Total Choosability of Multicircuits II

Alexandr V. Kostochka[1] and Douglas R. Woodall[2]*

[1]INSTITUTE OF MATHEMATICS
NOVOSIBIRSK, 630090, RUSSIA
E-mail: sasha@math.nsc.ru, AND
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ILLINOIS
URBANA, IL 61801, USA
E-mail: kostochk@math.uiuc.edu

[2]SCHOOL OF MATHEMATICAL SCIENCES
UNIVERSITY OF NOTTINGHAM
NOTTINGHAM, NG7 2RD, ENGLAND
E-mail: douglas.woodall@nottingham.ac.uk

Received June 5, 2000; revised October 29, 2001

DOI 10.1002/jgt.10030

Abstract: A multicircuit is a multigraph whose underlying simple graph is a circuit (a connected 2-regular graph). In this paper, the method of Alon and Tarsi is used to prove that all multicircuits of even order, and some regular and near-regular multicircuits of odd order have total choosability (i.e., list total chromatic number) equal to their ordinary total chromatic number. This completes the proof that every multicircuit has total choosability equal to its

Contract grant sponsor: Engineering and Physical Sciences Research Council; Contract grant number: GR/L54585 (to DRW for AVK); Contract grant sponsor: Russian Foundation for Fundamental Research; Contract grant number: 99-01-00581 (to AVK); Contract grant sponsor: UIUC Campus Research Grant (to AVK); Contract grant sponsor: INTAS; Contract grant number: 97-1001 (to DRW).

*Correspondence to: Douglas R. Woodall, School of Mathematical Sciences, University of Nottingham, Nottingham, NG7 2RD, England.
E-mail: douglas.woodall@nottingham.ac.uk

total chromatic number. In the process, the total chromatic numbers of all multicircuits are determined. J Graph Theory 40: 44–67, 2002

Keywords: *graph colouring; list-total-colouring conjecture; list chromatic number; list total chromatic number; total choosability; multicircuit*

1. INTRODUCTION

The main result proved in this paper, and the motivation for it, were discussed in [10]. This paper follows on from [10] and uses the same terminology. The specific conjecture stated in [9] and [10], which we can now state as a theorem, is the following.

Theorem 1. *If C is a multicircuit with n vertices, m edges and maximum degree Δ, then* $\mathrm{ch}''(C) = \chi''(C) = \tau(C)$, *where*

$$\tau(C) := \max\left\{\Delta + 1, \left\lceil \frac{m}{\lfloor \frac{1}{2} n \rfloor} \right\rceil, \left\lceil \frac{m+n}{\lfloor \frac{2}{3} n \rfloor} \right\rceil \right\}.$$

Here ch'' and χ'' denote the list and ordinary total chromatic numbers, respectively, and a *multicircuit* is a multigraph whose underlying simple graph is a circuit (a connected 2-regular graph). We call a multicircuit *semiregular* if it has at most one vertex with degree less than its maximum degree Δ; note that semiregular multicircuits of even order are regular. In [9] and [10], Theorem 1 was proved for all multicircuits with order $n \leq 5$, for all multicircuits that are not semiregular, and for all multicircuits C with maximum degree Δ such that $\tau(C) \geq \Delta + 2$; it thus remains to prove it for all semiregular multicircuits with order $n \geq 6$ such that $\tau(C) = \Delta + 1$. We do this mainly by using the method of Alon and Tarsi ([2], see also [1]). In fact, we shall include a proof for all multicircuits of even order, since it would not be any shorter to prove the result here just for semiregular multicircuits, and it seems interesting to have an alternative proof to that of [10] for the nonregular even cases.

It is clear in all cases that $\mathrm{ch}''(C) \geq \chi''(C) \geq \tau(C)$, and so it suffices to prove that $\mathrm{ch}''(C) \leq \tau(C)$. Let μ denote the minimum edge-multiplicity of a multicircuit C, that is, the smallest number of edges joining two consecutive vertices around C. It is easy to see that if C is a semiregular multicircuit of odd order n and minimum edge-multiplicity μ for which $\tau(C) = \Delta + 1$ then $n \geq 2\mu + 1$, since if $n = 2k + 1$ then

$$\Delta + 1 = \tau(C) \geq \frac{m}{\lfloor \frac{1}{2} n \rfloor} = \frac{(k+1)\mu + k(\Delta - \mu)}{k} = \Delta + \frac{\mu}{k},$$

and so $k \geq \mu$. Moreover, it is easy to check that $\tau(C) = \Delta + 2$ if $\Delta = 2$ and $n \not\equiv 0 \pmod 3$, or if C is semiregular with $(n, \Delta) = (7, 3)$, or if C is regular with

$(n, \Delta) = (7, 4)$ or $(10, 3)$. In view of these statements and the results of [9] and [10] mentioned earlier, it is not difficult to see that Theorem 1 follows from the following theorem, which is the main result of this paper.

