Total Choosability of Multicircuits I

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Abstract: A multicircuit is a multigraph whose underlying simple graph is a circuit (a connected 2-regular graph). In this pair of papers, it is proved that every multicircuit C has total choosability (i.e., list total chromatic number) $\mathrm{ch}''(C)$ equal to its ordinary total chromatic number $\chi''(C)$. In the present paper, the kernel method is used to prove this for every multicircuit that

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has at least two vertices with degree less than its maximum degree Δ. The result is also proved for every multicircuit C for which $\chi''(C) \geq \Delta + 2$.

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1. GENERAL INTRODUCTION

It has long been conjectured that every (multi)graph has edge choosability (i.e., list edge chromatic number—see the definitions in the next section) equal to its ordinary edge chromatic number. By analogy with this conjecture, several authors, including Borodin et al. [3], have conjectured that every multigraph has total choosability (i.e., list total chromatic number) equal to its ordinary total chromatic number; we call this the *List-Total-Colouring Conjecture (LTCC)*.

Having made such a conjecture, one naturally looks for a class of graphs—*any* class of graphs—for which it can be proved. In [6], we tried to prove it for multicircuits, where a *multicircuit* is a multigraph whose underlying simple graph is a circuit (a connected 2-regular graph). We shall always use n for the order and Δ for the maximum degree of a multicircuit.

In [6], we proved the LTCC for multicircuits with $n \leq 5$ and for multicircuits with even n that contain at least six suitably distributed vertices with degree less than Δ. However, we underestimated the difficulty of proving the LTCC, even for such a simple class of multigraphs, and in [6] we were really trying to prove a stronger conjecture concerning the choosability of inflations of total graphs of multicircuits. In the present work, we do not consider inflations at all; instead, we complete the more limited task of proving the LTCC for all multicircuits.

We use two essentially different approaches. In most of the present work, we use the now-familiar kernel method that was pioneered so successfully by Galvin [5], in order to prove the LTCC for all multicircuits of even order except those that are Δ-regular, and for all those of odd order except those that are Δ-regular or have $n-1$ vertices of degree Δ and one of lower degree.

In the successor work [7], we use the method of Alon and Tarsi ([2], see also [1]) to prove the LTCC for all multicircuits of even order and some regular and near-regular multicircuits of odd order. The results of [7], together with those of [6] and the present work, complete the proof of the LTCC for all multicircuits.

For nonregular multicircuits of even order, we thus have two entirely different proofs, one using the kernel method and one using the method of Alon and Tarsi. In fact, there are relatively few multicircuits for which the result does not follow from [6] and [7]. However, we believe that it is of interest to have two such different proofs, especially since the arguments in the present work are algorithmic, whereas the method of Alon and Tarsi used in [7] is nonconstructive.

2. DEFINITIONS AND PRELIMINARIES

Let $G=(V,E)$ be a multigraph with vertex-set $V(G)=V$ and edge-set $E(G)=E$. We say that G is *totally k-choosable* if, whenever each element $x\in V\cup E$ is given a set ("list") $L(x)$ of k "colours", we can choose a colour $c(x)\in L(x)$ for each element x so that no two adjacent vertices or adjacent edges have the same colour, and no vertex has the same colour as an edge incident with it; in this case we say loosely that G can be *totally coloured from its lists*. The *total choosability* or *list total chromatic number* $\mathrm{ch}''(G)$ of G is the smallest integer k such that G is totally k-choosable. The *choosability* or *list* (*vertex*) *chromatic number*, $\mathrm{ch}(G)$, and the *edge choosability*, *list edge chromatic number* or *list chromatic index*, $\mathrm{ch}'(G)$, are defined similarly in terms of colouring vertices alone, or edges alone, respectively; and so are the concepts of (*vertex-*)*k-choosability* and *edge-k-choosability*. The ordinary vertex, edge and total chromatic numbers of G are denoted by $\chi(G)$, $\chi'(G)$ and $\chi''(G)$. Of course, multiple edges are irrelevant to vertex-colourings. We shall denote the *simple* line graph and total graph of G by $L(G)$ and $T(G)$, respectively. Then $\mathrm{ch}'(G)=\mathrm{ch}(L(G))$, $\mathrm{ch}''(G)=\mathrm{ch}(T(G))$, $\chi'(G)=\chi(L(G))$, and $\chi''(G)=\chi(T(G))$.

In this terminology, the well-known List-Edge-Colouring Conjecture (LECC) asserts that $\mathrm{ch}'(G)=\chi'(G)$ for every multigraph G, and the LTCC, mentioned in the previous section, asserts that $\mathrm{ch}''(G)=\chi''(G)$ for every multigraph G. In the present work, we are concerned with the following more specialized conjecture which we made in [6].

Conjecture 1. *If C is a multicircuit with n vertices, m edges and maximum degree Δ, then $\mathrm{ch}''(C)=\chi''(C)=\tau(C)$, where*

$$\tau(C):=\max\left\{\Delta+1,\left\lceil\frac{m}{\lfloor\frac{1}{2}n\rfloor}\right\rceil,\left\lceil\frac{m+n}{\lfloor\frac{2}{3}n\rfloor}\right\rceil\right\}. \tag{1}$$

Note that if n is even then the middle term in (1) is not needed, since it cannot exceed $\Delta+1$; and if $n>4$ and $\Delta>2$ then there are only three multicircuits (with $(n,\Delta)=(7,3)$, $(7,4)$, and $(10,3)$) for which the last term in (1) is needed.

Conjecture 1 was proved in [6] for $n\le 5$. The main result of the present work is the following theorem.

Theorem 1. *Let C be a multicircuit with $n\ge 6$ vertices, m edges and maximum degree Δ.*

(i) *If n is even and C is not Δ-regular, then $\mathrm{ch}''(C)=\chi''(C)=\Delta+1$.*
(ii) *If n is odd and C has at least two vertices with degree less than Δ, then*

$$\mathrm{ch}''(C)=\chi''(C)=\max\left\{\Delta+1,\left\lceil\frac{m}{\lfloor\frac{1}{2}n\rfloor}\right\rceil\right\}.$$

Note that these results agree with Conjecture 1. It is clear in each case that $\text{ch}''(C) \geq \chi''(C) \geq \tau(C)$, where $\tau(C)$ is defined in (1), and so it suffices to prove that $\text{ch}''(C) \leq \tau(C)$. This we do in Theorems 2 and 3. Note that if n is even and C is not Δ-regular then C has at least two vertices with degree less than Δ; thus Theorem 1 follows from Theorems 2 and 3, whose hypotheses are satisfied because of the existence of these vertices. In essence, Theorem 2 proves the result for even n and Theorem 3 for odd n; however, Theorem 3 rests on Theorem 2, and some parts of the proof for odd n are covered by Theorem 2.

