\{x, x'\}$ be a light pair of T with $s = s(x, x')$, Then $g(T) \geq s(k - s) \geq k - 1$ and $1 \leq s \leq \frac{1}{2}(k - 4)$.*

Proof. Let B be the end-block of T with $x, x' \in V(B)$. By Proposition 3, B is a K_t where $t \geq 4$.

First, we claim that $g(T) \geq t(k - t)$. For a non-separating vertex v of T contained in B, we have $k - 1 = d^*(v) = d_T(v) + |H_v| = t - 1 + |H_v|$ and, therefore, $|H_v| = k - t$. Consequently, in case $B = T$, we have

$$g(T) = \sum_{x \in V(T)} |H_x| \geq t(k - t).$$

In case of $B \neq T$ we argue as follows. There is an end-block $B' \neq B$ of T, and, by Proposition 3, B' is a $K_{t'}$ and $t' \leq t$. Since $|H_v| \geq k - t' \geq k - t$ for every non-separating vertex v of T contained in B', we have

$$g(T) \geq (t - 1)(k - t) + (k - t') \geq t(k - t).$$

This proves the claim.

Next, we prove that $t(k - t) \geq s(k - s)$. Since $H_x \neq H_{x'}$ and $|H_x| = |H_{x'}| = k - t$, we have $1 \leq s \leq k - t$ and $2s \leq |H_x \cup H_{x'}| \leq |H|$. Because of $|H| \leq k - 4$, this implies that $s \leq \frac{1}{2}(k - 4)$. Consequently, $s(k - s) \geq k - 1$. Furthermore, $k - t = |H_x| \leq |H| - |H_x - H_{x'}| \leq (k - 4) - s$ implying $s < t$. From $0 < t - s$ and $k \leq t + s$ we conclude that $s(k - s) \leq t(k - t)$. Thus Proposition 4 is proved. □

Proposition 5. *For every component $T \in \mathcal{C}_2$, let $\{x_T, x'_T\}$ be a light pair of T with $s_T = s(x_T, x'_T)$. Then*

$$d(H) \geq \frac{k+4}{2} \sum_{T \in \mathcal{C}_2} s_T$$

and $|\mathcal{C}_2| \leq k-4$.

Proof. From Proposition 3 it follows that for every component $T \in \mathcal{C}_2$ there is a light pair $\{x_T, x'_T\}$ of T. Let $s_T = s(x_T, x'_T)$. Based on Proposition 4, we then infer that

$$\begin{aligned} d(H) &= \sum_{y \in H} d(y) \geq \sum_{T \in \mathcal{C}_2} g(T) \geq \sum_{T \in \mathcal{C}_2} s_T(k - s_T) \geq \sum_{T \in \mathcal{C}_2} s_T\left(k - \frac{k-4}{2}\right) \\ &= \frac{k+4}{2} \sum_{T \in \mathcal{C}_2} s_T. \end{aligned}$$

By Proposition 4, $g(T) \geq k-1$ and, therefore, $d(H) \geq |\mathcal{C}_2|(k-1)$. Because of (2), this implies $|\mathcal{C}_2| \leq (k(k-4))/(k-1)$ and, therefore, $|\mathcal{C}_2| \leq k-4$. This proves Proposition 5. □

3.3. A Good Colouring for the Components in $\mathcal{C}_2$

Now we are ready to show that there is a colouring $\varphi \in \mathcal{F}$ such that T is Φ_φ-colourable for all $T \in \mathcal{C}_2$. First, we prove the following result.

Proposition 6. *Let $T \in \mathcal{C}_2$ and let $\{x, x'\}$ be a light pair of T with $s = s(x, x')$. Suppose that $\varphi \in \mathcal{F}$ and T is not Φ_φ-colourable. Then*

$$\Phi(x) - \Phi(x') = \{\varphi(y) \,|\, y \in H_x - H_{x'}\} \tag{3}$$

and

$$\Phi(x') - \Phi(x) = \{\varphi(y) \mid y \in H_{x'} - H_x\}. \tag{4}$$

Proof. For $v \in V(T)$, we have $k-1 = d^*(v) = d_T(v) + |H_v|, |\Phi(v)| = k-1$, and $\Phi_\varphi(v) = \Phi(v) - \{\varphi(y) \mid y \in H_v\}$. Since $\varphi \in \mathcal{F}$, this implies that $|\Phi_\varphi(v)| = d_T(v)$ for all $v \in V(T)$ and (T, Φ_φ) is a bad pair.

Let B be the end-block of T with $x, x' \in V(B)$. Then, by Proposition 3, B is a K_t with $t \geq 4$. From Lemma 3 we infer that $\Phi_\varphi(x') = \Phi_\varphi(x) = \sum$. Consequently, $\Phi(v) = \sum \cup \{\varphi(y) \,|\, y \in H_v\}$ for $v \in \{x, x'\}$. Since $\varphi \in \mathcal{F}$, this implies (3) and (4). □

Proposition 7. *For every component $T \in \mathcal{C}_2$, let $\{x_T, x'_T\}$ be a light pair of T and let $s_T = s(x_T, x'_T)$ and $S_T = H_{x_T} - H_{x'_T}$. Then there is a colouring $\varphi \in \mathcal{F}$ such that*

$$\Phi(x_T) - \Phi(x'_T) \neq \{\varphi(y) \mid y \in S_T\}$$

for all $T \in \mathcal{C}_2$.

Proof. For $y \in H$, let $\sigma(y)$ denote the number of all components $T \in \mathcal{C}_2$ such that $y \in S_T$. Assume that $H = \{y_1, \ldots, y_m\}$ where $\sigma(y_1) \geq \cdots \geq \sigma(y_m)$. Let $H_0 = \emptyset$ and, for $i = 1, \ldots, m$, let $H_i = \{y_1, \ldots, y_i\}$.