Theorem 2. *Let C be a multicircuit with $n \geq 6$ vertices, maximum degree Δ and minimum edge-multiplicity μ. Suppose moreover that if $\Delta = 2$ then $n \equiv 0 \pmod 3$, and that if n is odd then C is semiregular and $n \geq 2\mu + 1$. Then $\mathrm{ch}''(C) \leq \Delta + 1$ except when C is semiregular and $(n, \Delta) = (7, 3)$, or C is regular and $(n, \Delta) = (7, 4)$ or $(10, 3)$.*

Theorem 2 is proved in Sections 3–8. The proof when $n = 7$ or 10 is rather tricky. The semiregular cases with $(n, \mu) = (7, 1)$, $(7, 2)$ or $(10, 1)$ are somewhat exceptional and are postponed until Section 8.

2. PRELIMINARIES

Before getting into anything more difficult, let us dispose of a simple but necessary lemma.

Lemma 1. *Let $L(n, r)$ (respectively $C(n, r)$) denote the number of ways of selecting r objects from n objects arranged in a line (respectively, arranged in a circle) in such a way that no two consecutive objects are selected. Let*

$$l(n) := \sum_{r \geq 0} (-1)^r L(n, r) \quad \text{and} \quad c(n) := \sum_{r \geq 0} (-1)^r C(n, r).$$

Then $l(n) = (2/\sqrt{3}) \sin((n + 2)\pi/3)$ and $c(n) = 2\cos(n\pi/3)$; that is,

if	n	$\equiv$	0	1	2	3	4	5	(mod 6)
then	$l(n)$	$=$	1	0	−1	−1	0	1	
and	$c(n)$	$=$	2	1	−1	−2	−1	1.	

Proof. It is an elementary exercise (see e.g., [4], p. 27) to prove that

$$L(n, r) = \binom{n - r + 1}{r} \quad \text{and}$$
$$C(n, r) = L(n - 1, r) + L(n - 3, r - 1) = \frac{n}{n - r}\binom{n - r}{r}.$$

In view of this, the result for $c(n)$ can be recognized as the so-called Hardy identity (see e.g., [3], p. 26–27, or (1.68) in [7]). The result for $l(n)$ is (1.75) in [7], and is also equivalent to the case $z = -1$ of Problem 1.35 in [12] and 1.42(g) in [11]. The two results can also be derived quite easily from each other, since it follows from the above equations that $c(n) = l(n - 1) - l(n - 3)$. ∎

We now summarize the method of Alon and Tarsi ([2], see also [1]), which was used also by Ellingham and Goddyn [5], Fleischner and Stiebitz [6] and Häggkvist and Janssen [8]. Let D_0 be an arbitrary orientation (the so-called *reference orientation*) of an undirected multigraph G. If D is any other orientation of G, let $a(D)$ be the number of edges that have opposite orientations in D and D_0, and define $\mathrm{sign}(D)$ to be 1 or -1 according as $a(D)$ is even or odd. We call D *positive* or *negative* if $\mathrm{sign}(D) = 1$ or $\mathrm{sign}(D) = -1$, respectively. Let $\mathcal{O}$ denote the set of all orientations of G in which every vertex v has the same outdegree $d^+(v)$ that it has in D_0 and let $\sigma(D_0) := \sum_{D\in\mathcal{O}} \mathrm{sign}(D)$. Suppose that every vertex v of G is given a list $L(v)$ of at least $d^+(v) + 1$ colors. The main result of Alon and Tarsi is that, if $\sigma(D_0) \neq 0$, then G can be colored from these lists.

Now let $C_1, \ldots, C_r$ be edge-disjoint complete subgraphs of G, and let $\mathcal{O}'$ denote the subset of all orientations in $\mathcal{O}$ that are acyclic on every clique C_i. A refinement of the basic method (see the discussion on page 13 of [1], or Proposition 2.3 of [8]) is to observe that $\sigma(D_0) = \sum_{D\in\mathcal{O}'} \mathrm{sign}(D)$; that is, in calculating $\sigma(D_0)$, we need only consider orientations that are acyclic on every C_i. The proof of these surprising facts, using the graph polynomial, is fascinating but it is explained in [1,2,8] and elsewhere and so we do not include it here.