We conclude the paper, in Theorem 4, by proving Conjecture 1 for every multicircuit C for which $\tau(C) \geq \Delta + 2$. These are precisely the multicircuits C for which $\chi''(C) \geq \Delta + 2$, as mentioned in the Abstract, although this equivalence will not be clear until the proof of Conjecture 1 is finally completed in [7].

Throughout the paper, C will be a multicircuit of order $n \geq 6$ with vertices $v_0, \ldots, v_{n-1}$, and H will be an induced subgraph of $T(C)$, with vertex-sets V_i and E_i $(i = 0, \ldots, n-1)$, where, for each i, $V_i = \emptyset$ or $\{v_i\}$, and E_i corresponds to (some) edges joining v_{i-1} and v_i in C. (Note this change from the terminology in [9], where edges with subscript i went between v_i and v_{i+1}.) Thus we can write $V(H) = \bigcup_{i=0}^{2n-1} Z_i$, where $Z_{2i-1} = E_i$ and $Z_{2i} = V_i$ for each i, and $z \in Z_i$ is adjacent to $z' \in Z_j$ $(z' \neq z)$ if and only if $|i - j| \leq 2$ (modulo $2n$). Some sets Z_i may be empty, but not all. For convenience, we shall refer to the direction of increasing subscripts as *clockwise*. Subscripts to V, E, and Z are to be taken modulo n, n and $2n$, respectively.

Suppose that a list of colours is assigned to each vertex of H. We say that a colour is *present on* Z_i if it belongs to the list of at least one vertex in Z_i; clearly this implies $Z_i \neq \emptyset$. If n is odd, a colour c will be called *exceptional* if it is present on every set E_i $(0 \leq i \leq n-1)$ and there exists a j such that $|E_{j-1}| = |E_j| = 1$ and either c is present on V_j or $V_j = \emptyset$; c is *nonexceptional* otherwise. Note that a colour with these properties is not called exceptional if n is even; in this case all colours are nonexceptional. Note also that by deleting vertices from H one cannot turn an exceptional colour into one that is nonexceptional *and present on every set* E_i.

For $0 \leq i \leq n-1$, let $f_i := |E_{i-1}| + |V_{i-1}| + |E_i|$ and let

$$f_i^+ := \begin{cases} f_i & \text{if } |E_{i-1}| \geq 1 \\ f_i + 1 & \text{if } |E_{i-1}| = 0. \end{cases}$$

Note that, if H is the whole of $T(C)$, then $f_i^+ = f_i = d(v_{i-1}) + 1$, where d denotes degree in C; thus $f_i^+ = f_i \leq \Delta + 1$ for every i, and by the hypotheses of Theorem 1 there are at least two values of i for which $f_i^+ + 1 = f_i + 1 \leq \Delta + 1$.

3. EVEN CASE

This section is devoted to a proof of the following theorem, which proves Theorem 1 (i) and also goes some way towards proving Theorem 1 (ii). It is stated in the terminology introduced in the previous section.

Theorem 2. *Let $n \geq 6$. Suppose that, for each i, each vertex in V_i is given a list of at least l_i colours and each vertex in E_i is given a list of at least l_i' colours, where $l_i \geq f_i^+$ and $l_i' \geq f_i$, and there are at least two values of i for which $l_i \geq f_i^+ + 1$ and $l_i' \geq f_i + 1$. Suppose moreover that if n is odd then no nonexceptional colour is present on every set E_i $(0 \leq i \leq n-1)$. Then the vertices of H can be coloured from their lists.*

Proof. If $l_i = f_i^+ + \varepsilon$ and $l_i' = f_i + \varepsilon$, then we say that i, E_i and V_i are *ε-special*, and 1-special indices and sets are called simply *special*.

We may suppose that H is a counterexample to the Theorem with as few vertices as possible and, subject to this condition, such that $\sum |E_i|$ is as small as possible. We may also suppose that every list of colours has the minimum size specified in the statement of the Theorem, and, for each i, that l_i and l_i' are equal to the lower bounds stated for them in the Theorem. Thus there are two 1-special values of i and $n-2$ 0-special values.

Let D be the digraph with $V(D) = V(H)$ in which $z \in Z_i$ is joined by an arc to $z' \neq z$ if and only if $z' \in Z_{i-2} \cup Z_{i-1} \cup Z_i$; so each edge of H between two vertices in the same set Z_i is replaced by two oppositely oriented arcs in D, and every other edge of H is oriented anticlockwise. Note that, in D, each vertex in E_i has outdegree $f_i - 1$ and each vertex in V_i has outdegree at most $f_i^+ - 1$, for each i, and so,

for each vertex v, the number of colours in the list of v is strictly greater than its outdegree. (2)

For a colour c, let D_c denote the subdigraph of D induced by the set of all vertices with colour c in their lists. Recall that a *kernel* of a digraph D is a set K of nonadjacent vertices such that every vertex in $V(D)\backslash K$ is joined by an arc to at least one vertex in K. By a *pseudokernel* of D_c, we shall mean a set K of nonadjacent vertices such that if $v \in V(D_c)\backslash K$ and v is *not* joined to a vertex in K, then $v \in V_i$ for some i with the following properties: $E_{i-1} \cap K \neq \emptyset$ (which implies that c is present on both E_{i-1} and V_i), and $|E_{i-1}| \neq 1$. If D_c has a kernel or pseudokernel K, then we call *K a (pseudo)kernel of (the colour-class of) c*. In this case, if we give colour c to all vertices in K, delete these vertices from H and delete c from the lists of all other vertices that contain it, then we obtain a smaller graph H' satisfying all the hypotheses of the Theorem, since f_i decreases by 1 whenever c was present on E_i and f_i^+ decreases by 1 whenever c was present on V_i and v_i is not deleted. Moreover, if H' can be coloured from its lists then so can H. It follows from the minimality of H that

no colour-class has a kernel or pseudokernel. (3)

The following algorithm attempts to construct a kernel $K = \{k_1, \ldots, k_r\}$ of D_c.

Algorithm Ker. Choose k_1 to be a vertex in some nonempty set $Z_s \cap D_c$. Having chosen $k_1, \ldots, k_j$, where $k_j \in Z_p$, say, choose $k_{j+1} \in Z_q \cap D_c$ where Z_q is the first set after Z_{p+2} in clockwise order such that $Z_q \cap D_c \neq \emptyset$. If k_{j+1} is adjacent or equal to k_1, set $r := j$ and stop. We note that if this happens and $k_{j+1} \in Z_q$ then $q = s - 2$ or $s - 1$ or s (mod n), and K is a kernel if and only if $q = s$. If $k_r \in Z_t$ then Z_s and Z_t are called the *starting set* and *finishing set*, respectively.

We now see that

$$\text{for each } i\text{, every colour } c \text{ is present on at least one of } Z_{i-1} \text{ and } Z_i, \qquad (4)$$

since otherwise we could construct a kernel of D_c by Algorithm Ker, taking as starting set the first set Z_s in clockwise order after Z_i such that $Z_s \cap D_c \neq \emptyset$; and this would contradict (3).