A Φ-colouring φ of $G^*[H_i]$ is said to be *good*, if $\varphi(y) \neq \varphi(y')$ for every two distinct vertices $y, y' \in H_i$ and

$$\Phi(x_T) - \Phi(x'_T) \neq \{\varphi(y) \mid y \in S_T\}$$

for all $T \in \mathcal{C}_2$ with $S_T \subseteq H_i$. We show by induction that, for $i = 0, \ldots, m$, there is a good Φ-colouring of $G^*[H_i]$. This is evident for $i = 0$.

Now assume that $1 \leq i \leq m$ and there is a good Φ-colouring φ of $G^*[H_{i-1}]$. Let $\mathcal{T}$ denote the set of all components $T \in \mathcal{C}_2$ such that $S_T \subseteq H_i$ and $y_i \in S_T$.

Let $\Gamma = \Phi(y_i) - \{\varphi(y) \mid y \in H_{i-1}\}$. A colour $\alpha \in \Gamma$ is called *feasible* for $T \in \mathcal{T}$ if

$$\Phi(x_T) - \Phi(x'_T) \neq \{\varphi(y) \mid y \in S_T, y \neq y_i\} \cup \{\alpha\}.$$

If a colour $\alpha \in \Gamma$ is not feasible for a component $T \in \mathcal{T}$, then, because of $\Gamma \cap \{\varphi(y) \mid y \in S_T, y \neq y_i\} \neq \emptyset$, every colour in $\Gamma - \{\alpha\}$ is feasible for T. Let Γ_1 be the set of all colours $\alpha \in \Gamma$ such that α is not feasible for some component $T \in \mathcal{T}$ and let $\Gamma_2 = \Gamma - \Gamma_1$. We claim that $\Gamma_2 \neq \emptyset$.

To prove this, let $c = |\Gamma_2|$. Clearly, $c = k - 1 - (i - 1) - |\Gamma_1| \geq k - i - |\mathcal{T}| \geq k - i - \sigma(y_i)$ and, by Proposition 5, $\sigma(y_i) \leq |\mathcal{C}_2| \leq k - 4$. Therefore, in case of $1 \leq i \leq 3$, we have $c \geq 1$. If $i \geq 4$, then we argue as follows. First, we infer from (1) that $i \leq |H| \leq k - 4$ and, therefore, $k \geq 8$. Clearly, if $\sigma(y_i) < k - i$, then $c \geq 1$. Otherwise, $\sigma(y_1) \geq \cdots \geq \sigma(y_i) \geq k - i$ and we conclude that

$$\sum_{T \in \mathcal{C}_2} s_T = \sum_{y \in H} \sigma(y) \geq i(k - i) \geq 4(k - 4).$$

Because of Proposition 5 and $k \geq 8$ this implies that

$$d(H) \geq \frac{k + 4}{2}(4(k - 4)) > k(k - 4),$$

a contradiction to (2). Therefore, $c \geq 1$ and the claim is proved.

Consequently, there is a colour $\alpha \in \Gamma$ such that α is feasible for all $T \in \mathcal{T}$. Then the colouring φ' with $\varphi'(y) = \varphi(y)$ for $y \in H_{i-1}$ and $\varphi'(y_i) = \alpha$ is a good Φ-colouring for $G^*[H_i]$. Thus Proposition 7 is proved. □

By Proposition 3, for every component $T \in \mathcal{C}_2$ there exists a light pair. Then we conclude from Propositions 6 and 7 that there is a colouring $\varphi \in \mathcal{F}$ such that T is Φ_φ-colourable for all components T of $G[L]$ that belong to $\mathcal{C}_2$. By Proposition 2, this implies that $G[L]$ is Φ_φ-colourable. Consequently, G is Φ-colourable, a contradiction to (a). This contradiction proves Theorem 2.

4. CONCLUDING REMARKS

We are not able to give a complete description of the extremal cases for the bound establish in Theorem 2. A graph G is called a *Hajós graph* of order $2k-1$ if the vertex set of G consists of three non-empty pairwise disjoint sets A, B_1, B_2 with $|B_1| + |B_2| = |A| + 1 = k - 1$ and two additional vertices a, b such that $G[A]$ and $G[B_1 \cup B_2]$ are two complete subgraphs of G not joined by any edge, $N_G(a) = A \cup B_1$ and $N_G(b) = A \cup B_2$. Dirac [7] proved that $d(G) = (k-1)n + k - 3$ for a k-colour-critical graph G on $n \geq k+2$ vertices and with $k \geq 4$ if and only if G is a Hajós graph of order $2k-1$. We do not know any other critical hypergraph for which the bound in Theorem 2 is sharp. On the contrary, for critical hypergraphs G having a large number of vertices the bound for *d(G)* given in Theorem 2 can be improved essentially (see [15]).

Consider a Φ-critical hypergraph G on n vertices where Φ is a list for G with $|\Phi(v)| = k-1$ for every $v \in V(G)$. If G is r-uniform, i.e. $|e| = r$ for every $e \in E(G)$, then $d(G) = r|E(G)|$ and, therefore, Theorem 2 implies that

$$|E(G)| \geq \frac{1}{r}((k-1)n + k - 3).$$

Using probabilistic arguments, the authors have been able to improve this bound significantly provided that $r \geq 3$ and k is large enough. In [16] the following result was proved.

Theorem 3. *Let G be a hypergraph on n vertices and without ordinary edges, and let Φ be a list for G with $|\Phi(v)| = k-1$ for every $v \in V(G)$. If G is Φ-critical, then $|E(G)| \geq (k-1)(1 - 3/\sqrt[3]{k-1})n$.*

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