To explain how we apply this method, we first need to introduce our terminology for multicircuits. Throughout the paper, C will be a multicircuit of order $n \geq 6$, with vertex-set $V(C) = \{v_0, \ldots, v_{n-1}\}$ and edge-sets $E_0, \ldots, E_{n-1}$, where E_i is the set of edges joining v_{i-1} and v_i in C. (Subscripts to v and E should be interpreted modulo n throughout.) Let $E_i = \{e_{i,1}, \ldots, e_{i,\mu_i}\}$, where $\mu_i = |E_i|$. Let G be obtained from the simple total graph $T(C)$ of C by doubling every edge that joins two vertices in the same set E_i; that is, replacing it by two parallel edges. So G has vertex-set $V(C) \cup \bigcup_{i=0}^{n-1} E_i$, with a single edge $v_i v_{i+1}$ for each i, a single edge xv_i whenever $x \in E_i \cup E_{i+1}$, a single edge xy whenever $x \in E_i$ and $y \in E_{i+1}$, and two parallel edges xy whenever $x, y \in E_i$ $(0 \leq i \leq n-1)$.

For convenience, we shall refer to the direction of increasing subscripts as *clockwise*. It might seem natural to take a clockwise orientation of G as our reference orientation, but it makes little difference in practice, and so for consistency with [10] we shall take an anticlockwise reference orientation. However, we need to describe the orientation more carefully than in [10], since reversing the directions of the edges in a directed 2-cycle will give a different orientation in the sense of Alon and Tarsi, even though the new orientation is isomorphic to the old one and is entirely equivalent to it for all purposes discussed in [10].

We can write G as the edge-disjoint union of graphs Γ and $C_0, \ldots, C_{n-1}$, where Γ is the subgraph of G (an n-cycle) induced by $V(C)$, and C_i is a clique with vertex-set $V(C_i) = E_i \cup \{v_i\} \cup E_{i+1}$ and order $\mu_i + \mu_{i+1} + 1$ $(0 \leq i \leq n-1)$. By a *proper* orientation of G, we shall mean an orientation that is acyclic on each clique C_i. Note that these cliques are not vertex-disjoint; an acyclic orientation of the edges of C_i corresponds to an ordering of its vertex-set $V(C_i)$, but the orderings of the vertex-sets of overlapping cliques in a proper orientation need not agree on their intersection. In fact, we shall soon impose conditions requiring the ordering of E_i induced by the acyclic orientation of C_i to be the opposite of the ordering induced by the acyclic orientation of C_{i-1}.

A proper orientation D of G can thus be specified by the orientation $D(\Gamma)$ that it induces on Γ, and the induced orderings $D(C_i)$ of the vertex-sets of the n cliques C_i. We shall write

$$D(C_i) = (z_i^{-\mu_i}, z_i^{-\mu_i+1}, \ldots, z_i^{-1}, z_i^0, z_i^1, \ldots, z_i^{\mu_{i+1}}),$$

and refer to the vertex z_i^j as being *in position* j in $D(C_i)$ $(-\mu_i \leq j \leq \mu_{i+1})$, where the edges of C_i are oriented in D from vertices in higher positions towards those in lower positions. In diagrams, we shall represent the orderings $D(C_i)$ vertically, with position 0 of each $D(C_i)$ being on the same horizontal line, with position 1 on the line above and position -1 on the line below.

A *good* orientation D of G is a proper orientation in which, for each i $(i = 0, \ldots, n-1)$,

$$\begin{aligned} d^+(v_i) &= \mu_i + 1 \quad \text{and} \\ d^+(x) &= \mu_{i-1} + \mu_i \quad \text{for each } x \in E_i, \end{aligned} \tag{1}$$

where d^+ denotes outdegree in D. Since the vertex in position j in $D(C_i)$ has outdegree $\mu_i + j$ in the induced acyclic orientation of C_i, we can interpret (1) as saying that, for each i,

$$\begin{aligned} &v_i \text{ is in position } 1, 0 \text{ or } -1 \text{ in } D(C_i) \text{ according as} \\ &v_i \text{ has outdegree } 0, 1 \text{ or } 2 \text{ in } D(\Gamma) \end{aligned} \tag{2}$$

and

$$\begin{aligned} &\text{if a vertex of } E_i \text{ is in position } j \text{ in } D(C_i) \\ &\text{then it is in position } -j \text{ in } D(C_{i-1}). \end{aligned} \tag{3}$$