We now proceed by a sequence of claims. Let the total number of colours that are present on $V(H)$ be χ.

Claim 3.1. $\chi \geq 3$.

Proof. Let i be one of the two special indices. By (4), E_{i-1} and V_{i-1} are not both empty, nor are V_{i-1} and E_i, nor are E_i and V_i. So $f_i \geq 2$ unless $E_{i-1} = E_i = \emptyset$, and $f_i^+ \geq 2$ always. If $E_i \neq \emptyset$ then $\chi \geq f_i + 1 \geq 3$, and if $V_i \neq \emptyset$ then $\chi \geq f_i^+ + 1 \geq 3$. This proves Claim 3.1. ■

Claim 3.2. *Not all sets E_i are empty, nor are all sets V_i empty.*

Proof. If all sets E_i are empty, then every colour is present on every set V_i by (4). This implies that no set V_i is empty, so that $f_i^+ = 2$ for every i. But then the special sets V_i should have more colours than other sets V_i, a contradiction.

If all sets V_i are empty, then every colour is present on every set E_i by (4). If n is even then every colour c has a kernel, made up of vertices in $E_i \cap D_c$ for all even i, and this contradicts (3). If n is odd then, by the hypotheses of the Theorem, every colour c is exceptional. The definition of an exceptional colour implies the existence of a j such that $|E_{j-1}| = |E_j| = 1$, which means that $f_j = 2$. Since every colour is present on E_j, $\chi \leq 3$. By Claim 3.1, $\chi = 3$. Thus $l_i' = 3$ for each i, since if $|E_i| \geq 2$ then $3 = \chi \geq l_i' \geq f_i \geq 3$, and if $|E_i| = 1$ then $l_i' = \chi = 3$ because every colour is present on the one edge in E_i. But then, since there are two special values of i, $2 + 2|E(H)| = 2 + \sum_{i=0}^{n-1} f_i = \sum_{i=0}^{n-1} l_i' = 3n$, which is impossible because the LHS is even and the RHS is odd. This contradiction proves Claim 3.2. ■

Claim 3.3 *Let c be a nonexceptional colour that is present on at least one set E_i. Then there exists an i such that c is present on both E_{i-1} and V_i.*

Proof. Suppose the Claim is false, so that if c is present on E_{i-1} then it is not present on V_i and so it is present on E_i by (4). Then c is present on every set E_i and on no set V_i. Since c is nonexceptional, the hypotheses of the Theorem imply that n is even. Thus D_c has a kernel, made up of vertices in $E_i \cap D_c$ for all even i, and this contradicts (3). ■

Claim 3.4. *Every colour c is present on at least one set E_i.*

Proof. Suppose c is present on no set E_i. By (4), c is present on every set V_i. If n is even, then D_c has a kernel, made up of vertices v_i for all even i, and this contradicts (3). Thus n is odd.

Let c' be a colour that is present on some set E_i, which must exist by Claim 3.2. Then there exists an i such that c' is present on E_{i-1} and V_i; this follows from Claim 3.3 if c' is nonexceptional, and it follows from the definition of an exceptional colour otherwise, since (in the present Claim) no set V_i is empty. Apply Algorithm Ker to $D_{c'}$ with V_i as the starting set to construct a set K'; give colour c' to the vertices in K', delete them from H, and delete colour c' from all lists containing it, to form a graph H^* with corresponding digraph D^*. The finishing set is one of E_{i-2}, V_{i-2}, and E_{i-1}. If it is E_{i-1} then K' is a kernel of $D_{c'}$, contradicting (3). If it is V_{i-2} then H^* satisfies all the hypotheses of the Theorem except that vertices of E_i may have lost a colour from their lists but f_i has not decreased; so give v_{i-1} colour c and delete it, which decreases f_i without changing any remaining lists; by the minimality of H, the resulting graph can be coloured from its lists, and hence so can H. This contradiction shows that the finishing set must be E_{i-2}. Then H^* satisfies all the hypotheses of the Theorem except that vertices in V_{i-1} and E_i may have lost a colour from their lists without f_{i-1}^{+} or f_i decreasing.

By (4) and for reasons of parity, we see that there exists a $j \neq i$ such that $v_j \in K'$. Therefore there exists a j such that $v_j \in K'$ and $v_h \notin K'$ for all $h \in \{j+1, \ldots, i-1\}$. Then K' contains vertices in each of the sets E_{j+2}, $E_{j+4}, \ldots, E_{i-2}$, which shows that the sequence $\{j+1, \ldots, i-1\}$ has an odd number of terms. Apply Algorithm Ker to D_c^* (in an obvious terminology) with V_{j+1} as the starting set. Since c is not present on any set E_h, this will construct a kernel K of D_c^* with $v_{i-1} \in K$. Give colour c to all vertices in K, delete them from H^* and delete colour c from all lists containing it. The resulting graph $H^{*\prime}$ satisfies all the hypotheses of the Theorem, since we have deleted one vertex v_{i-1} that had too few colours in H^*, and in the process we have reduced f_i without removing a further colour from the list of any vertex in E_i. Thus $H^{*\prime}$ can be coloured from its lists, and therefore so can H. This contradiction completes the proof of Claim 3.4. ■

The following Claim is crucial: it is what makes the whole proof work.

Claim 3.5. *If some colour c is present on both E_{i-1} and V_i, then for at least one $j \in \{i-1, i+1\}$ the following hold: $|E_{j-1}| = |V_j| = 1$ and c is present on both E_{j-1} and V_j.*

Proof. Suppose the Claim is false, so that if c is present on E_{i-2} and V_{i-1} then $|E_{i-2}| \geq 2$ and if c is present on E_i and V_{i+1} then $|E_i| \geq 2$. Apply Algorithm Ker to D_c with V_i as the starting set to construct a set K. The finishing set is one of E_{i-2}, V_{i-2}, and E_{i-1}. If it is E_{i-1} then K is a kernel of D_c. If it is E_{i-2} or V_{i-2} and c is not present on E_i then K is a pseudokernel or a kernel. If it is E_{i-2} or V_{i-2}

and c is present on E_i then we obtain a pseudokernel or a kernel by removing v_i from K and inserting instead a vertex of $E_i \cap D_c$. In each case, we obtain a contradiction to (3) and this completes the proof of the Claim. ■

Claim 3.6. *For each nonexceptional colour c, there exists a $j = j(c)$ such that $|E_{j-1}| = |E_j| = 1$ and c is present on E_j and V_j.*

Proof. By Claims 3.3 and 3.4, there is an i such that c is present on both E_{i-1} and V_i. By Claim 3.5, we can choose i so that $|E_{i-1}| = 1$, and by Claim 3.5 again we find a j such that c is present on E_{j-1}, E_j, V_j, and V_{j+1}, and $|E_{j-1}| = |E_j| = 1$. ■

Recall that χ is the total number of colours that are present on $V(H)$.

Claim 3.7. $\chi \leq 5$.

Proof. Note first that if c is an exceptional colour, then by definition there is a j such that $|E_{j-1}| = |E_j| = 1$ and either c is present on V_j or $V_j = \emptyset$. If $V_j = \emptyset$ then every colour is present on E_j by (4), and since $f_j \leq 3$ it follows that $\chi \leq 4$. Thus we may suppose that $V_j \neq \emptyset$, so that c is present on both E_j and V_j (since an exceptional colour is present on every set E_i by definition). It follows that for every colour c, including the exceptional colours, there exists a $j = j(c)$ satisfying the conclusion of Claim 3.6. For a fixed such j, let A be the set of colours present on E_j and B the set of colours present on V_j, and assume that j is ε-special ($\varepsilon \in \{0, 1\}$). Then $|A| = |B| = f_j + \varepsilon$. Since $f_j \leq 3$ and every colour must be present on either E_j or V_j by (4), the number of colours c with $j(c) = j$ is at most $|A \cap B| = |A| + |B| - |A \cup B| \leq 6 + 2\varepsilon - \chi$. If $\chi \geq 6$ then this is at most 2ε, and since there are only two 1-special values of j there can be at most four colours in total, a contradiction. ■

We now consider a number of cases, and obtain a contradiction in each case.

Case 1. $\chi \geq 4$ *and* $E_h = \emptyset$ *for some* h.

By Claim 3.2, not all sets E_h are empty, and so we may choose i so that $E_i = \emptyset$ and $E_{i-1} \neq \emptyset$. Then every colour c that is present on E_{i-1} is also present on V_i by (4), and so, by Claim 3.5, $|E_{i-2}| = |V_{i-1}| = 1$ and c is present on E_{i-2} and V_{i-1}. By (4), every colour is present on V_{i-1}, and so V_{i-1} has $\chi \geq 4$ colours. If $|E_{i-1}| = 1$, then we can colour all vertices of H clockwise starting at V_i and finishing with V_{i-1}, since (2) holds in D and we can choose a colour for V_{i-1} that is different from the colours given to V_{i-2}, E_{i-1}, and V_i. Therefore $|E_{i-1}| \geq 2$. Applying Claim 3.5 again to E_{i-2} and V_{i-1}, we find that $|E_{i-3}| = |V_{i-2}| = 1$. Moreover, every colour is present on V_i and E_{i-1} (since $f_{i-1} = f^+_{i-1}$ and every colour is present on V_{i-1}), and so by Claim 3.5 every colour is present on E_{i-2} and V_{i-1} and therefore on E_{i-3} and V_{i-2}. Thus we can colour all vertices of H clockwise starting at V_i and finishing at V_{i-1}; when we reach E_{i-2}, we give it a colour different from the colours given to E_{i-3}, V_{i-3}, and V_i, so that the colour of

V_i can be used on V_{i-2} or E_{i-1} and is not used on V_{i-1}. This contradiction shows that Case 1 cannot arise.

Case 2. $\chi \geq 4$ *and* $V_h = \emptyset$ *for some* h.

By Claim 3.2, not all sets V_h are empty, and so we can choose i so that $V_{i-1} = \emptyset$ and $V_i \neq \emptyset$. Then every colour c that is present on V_i is also present on E_{i-1} by (4), and so, by Claim 3.5, $|E_i| = |V_{i+1}| = 1$ and c is present on V_{i+1}. By (4), every colour is present on E_{i-1} and E_i, and hence on V_i since $l_i \geq l'_i$ always; and so $f_i \geq \chi - 1 \geq 3$ (with equality only if i is special). Since $V_{i-1} = \emptyset$ and $|E_i| = 1$, it follows that $|E_{i-1}| \geq 2$. Applying Claim 3.5 again to E_i and V_{i+1}, we find that $|E_{i+1}| = 1$, which means that $f_{i+1} = 3$; and since every colour is present on E_{i-1} and V_i, therefore every colour is present on E_i and V_{i+1}. Since $\chi \geq 4$ and $f_{i+1} = 3$, therefore $i+1$ must be special. Choose an arbitrary colour c that is present on E_{i+2} (which must exist since $E_{i+2} \neq \emptyset$ after Case 1), and apply Algorithm Ker to D_c with E_{i+2} as the starting set to construct a set K. Give colour c to the vertices in K, delete them from H and delete colour c from all remaining lists to form a graph H^*. The finishing set is either E_i or V_i. If it is V_i then K is a kernel of D_c, contradicting (3). So it must be E_i. Now (2) still holds in H^*, since the only vertex that has lost a colour without decreasing its outdegree is v_{i+1}, which had a spare colour since it was special. So we can colour all vertices of H^* clockwise starting from v_i, and this contradiction shows that Case 2 cannot arise.

Case 3. $\chi \geq 4$ *and there exists a* j *with the following properties:* $j+2$ *is special and* $j+3$ *is not special,* $|E_i| = 1$ *for all* i *except that possibly* $|E_j| = 2$*, and if* $|E_j| = 2$ *then* $j-1$ *is special. (Note that this covers the case in which* $\chi \geq 4$ *and* $|E_i| = 1$ *for every* i*.)*

For convenience, assume $j = 0$. We may suppose by Case 2 that $|V_i| = 1$ for every i, so that $f_1 \geq 3$ and $f_2 = f_3 = 3$. Thus there are at least three colours present on each of E_1 and V_1, exactly four colours on each of E_2 and V_2, and exactly three on each of E_3 and V_3. By Claim 3.7 and (4), $\chi \leq 5$, and E_3 and V_3 have exactly $6-\chi$ colours in common. So $3-(6-\chi) = \chi - 3 \geq 1$ colours are present on V_3 but not E_3. We proceed as follows:

Step 1. Choose a colour c that is present on V_3 but not E_3. Apply Algorithm Ker to D_c with V_3 as the starting set, give colour c to all the vertices in the set K constructed by the algorithm, delete these vertices and delete colour c from all other vertices to obtain a graph H^* with corresponding digraph D^*. Note that the finishing set is E_1, since the other two possibilities, V_1 and E_2, would both imply that K was a kernel, contradicting (3).
Step 2. Choose a colour c' that is present on E_3 in H^*. Apply Algorithm Ker again to $D^*_{c'}$ (in an obvious terminology) with E_3 as the starting set, give colour c' to the vertices in the resulting set K', and delete these vertices and the colour c' from all other vertices to obtain a graph $H^{*\prime}$ with corresponding digraph $D^{*\prime}$.

Step 3. Try to colour the remaining vertices clockwise starting with E_4.