For our reference orientation of G, we take the orientation D_0 in which Γ has the anticlockwise orientation (that is, each edge $v_{i-1}v_i$ is directed from v_i towards v_{i-1}), and

$$D_0(C_i) = (e_{i,\mu_i}, \ldots, e_{i,1}, v_i, e_{i+1,1}, \ldots, e_{i+1,\mu_{i+1}})$$

for each i; clearly this is a good orientation. Also, since $\mu_i + 1 \leq \mu_{i-1} + \mu_i \leq \Delta$ for each i, every vertex of G has outdegree at most Δ in D_0; and so if we can prove that $\sigma(D_0) \neq 0$ then it will follow by the method of Alon and Tarsi described above that $\mathrm{ch}''(C) \leq \Delta + 1$.

A *basic* orientation of G is a good orientation D such that, for each i, v_i is in position 0 in $D(C_i)$, that is, $z_i^0 = v_i$. So D_0 is a basic orientation. The *pattern* of a proper orientation D is obtained by replacing each element of E_i in $D(C_i)$ by E_i and each element of E_{i+1} by E_{i+1}, for each i; in other words, it specifies whether the vertex in each position in each $D(C_i)$ is v_i or belongs to E_i or to E_{i+1}, but in the last two cases it does not specify *which* vertex of E_i or E_{i+1} it is. A *good* pattern or *basic* pattern is the pattern of a good orientation or a basic orientation, respectively. A nonbasic good pattern uniquely determines the orientation $D(\Gamma)$ of Γ but a basic pattern does not, since by (2) it implies only that each vertex v_i has outdegree 1 in $D(\Gamma)$ and so Γ can have either the clockwise or the anti-clockwise orientation; a *full* pattern consists of a pattern together with a possible orientation for Γ.

Lemma 2. *All the good orientations corresponding to the same full pattern have the same sign.*

Proof. Let D and D' be good orientations corresponding to the same full pattern. Suppose that $x, y \in E_i$ and, in D, $z_{i-1}^{-j} = z_i^j = x$ and $z_{i-1}^{-k} = z_i^k = y$, where $j < k$. Suppose we interchange x and y so that $z_{i-1}^{-j} = z_i^j = y$ and $z_{i-1}^{-k} = z_i^k = x$. For each h $(j < h < k)$ this reverses the orientations of the two arcs xz_{i-1}^{-h} and $z_{i-1}^{-h}y$ in C_{i-1} and the two arcs yz_i^h and $z_i^h x$ in C_i, and it also reverses the orientations of the arc xy in C_{i-1} and the arc yx in C_i; it thus reverses an even number of arcs. Since D can be converted into D' by a finite sequence of interchanges of this type, it follows that D and D' have the same sign. ■

In view of Lemma 2, if P is a good full pattern then we can define $\operatorname{sign}(P)$ to be the sign of any good orientation with pattern P. Let $\rho(D_0) := \sum \operatorname{sign}(P)$, where the sum is over all distinct good full patterns P; note that these correspond precisely to orientations D in which every vertex v has the same outdegree as in D_0.

Lemma 3. $\sigma(D_0) = \left(\prod_{i=0}^{n-1} \mu_i!\right)\rho(D_0)$.

Proof. Let P be a good full pattern. The μ_i vertices of each set E_i can be assigned to the positions marked E_i in $D(C_i)$ in $\mu_i!$ ways, and by (3) this assignment uniquely determines the positions of these vertices in $D(C_{i-1})$ as well. Thus the number of good orientations corresponding to P is $\prod_{i=0}^{n-1} \mu_i!$, and they all have sign $\operatorname{sign}(P)$ by Lemma 2. The result follows. ■

In order to count the basic patterns, we need the concept, already introduced, of the minimum edge-multiplicity $\mu := \min\{\mu_i : 0 \le i \le n-1\}$. For even n we need also the analogous minimum even and odd edge-multiplicities

$$\mu_{\mathrm{e}} := \min\{\mu_i : i \text{ is even}\} \quad \text{and} \quad \mu_{\mathrm{o}} := \min\{\mu_i : i \text{ is odd}\}.$$

Lemma 4. *The number of basic patterns is* $\binom{\mu_{\mathrm{e}}+\mu_{\mathrm{o}}}{\mu_{\mathrm{o}}}$ *if n is even and 2^μ if n is odd.*

Proof. Suppose first that n is even. Let D be a basic orientation. Suppose that i is even, $1 \le j \le \mu_{i+1}$, and the vertex z_i^j in position j in $D(C_i)$ is an element of E_i

(not E_{i+1}, as it would be in D_0). Then $z_{i+1}^{-j} \notin E_{i+1}$ (since, by (3), this would require $z_{i+1}^{-j} = z_i^j$), and so $z_{i+1}^{-j} \in E_{i+2}$. Thus, by (3) again, $z_{i+1}^{-j} = z_{i+2}^j$, which means that z_{i+2}^j and $j \leq \mu_{i+3}$ (since otherwise the element z_{i+2}^j would not exist). Continuing in this way, we find that $z_h^j \in E_h$ and $j \leq \mu_{h+1}$ for every even h, so that $j \leq \mu_{\mathrm{o}}$.