We will easily be able to complete the colouring provided that (2) still holds in $D^{*\prime}$. It certainly holds if Step 2 finishes on V_1. The only other possibility is that Step 2 finishes on V_0. In this case V_1, E_2, and V_2 are isolated from the rest of the graph and are left with at least 1, 2, and 2 colours, respectively. If the two colours left on E_2 are not the same as the two colours left on V_2, then we can easily colour these three vertices before colouring all remaining vertices as in Step 3.

The only problem arises if E_2 and V_2 are left with the same two colours. This means that they started with the same list of four colours. By (4), therefore $\chi = 4$. In this case, we forget the above colouring and start again from scratch. It is easy to see that if $|E_i| = |V_i| = 1$ then the lists permit the elements of $E_i \cup V_i$ to be coloured with any pair of colours (although we cannot specify which vertex gets which colour of the pair), since by (4) each colour appears in at least one of the two lists, and each list contains at least one colour of the pair since $\chi = 4$. Let the four colours be labelled a, b, c, d, chosen so that b is present on E_3 and a on V_3.

Suppose first that $|E_0| = 1$. Then colour E_i and V_i with a and b whenever i is even and with c and d whenever i is odd. If n is even then this colouring is proper. If n is odd then recolour E_0 and V_0 with c and d, E_1 and V_1 with a and b, E_2 with the same colour as V_3 and V_2 with the same colour as E_1, which is possible since all four colours are present in the lists of E_2 and V_2.

If $|E_0| = 2$ then $n - 1$ is special by hypothesis and $f_0 = f_1 = 4$, and so all four colours are present on all elements of E_{n-1}, V_{n-1}, E_0, V_0, E_1, V_1, E_2, and V_2. Give these elements colours a, b, $\{c, d\}$, a, b, c, a, b, respectively. If n is odd, colour the elements of all remaining sets E_i and V_i with a and b if i is even and with c and d if i is odd. If n is even, interchange even and odd in this description, then recolour the elements of V_2, E_3, and V_3 with d, b, and a, respectively. In all cases, we have the required proper list colouring, and this disposes of Case 3.

Since Case 3 includes the case in which $\chi \geq 4$ and $|E_i| = 1$ for each i, the following case will complete the discussion of all possibilities with $\chi \geq 4$.

Case 4. $\chi \geq 4$ *and* $|E_i| \geq 2$ *for some* i.

We may suppose by Cases 1 and 2 that no set E_i or V_i is empty, so that $f_i \geq 3$ for each i, and since $\chi \leq 5$ by Claim 3.7 it follows that for every i there is a colour c that is present on both E_{i-1} and V_i. Choose j so that $|E_j| \geq 2$ and $|E_{j-1}| = 1$. Then Claim 3.5 implies that $|E_{j-2}| = 1$, and every colour present on E_{j-1} and V_j is also present on E_{j-2} and V_{j-1}. Since $f_j \geq 4$ and $f_{j-1} = 3$, there are at least $4 + 3 - \chi \geq 2$ colours common to E_{j-1}, V_j, and V_{j-1}.

Suppose first that $j - 1$ is not special. Since $f_{j-1} = 3$ and every colour is present on E_{j-1} or V_{j-1} by (4), therefore $\chi = 4$, which means that every colour is present on V_j. Therefore there are three colours common to E_{j-1}, V_j, and V_{j-1}, and this means that the fourth colour does not appear on E_{j-1} or V_{j-1}, which contradicts (4). This shows that $j - 1$ is special.

Therefore there are at least $4+4-\chi \geq 3$ colours common to E_{j-1}, V_j, and V_{j-1}. If $|E_{j+1}| \geq 2$ then $f_{j+1} \geq 5$ and so all five colours are present on V_{j+1}. Thus there are at least four colours common to E_j and V_{j+1}, and by Claim 3.5 these are present on all of E_{j-1}, V_j, and V_{j-1}. This means that the fifth colour cannot be present on E_{j-1} or V_{j-1}, contrary to (4). This contradiction shows that $|E_{j+1}| = 1$. By the same argument, if $|E_j| \geq 3$ then all five colours are present on V_j and so again there are four colours common to E_{j-1}, V_j, and V_{j-1}. This gives the same contradiction, showing that $|E_j| = 2$.

Since $f_{j+1} = 4$ and $f_{j+2} \geq 3$, there are at least $4+3-\chi \geq 2$ colours common to E_{j+1} and V_{j+2}. By Claim 3.5, $|E_{j+2}| = 1$ and these $7-\chi$ colours are present on E_{j+2} and V_{j+3}. Thus $f_{j+2} = 3$ and there are at least $7-\chi$ colours common to E_{j+2} and V_{j+2}. If $j+2$ is not special then there are at least $\chi-(6-7+\chi) = 1$ colours not present on either of E_{j+2} and V_{j+2}, contradicting (4). It follows that $j+2$ is special.

Since there are only two special indices and $n \geq 6$, it follows from what we have proved so far in Case 4 that if there is an $i \neq j$ such that $|E_i| \geq 2$ then $i = j+3$, $|E_{j+3}| = 2$, and $n = 6$. We may assume that this is the case, since otherwise the result holds by Case 3. Therefore $n = 6$ and $l_i = l'_i \geq 4$ for each i, since $f_i \geq 4$ for four values of i and the remaining two values are special. It follows that there is some colour c that is present on all of V_j, E_{j+2}, V_{j+3}, and $E_{j+5} = E_{j-1}$, and then D_c has a kernel, contrary to (3). This contradiction completes Case 4.

Case 5. $\chi \leq 3$.

In this case, we drop the hypothesis that $n \geq 6$ and assume only that $n \geq 3$. We shall not use any claims except for Claim 3.1. By Claim 3.1, $\chi = 3$. Also $f_i \leq 3$ for each i, and $f_i \leq 2$ if i is special. We first show that $|E_i| \leq 1$ for each i. Clearly $|E_i| \leq 2$, since otherwise $f_i \geq 4$ (since $|E_{i-1}| + |V_{i-1}| \geq 1$ by (4)). Suppose $|E_i| = 2$.

Suppose first that $V_{i-1} \neq \emptyset$. Then $f_i = 3$, $E_{i-1} = \emptyset$ (otherwise $f_i \geq 4$), i is not special (otherwise $l'_i = f_i + 1 = 4$), and $V_i = \emptyset$ (since otherwise v_i should have at least $f_i^+ = 4$ colours by hypothesis). Form H' from H by moving one vertex from E_i to V_i. This does not destroy any special indices, and so H' satisfies the hypotheses of the Theorem. It is not a counterexample, by the choice of H at the start of the proof. Hence we can colour H' from its lists, and the same colouring works for H.

This contradiction shows that $V_{i-1} = \emptyset$. It follows from (4) that every colour is present on E_{i-1} and so $E_{i-1} \neq \emptyset$. Since $f_i \leq 3$, it follows that $|E_{i-1}| = 1, f_i = 3$, and i is not special.

Suppose that $V_i \neq \emptyset$. Then $E_{i+1} = \emptyset$ since otherwise $f_{i+1} \geq 4$, and i and $i+1$ are not special. Form H' from H by moving one vertex from E_i to V_{i-1}. In H', $i+1$ is special, and the only index that can be special in H but not H' is $i-1$; thus, as before, H' satisfies the hypotheses of the Theorem and is not a counterexample to it. Hence we can colour H' from its lists, and the same colouring works for H.

This contradiction implies that $V_i = \emptyset$, which implies that $|E_{i+1}| = 1$ and $i+1$ is not special for exactly the same reason that $|E_{i-1}| = 1$ and i is not special. If $E_{i-2} = \emptyset$, then (4) implies that all three colours are present on v_{i-2}, which is adjacent to only v_{i-3} and the one vertex in E_{i-1}; thus by (2) we can colour all the vertices of H clockwise, starting with E_{i-1} and ending with v_{i-2}.

So we may assume that $E_{i-2} \neq \emptyset$. If $n = 4$ then the two special indices are $i-1$ and $i-2 = i+2$. Thus $|E_{i-2}| = 1$ and all sets V_j are empty. The five vertices of H are now easily coloured.

So we may assume that $n \geq 5$. Note that every colour is present on each of the four elements in $E_{i-1} \cup E_i \cup E_{i+1}$, and on v_{i+1} as well if it exists. Form H' from H by deleting the sets V_{i-1}, E_i, and V_i, and identifying E_{i+1} with E_{i-1}. This does not destroy any special indices, since neither i nor $i+1$ was special in H, and if $i-1$ was special in H (so that $|E_{i-2}| = 1$ and $V_{i-2} = \emptyset$) then it remains special in H'. Thus H' satisfies the hypotheses of the Theorem. Since H' is smaller than H, it is not a counterexample to the Theorem. Therefore H' can be coloured from its lists, and this colouring can be transferred to H by giving E_{i+1} the same colour as E_{i-1} and giving the elements of E_i the other two colours. This contradiction finally shows that $|E_i| \leq 1$ for each i.

By hypothesis, there are two values of i that are special. Suppose there is only one empty set Z_j. If $Z_j = E_i$ then $f_i = f_i^+ = f_{i+1} = 2$ but $f_{i+1}^+ = 3$ and $f_h = f_h^+ = 3$ for all other indices h; thus only i can be special, contrary to the hypothesis. If $Z_j = V_i$ then $f_{i+1} = 2$ but $f_h = 3$ for all other indices h; thus only $i+1$ can be special, contrary to the hypothesis. It follows that there are at least two empty sets Z_j. By (4) they are not consecutive. Suppose that the sets $Z_{j+1}, \ldots, Z_k$ are all nonempty but that $Z_j = Z_{k+1} = \emptyset$. Let the three colours be $1, 2, 3$. Every colour is present on every set $Z_{j+1}, \ldots, Z_k$ except that possibly only two colours are present on Z_{j+2}. (Every colour is present on Z_{j+1} by (4).) Suppose w.l.o.g. that 2 and 3 are present on Z_{j+2}. Colour Z_{j+i} with colour i (mod 3) for each i $(1 \leq i \leq k-j)$, colour Z_{k+2} with the same colour as Z_{k-1}, and then continue the colouring by giving each nonempty set Z_i a colour from its list that is different from any colours assigned to Z_{i-1} and Z_{i-2}. The only way in which this can fail to give a valid list colouring of H is if Z_{j-1} gets colour 1. In that case, we obtain a valid list colouring of H by increasing the colour of every set $Z_{j+1}, \ldots, Z_k$ by one. This contradiction completes the proof of Theorem 2. ■

4. ODD CASE

We now prove the corresponding more general result for odd n, which completes the proof of Theorem 1 (ii).

Theorem 3. *Suppose that n is odd $(n \geq 7)$ and that all the hypotheses of Theorem 2 are satisfied, except that nonexceptional colours are permitted to be* *present on every set E_i, but if any nonexceptional colour is present on every set E_i $(0 \leq i \leq n-1)$ then*

$$\sum_{i=0}^{n-1} l'_i \geq \frac{mn}{\lfloor \frac{1}{2} n \rfloor} \tag{5}$$

where $m = \sum_{i=0}^{n-1} |E_i|$. Then the vertices of H can be coloured from their lists.

Proof. Let H be a minimal counterexample as before, with two 1-special values of i and $n-2$ 0-special values. We may clearly suppose that some nonexceptional colour c is present on every set E_i, since otherwise the result follows immediately from Theorem 2.

Let us call a colour c *very exceptional* if it is present on every set E_i $(0 \leq i \leq n-1)$ and there exists a j such that $|E_{j-1}| = |E_j| = 1$ and c is present on V_j. Clearly every very exceptional colour is exceptional. By the previous paragraph, there is a colour c that is present on every set E_i and is not very exceptional; choose such a c and keep it fixed throughout the proof.

Claim 4.1. *There is no set V_j such that c is present on V_j.*

Proof. Suppose c is present on v_j for some j. If $|E_i| \geq 2$ for every i, then we form a pseudokernel K by choosing a vertex with c in its list from each of the sets $V_j, E_{j+2}, E_{j+4}, E_{j+6}, \ldots, E_{j-1}$. Suppose we give colour c to all vertices in K, delete these vertices and delete c from all remaining lists. Then the LHS of (5) goes down by n and so does the RHS, since m goes down by $\lfloor \frac{1}{2} n \rfloor$. All the other conditions are satisfied since K is a pseudokernel. Thus the remaining graph can be coloured from its lists, and hence so can H, a contradiction.

So we may suppose that $|E_i| = 1$ for at least one i. If we can manage to delete such an E_i, then there will no longer be a colour present on every set E_i and so we can forget about (5). Let I be the set of indices i such that $|E_i| = 1$ and c is present on V_{i+1}. If $I = \emptyset$ then we can proceed exactly as in the previous paragraph. So we may suppose $I \neq \emptyset$. Since c is not very exceptional, I does not contain two consecutive indices. Therefore there must exist an $i \in I$ such that none of $i-2$, $i-1$, and $i+1$ is in I (since if $i-2 \in I$ whenever $i \in I$ then clearly I would contain consecutive elements). Choose such an i and apply a modified form of Algorithm Ker to D_c with V_{i+1} as the starting set to construct a set K': follow the rules of Algorithm Ker, except that whenever we put into K' a vertex in a set E_j such that $j \notin I$, then the next element we choose must be in E_{j+2} and not in V_{j+1}. Since $i-2 \notin I$, the finishing set cannot be V_{i-1}, and so it must be E_{i-1} or E_i. If it is E_{i-1} then delete v_{i+1} from K' and replace it with an element of $E_{i+1} \cap D_c$; since neither $i-1$ nor $i+1$ is in I, this will create a pseudokernel K. For reasons of parity, K must contain a vertex in at least one set V_j, and by its construction it therefore contains the unique element of some set E_h such that $h \in I$. If the finishing set is E_i then K' is a pseudokernel; set $K := K'$ and note that K contains the unique element of E_h where $h = i \in I$. In either case, if we give colour c to all vertices of K, delete them, and delete c from all lists, then the resulting graph H^* will satisfy all the hypotheses of the Theorem, since now the set E_h is empty and so (5) is irrelevant, and all other hypotheses are satisfied since K is a pseudokernel. Thus H^* is a smaller counterexample to the Theorem than H, and this contradiction proves Claim 4.1. ■