In a similar way, if $1 \leq j \leq \mu_{i+1}$ and $z_i^j \in E_i$ for some odd i, then this holds for every odd i and so $j \leq \mu_{\mathrm{e}}$. What this shows is that in every basic pattern, the vertices z_i^j with i even and $\mu_{\mathrm{o}} < j \leq \mu_{i+1}$ are all in E_{i+1}, and so are the vertices z_i^j with i odd and $\mu_{\mathrm{e}} < j \leq \mu_{i+1}$. It also shows that, for every i, the values of $j > 0$ for which $z_i^j \in E_i$ are the same as the values of $j > 0$ for which $z_{i+2}^j \in E_{i+2}$ and the values of $j > 0$ for which $z_{i+2}^j \in E_{i+2}$. So the basic patterns correspond to the $\binom{\mu_{\mathrm{e}}+\mu_{\mathrm{o}}}{\mu_{\mathrm{o}}}$ partitions of the set $\{-\mu_{\mathrm{e}}, \ldots, -1, 1, \ldots, \mu_{\mathrm{o}}\}$ into two disjoint subsets A, B with $|A| = \mu_{\mathrm{e}}$ and $|B| = \mu_{\mathrm{o}}$, in the following way: if i is even, then $z_i^j \in E_i$ if $j \in A$ and $z_i^j \in E_{i+1}$ if $j \in B$, while if i is odd then $z_i^j \in E_i$ if $-j \in B$ and $z_i^j \in E_{i+1}$ if $-j \in A$. This proves the result for even n.

If n is odd, the argument is similar, except that there is no distinction between even and odd values of i, and μ_{e} and μ_{o} are both replaced by μ. The argument shows that, for every i, if $\mu < j \leq \mu_{i+1}$ then $z_i^j \in E_{i+1}$, and the values of $j > 0$ for which $z_i^j \in E_i$ are the same as the values of $j > 0$ for which $z_i^{-j} \in E_{i+1}$. So the basic patterns correspond to the 2^μ partitions of the set $\{1, 2, \ldots, \mu\}$ into two disjoint subsets A, B of arbitrary cardinality, in the following way: if $1 \leq j \leq \mu$, then $z_i^j \in E_i$ and $z_i^{-j} \in E_{i+1}$ if $j \in A$, while $z_i^j \in E_{i+1}$ and $z_i^{-j} \in E_i$ if $j \in B$. This completes the proof of Lemma 4. ■

In contrast with Lemma 2, the following lemma works only if n is even.

Lemma 5. *Suppose that n is even. Then all basic orientations have the same sign.*

Proof. Let D be a basic orientation. We shall show that D has sign 1, that is, it has the same sign as the reference orientation D_0.

In Γ, every vertex v_i has outdegree 1 in both D and D_0, by (2), and so an even number of edges of Γ (either 0 or n) change direction between D_0 and D.

Every vertex x of E_i is joined by edges to exactly two vertices of Γ, namely v_{i-1} and v_i. By (3), in each of D and D_0, one of these edges is oriented towards x and the other away from x. Thus an even number of edges between Γ and $V(G) - \Gamma$ change direction between D_0 and D.

For each two vertices $x, y \in E_i$, there are exactly two edges of G between them, one in C_{i-1} and one in C_i. By (3) again, in each of D and D_0, one of these edges is oriented from x to y and the other from y to x. Thus an even number of edges of this type change direction between D_0 and D.