Claim 4.2. *Not every set V_i is empty.*

Proof. Suppose $V_i = \emptyset$ for every i. Then we have only the vertices in the sets E_i to colour. In this case, we do not need the special values of i; (2) and (5) suffice. This stronger result follows by the argument but not quite from the result of Lemma 1.1 in [9]. Since $\sum_{i=1}^{n-1} f_i = 2m$, it follows from (5) that there exists an i for which $l'_i > f_i$. Give colour c to one vertex in each of sets $E_{i+1}, E_{i+3}, \ldots, E_{i-2}$, delete the coloured vertices, and delete colour c from all remaining lists. Then the LHS of (5) goes down by n and so does the RHS, since m goes down by $\lfloor \frac{1}{2}n \rfloor$. For each $j \neq i$, l'_j, and f_j decrease by 1, and so does l'_i, but f_i remains unchanged. But since $l'_i \geq f_i + 1$ to begin with, all the hypotheses of the Theorem remain true for this smaller graph, which is therefore a smaller counterexample than H. This contradiction proves Claim 4.2. ■

In completing the proof of Theorem 3, we shall often colour $\lfloor \frac{1}{2}n \rfloor$ vertices of H with colour c and some vertices with another colour c'. It is to be understood that we delete the coloured vertices from H, delete the colour c from all remaining lists, and delete c' from only the lists of those remaining vertices that are adjacent to a vertex that has been coloured with c', to form a smaller graph H^*. We shall say that a vertex in E_i *loses* or *gains in total* if $l'_i - f_i$ decreases or increases as a result of this process, and similarly for a vertex in V_i with $l_i - f_i^+$. Expressions such as "c-loses" will be used as a shorthand for "loses as a result of the colouring with c." If no vertex loses in total and *either* some set E_i becomes empty *or* (5) still holds, then H^* satisfies all the hypotheses of the Theorem, which gives rise to the required contradiction. To ensure that (5) holds, it suffices to ensure that, for some l, there are at most l sets E_i containing vertices that lose colour c' from their lists, and we assign colour c' to vertices in at least $\frac{1}{2}l$ of these sets; then the effect of operating with c' alone is that the LHS of (5) goes down by at most l and the RHS by at least $\frac{1}{2}ln/\lfloor \frac{1}{2}n \rfloor > l$.

By Claim 4.1, we may suppose that c is not present on any set V_j. By Claim 4.2, there is a colour c' that is present on some set V_j. Choose such a c' and keep it fixed for the remainder of the proof.

Claim 4.3. *If c' is present on V_j then it must be present on at least one of E_j and E_{j+1}.*

Proof. Suppose not. Then we give colour c' to v_j and colour c to one vertex in each of the sets $E_{j+2}, E_{j+4}, \ldots, E_{j-1}$. The only vertex that can c'-lose is v_{j-1}, which c-gains, and the only vertices that can c-lose are those in E_{j+1}, which

c'-gain. Therefore nothing loses in total, and we have a contradiction as explained in the previous paragraph. ■

Claim 4.4. *There is some i such that c' is not present on E_i.*

Proof. Suppose c' is present on every set E_i. Since c' is present on some set V_j by hypothesis, Claim 4.1 gives a contradiction if c' is not very exceptional. Therefore c' is very exceptional, which means that there exists a j such that $|E_{j-1}| = |E_j| = 1$ and c' is present on V_j. Let h be the first index in the sequence $j+2, j+4, \ldots$ such that $|E_h \cap (D_c \cup D_{c'})| = 1$; possibly $E_h = E_{j-1}$. Let K be a set comprising one vertex with c in its list taken from each of the sets E_{h+1}, $E_{h+3}, \ldots, E_j, E_{j+2}, \ldots, E_{h-2}$, where K is chosen so that $(E_i \backslash K) \cap D_{c'} \neq \emptyset$ for $i = j+2, j+4, \ldots, h-2$. Let $H^* := H - K$ with corresponding digraph D^*. Construct a set K' containing one vertex with c' in its list taken from each of the sets $V_j, E_{j+2} \backslash K, E_{j+4} \backslash K, \ldots, E_{h-2} \backslash K, E_h$, and then continuing the construction by applying Algorithm Ker to $D^*_{c'}$ (in an obvious terminology), starting with the existing vertex of K' in E_h and stopping when we reach E_{j-2} or V_{j-2} or E_{j-1}. Give colour c to the elements of K and c' to the elements of K'. Since c is not present on any set V_i, if a vertex in a set V_i does not c-gain then either $i = h$ or the *only* vertex of E_{i-1} was put into K; in either case, K' contains a vertex of $V_{i-1} \cup E_i$, and so v_i does not c'-lose. The only vertex in a set E_i that can c-lose is the unique vertex of $E_h \cap (D_c \cup D_{c'})$, which is deleted when we colour K'. The only vertex in a set E_i that can c'-lose is the unique vertex of E_j, which is deleted when we colour K. Therefore nothing loses in total, (5) is irrelevant since E_j has become empty, and we have the required contradiction. ■

From Claims 4.3 and 4.4, it follows that there exist indices h and j such that c' is present on all of

$$E_{h+1}, E_{h+2}, \ldots, E_j \tag{6}$$

but is not present on E_h or E_{j+1}; possibly $j+1 = h \pmod{n}$. Let $l := j - h$ if this is positive, otherwise $l := n + j - h$. We refer to (6) as an *E-segment of length l.*

Suppose first that there is an E-segment of even length l, labelled as in (6). There are two cases, depending on whether c' is or is not present on V_h. If c' is present on V_h, then give colour c' to one vertex in each of the sets V_h, E_{h+2}, $E_{h+4}, \ldots, E_j$ and colour c to one vertex in each of the sets $E_{j+2}, E_{j+4}, \ldots$, $E_{h-1}, E_{h+1}, \ldots, E_{j-1}$. The only vertices that can c'-lose are $v_{h-1}, v_{h+3}, v_{h+5}, \ldots$, v_{j-1}, all of which (if they exist) c-gain. The only vertices that can c-lose are those in E_{j+1}, which c'-gain. So nothing loses in total. Since the only sets E_i containing vertices that lose colour c' from their lists are the l sets in (6), and we have given colour c' to $\frac{1}{2}l$ of these vertices, it follows that (5) still holds, and we have the required contradiction.