All edges of G not so far considered join E_i and E_j for some i and j such that $|i - j| = 1$. These edges form a bipartite graph F with partite sets X, Y, where $X = \bigcup\{E_i : i \text{ is even}\}$ and $Y = \bigcup\{E_i : i \text{ is odd}\}$. By the previous two paragraphs, in each of D and D_0, each vertex $x \in E_i$ has one edge directed out of it towards Γ and $\mu_i - 1$ edges directed out of it towards other vertices in E_i, and so by (1) it must have μ_{i-1} edges directed out of it in F. Thus the sum of the outdegrees of all vertices of X in F is the same in D as it is in D_0. Since reversing the direction of one edge of F changes the parity of this sum, an even number of edges of F must change direction between D and D_0. Thus an even number of edges of G change direction between D and D_0, which means that D and D_0 have the same sign. ■

Finally, before applying these results, we introduce the concept of a 3-*switching*. If for some i, v_i is in position 1 in $D(C_i)$, v_{i-1} is in position -1 in $D(C_{i-1})$, some vertex $x \in E_i$ is in position 0 both in $D(C_i)$ and $D(C_{i-1})$, and the edge $v_{i-1}v_i$ is directed from v_{i-1} towards v_i in $D(\Gamma)$, then reversing the directions of the three arcs v_ix, xv_{i-1}, and $v_{i-1}v_i$ will move both v_i and v_{i-1} into position 0 in $D(C_i)$ and $D(C_{i-1})$, respectively. A similar remark applies if v_i is in position -1 in $D(C_i)$, v_{i-1} is in position 1 in $D(C_{i-1})$, and the edge $v_{i-1}v_i$ is directed from v_i towards v_{i-1} in $D(\Gamma)$; in this case we reverse the directions of the three arcs $v_{i-1}x$, xv_i, and v_iv_{i-1} to get v_i and v_{i-1} into position 0. We refer to either of these operations as a 3-*switching*. Note that a 3-switching changes the sign of the orientation. We call a good orientation *reduced* if it is not possible to carry out any 3-switchings, and the pattern of a reduced orientation is a *reduced* pattern.

Lemma 6. *A good orientation is reduced if and only if it is basic.*

Proof. It is clear that all basic orientations are reduced. Suppose that D is a nonbasic good orientation. Then some vertex v_i is not in position 0 in $D(C_i)$. By (2), either v_i is in position 1 in $D(C_i)$ and has outdegree 0 in $D(\Gamma)$, or else v_i is in position -1 in $D(C_i)$ and has outdegree 2 in $D(\Gamma)$. Suppose the former. (The other case is similar.) The vertex z_i^0 in position 0 in $D(C_i)$ belongs to either E_i or E_{i+1}; suppose it is an element x of E_i. (Again, the other case is similar.) Then $z_{i-1}^0 = z_i^0 = x$ by (3), and so v_{i-1} is not in position 0 in $D(C_{i-1})$. Since v_i has outdegree 0 in $D(\Gamma)$, the edge $v_{i-1}v_i$ of Γ is directed from v_{i-1} towards v_i, and so v_{i-1} has outdegree at least 1 in $D(\Gamma)$. It follows from (2) that v_{i-1} has outdegree 2 in $D(\Gamma)$ and is in position -1 in $D(C_{i-1})$. Then we can carry out a 3-switching of the first type described above. Thus a nonbasic orientation is not reduced. This completes the proof of the Lemma. ■

In some later sections, we shall modify G slightly so that Lemma 6 is no longer true. However, we will still be able to use the method of proof of this Lemma to show that, in a reduced orientation, all but a small number of vertices v_i are in position 0 in $D(C_i)$.

3. CASE $n \equiv 0 \pmod 6$

In this section, we shall use G and D_0 exactly as defined in the previous section. We wish to evaluate $\sigma(D_0)$, or more directly $\rho(D_0)$, by considering all the (patterns of) good orientations. We can do this by an appropriate consideration of just the basic orientations, since, by the argument of Lemma 6, every nonbasic good orientation can be reduced to a unique basic orientation. However, not every basic orientation corresponds to the same number of nonbasic orientations, and so we need to classify the basic orientations into types according to the patterns of the elements in positions 0, 1 and -1 of the orderings $D(C_i)$. There are four essentially different patterns, for each of which there are two different orientations of Γ, as shown in Fig. 1. (For comparison with later figures, we label the four patterns in Fig. 1, rather incongruously, as types 1, 2, 7, and 8.) Here each isolated black blob represents a vertex v_i in position 0 in $D(C_i)$. The vertex z_i^1 above it, in position 1 in $D(C_i)$, may be in E_i or in E_{i+1}. If it is in E_i then it equals z_{i-1}^{-1}, and a line sloping down to the left is shown connecting the equal elements z_i^1 and z_{i-1}^{-1}. If however $z_i^1 \in E_{i+1}$, then $z_i^1 = z_{i+1}^{-1}$ and a line sloping down to the right is shown connecting these equal elements. The reference orientation D_0 is of type 1A.