So suppose that there is an E-segment of even length, labelled as in (6), and that c' is not present on V_h. Give colour c' to one vertex in each of the sets $E_{h+1}, E_{h+3}, \ldots, E_{j-1}$; note that this includes $\frac{1}{2}l$ vertices of the E-segment (6). Delete these vertices and delete colour c' from the lists of all remaining vertices in this E-segment to form a graph H^* with corresponding digraph D^*. We may suppose that some vertex c'-loses, since otherwise we have the required contradiction immediately, without using colour c at all. The only vertices that can c'-lose are $v_{h+2}, v_{h+4}, \ldots, v_j$; let v_i be the first of these that c'-loses. Note that v_i c'-loses because the *only* vertex of E_{i-1} has been given colour c' and deleted. So form a set K consisting of one vertex of D_c^* in each of the sets $E_i, E_{i+2}, \ldots,$ $E_j, E_{j+2}, \ldots, E_{h-1}$ and then continuing the construction by applying Algorithm Ker to D_c^*, starting with the existing vertex of K in E_{h-1} and stopping when we reach E_{i-3} or E_{i-2}. Then all the vertices that c'-lose also c-gain, and no vertex c-loses since the set E_{i-1} has already become empty. Thus no vertex loses in total, and we have the required contradiction.

So we may suppose that there are no E-segments of even length. For reasons of parity, there must be an E-segment of odd length l, labelled as in (6), such that c' is present on neither E_{j+1} nor E_{j+2}, and then by Claim 4.3, c' cannot be present on V_{j+1} either. So give colour c to one vertex in each of the sets $E_{j+2}, E_{j+4}, \ldots,$ $E_h, E_{h+2}, \ldots, E_{j-1}$ and colour c' to one vertex in each of the sets E_{h+1}, $E_{h+3}, \ldots, E_j$ (note that this includes more than $\frac{1}{2}l$ vertices of the E-segment (6)). The only vertices that can c'-lose are $v_h, v_{h+2}, \ldots, v_{j-1}$, all of which (if they exist) c-gain, and the only vertices that can c-lose are those in E_{j+1}, which c'-gain. So nothing loses in total, and this contradiction finally completes the proof of Theorem 3. ■

5. CONJECTURE 1 FOR $\tau(C) \geq \Delta + 2$

We conclude this work by proving that Conjecture 1 holds whenever $\tau(C) \geq \Delta + 2$. This follows from the following theorem.

Theorem 4. *If C is a multicircuit with n vertices, m edges, and maximum degree Δ, then* $\mathrm{ch}''(C) \leq \tau'(C)$*, where*

$$\tau'(C) := \max\left\{\Delta + 2, \left\lceil \frac{m}{\lfloor \frac{1}{2}n \rfloor} \right\rceil \right\}.$$

Proof. Let C have vertex-set $V(C) = \{v_0, \ldots, v_{n-1}\}$ and edge set $E(C) = E_0 \cup \cdots \cup E_{n-1}$, where E_i is now the set of all edges joining v_{i-1} and v_i. Suppose every element x of $V(C) \cup E(C)$ is given a list $L(x)$ of at least $\tau'(C)$ colours. By ([9], Theorem 1), the edges of C can be coloured from these lists. Let $c(e)$ be the colour assigned to edge e; let $C_i := \{c(e) : e \in E_i \cup E_{i+1}\}$, which is the set of colours used on edges incident with v_i; and let $L'(v_i) := L(v_i) \setminus C_i$.

Then $|L'(v_i)| \geq 2$ for each i. It is well known (and easy to prove, [4,8]) that the vertices v_i can be coloured from the lists $L'(v_i)$ unless n is odd and all the lists are identical and of cardinality 2, say $L'(v_i) = \{\alpha, \beta\}$ for each i. Suppose this is the case. Then C is Δ-regular, and $|C_i| = \Delta$ and $L(v_i) = C_i \cup \{\alpha, \beta\}$ for each i.

If α or β belongs to the list of some edge e, then we can recolour e with α or β and we will be able to extend the new colouring to the vertices of C. Thus for every edge $e \in E_i$, we may suppose that $\alpha, \beta \notin L(e)$, and so there are at least two colours in $L(e) \backslash L(v_{i-1})$. Choose such a colour, for each edge e, and call it $\gamma(e)$. If no edge of E_{i+1} is coloured with $\gamma(e)$, then we can recolour e with $\gamma(e)$ and we will be able to extend the new colouring to the vertices of C. Thus we may suppose that, for each i and each edge $e \in E_i$, there is an edge $g(e) \in E_{i+1}$ that has been coloured with $\gamma(e)$.

Now consider the edges of C as vertices of the line graph $L(C)$. Starting at an arbitrary vertex x, follow the path in which, whenever you are at a vertex y, you move to $g(y)$. Sooner or later you must arrive at a vertex you have been to before, thereby completing a directed circuit D. If D goes k times round $L(C)$, then $|E_i \cap V(D)| = k$ for every i. For every $e \in V(D)$, recolour e with $\gamma(e)$. Since $\gamma(e) \notin L(v_{i-1}) = C_{i-1} \cup \{\alpha, \beta\}$, for each $e \in E_i$, the new colouring is a proper vertex colouring of $L(C)$, giving a proper edge colouring of C. But now every vertex v_i of C has an incident edge e with a colour $\gamma(e) \notin L(v_i)$, and so we have at least three colours available with which to colour v_i. The vertices can now be coloured greedily, and this completes the proof. ■

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REFERENCES

[1] N. Alon, Restricted colorings of graphs, Surveys in Combinatorics, 1993, London Math Soc Lecture Note Series 187, K. Walker (Editor), Cambridge University Press, Cambridge, UK, 1993, pp. 1–33.

[2] N. Alon and M. Tarsi, Colorings and orientations of graphs, Combinatorica 12 (1992), 125–134.

[3] O. V. Borodin, A. V. Kostochka, and D. R. Woodall, List edge and list total colourings of multigraphs, J Combin Theory Ser B 71 (1997), 184–204.

[4] P. Erdős, A. L. Rubin, and H. Taylor, Choosability in graphs, Proc West Coast Conference on Combinatorics, Graph Theory and Computing, Arcata, 1979, Congr Numer 26 (1980), 125–157.

[5] F. Galvin, The list chromatic index of a bipartite multigraph, J Combin Theory Ser B 63 (1995), 153–158.

[6] A. V. Kostochka and D. R. Woodall, Choosability conjectures and multicircuits, Discrete Math 240 (2001), 123–143.

[7] A. V. Kostochka and D. R. Woodall, Total choosability of multicircuits II, J Graph Theory 40 (2002), 44–67.

[8] V. G. Vizing, Vertex colouring with given colours (in Russian), Metody Diskret Analiz 29 (1976), 3–10.

[9] D. R. Woodall, Edge-choosability of multicircuits, Discrete Math 202 (1999), 271–277.