We must calculate how many good orientations we can obtain from basic orientations of each of these types by applying reverse 3-switchings. For basic orientations of types 1A and 2B, the answer is 1; no reverse 3-switchings are possible. For basic orientations of types 1B and 2A, we can apply reverse 3-switchings to any subset of the edges of Γ, provided that we do not attempt to reverse two consecutive edges (since once one edge has been reversed, as in Fig. 2, it is no longer possible to reverse the adjacent edges). Since each reverse 3-switching changes the sign of the orientation, the sum of the signs of all the orientations that we can obtain from a given basic orientation of either of these two types is $c(n)$, as in Lemma 1.

For basic orientations of types 7A, 7B, 8A, and 8B, we can apply reverse 3-switchings to any subset of the edges of Γ that correspond to the crosses in the diagram, as shown in Fig. 2. There are thus $2^{n/2}$ possible switchings, of which half have sign 1 and half have sign -1. Their net contribution to $\sigma(D_0)$ is thus 0.

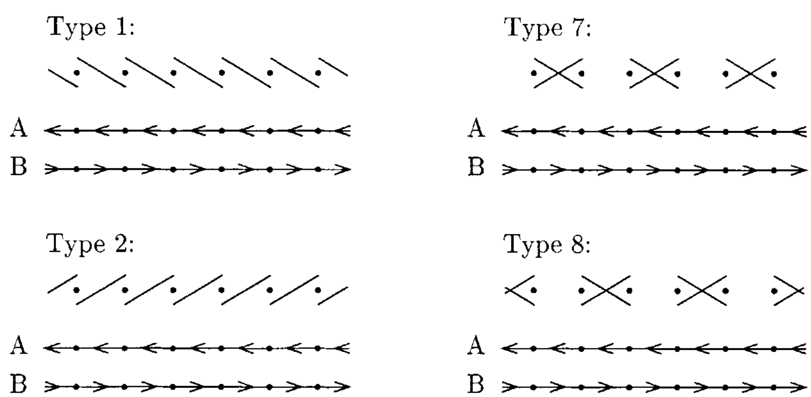

FIGURE 1. Basic orientations of G.

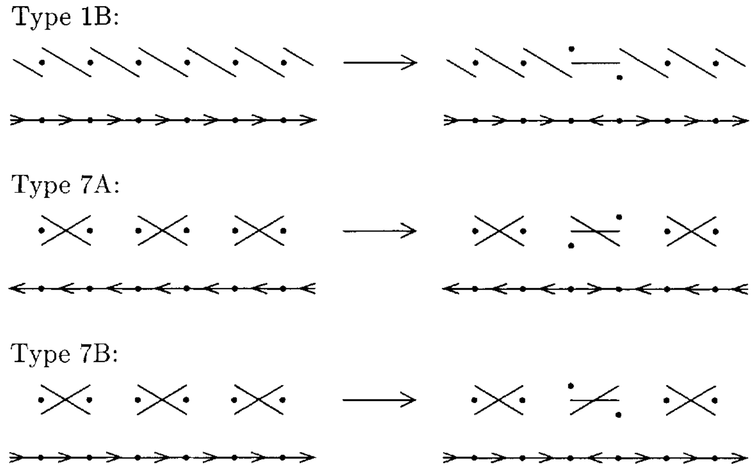

FIGURE 2. Sample reverse 3-switchings.

In fact, whenever the diagram representing a type of reduced orientation contains a cross $>\!\!<$, the contribution of that type to $\sigma(D_0)$ will be 0. For, it must be possible to make a reverse 3-switching of the corresponding edge of Γ, without interfering with anything else that we do, and the contributions of the orientations in which this switching is made will exactly cancel out the contributions of those in which it is not made.

In view of Lemma 3, we shall consider basic patterns rather than basic orientations. By the argument of Lemma 4, the basic patterns of type 1 correspond to partitions of the set $\{-\mu_e, \ldots, -1, 1, \ldots, \mu_o\}$ into two disjoint subsets A, B with $|A| = \mu_e$ and $|B| = \mu_o$ such that $-1 \in A$ and $1 \in B$. The basic patterns of type 2 are similar except that $1 \in A$ and $-1 \in B$. In each case the number of basic patterns is $\binom{\mu_e+\mu_o-2}{\mu_o-1}$. This is the number of full basic patterns of each of types 1A, 1B, 2A, and 2B, and so $\sigma(D_0)$ is a nonzero multiple of $1 + c(n)$; to be precise,

$$\sigma(D_0) = \left(\prod_{i=0}^{n-1} \mu_i!\right)\rho(D_0) \quad \text{where } \rho(D_0) = \binom{\mu_e + \mu_o - 2}{\mu_o - 1} 2(1 + c(n)).$$

For even values of n, we see from Lemma 1 that $1 + c(n) = 3$ if $n \equiv 0 \pmod 6$ and $1 + c(n) = 0$ otherwise. Thus we can deduce that $\sigma(D_0) \neq 0$, and Theorem 2 is proved, for all multicircuits of order $n \equiv 0 \pmod 6$.

4. CASE $n \equiv 2 \pmod 6$

It is not surprising that the argument of the previous section failed to prove the result for even $n \not\equiv 0 \pmod 6$, since we did not assume there that $\Delta > 2$, and if

$\Delta = 2$ then it is not true that $\mathrm{ch}''(C) \leq \Delta + 1$ when $n \not\equiv 0 \pmod 3$. To prove the result for $n \equiv 2 \pmod 6$, we shall therefore assume that $\Delta \geq 3$. Note that $\mu_o + \mu_e \leq \Delta$, since every vertex has degree at least $\mu_o + \mu_e$ in G. Assume without loss of generality that

$$\mu_1 = \mu_o \leq \mu_e, \tag{4}$$

so that $\mu_1 \leq \Delta - 2$; this follows because $\mu_1 \leq \Delta - \mu_e \leq \Delta - \mu_1 \leq \Delta - 2$ if $\mu_1 \geq 2$, and it is obvious if $\mu_1 = 1$ since $\Delta \geq 3$.

Form $G^{(1)}$ from G and $\Gamma^{(1)}$ from Γ by adding an extra edge between v_0 and v_1. Form the reference orientation $D_0^{(1)}$ from D_0 by orienting this extra edge from v_1 to v_0, in the same direction as the existing edge between v_0 and v_1. Note that v_1 has outdegree $\mu_1 + 2 \leq \Delta$ in $D_0^{(1)}$, and so if we can prove that $\sigma(D_0^{(1)}) \neq 0$ then it will follow as before that $\mathrm{ch}''(C) \leq \Delta + 1$.

Let a proper orientation D of $G^{(1)}$ be called *good* if every vertex v has the same outdegree in D as in $D_0^{(1)}$, so that D makes nonzero contribution to $\sigma(D_0^{(1)})$. As before, a good orientation D is *basic* if every vertex v_i is in position 0 in $D(C_i)$, and *reduced* if no 3-switching is possible to move two vertices from positions ± 1 to position 0. Lemmas 2–5 still hold, but Lemma 6 does not; that is, it is not now true that every reduced good orientation is basic. However, the argument of Lemma 6 shows that every good orientation can be reduced by 3-switchings to an orientation D such that, for each $i \in \{3, 4, \ldots, n-2\}$, v_i is in position 0 in $D(C_i)$. It is now not difficult to classify the reduced orientations into types according to the positions of v_{n-1}, v_0, v_1, and v_2 and the patterns of the elements in positions 0, 1, and -1 of the orderings $D(C_i)$. The different types are shown in Fig. 3. As before, the reference orientation is of type 1A. For each of types 1B, 2B, 7B, and 8B, only one of two examples is shown, the other being obtained by orienting the 2-cycle of $\Gamma^{(1)}$ in the opposite direction. We can ignore types 7–11, since (when $n \geq 6$) each of them contains the cross $\rtimes\!\ltimes$ above a single edge of Γ, and these types therefore make zero contribution to $\sigma(D_0^{(1)})$, as explained in the previous section. We must consider types 1–6.

Types 1A, 1B, 2A, and 2B are basic orientations, and by Lemma 5 they all have the same sign. For convenience, let e_i denote the sloping edge of E_i shown in the diagrams for types 1 and 2; so $e_i = z_{i-1}^{1} = z_i^{-1}$ in type 1, and $e_i = z_{i-1}^{-1} = z_i^{1}$ in type 2. Then we can convert an orientation of type 2A into one of type 3 by reversing a 7-cycle $v_2 v_1 v_0 v_{n-1} e_0 e_1 e_2$ and a 3-cycle $v_1 v_0 e_1$. This gives a bijection between orientations of types 2A and 3. By the argument in the previous section, there are thus exactly $\binom{\mu_e + \mu_o - 2}{\mu_o - 1}$ full patterns of each of types 1A, 1B, 2A, 2B, and 3, all with the same sign as $D_0^{(1)}$. (There are also the same number of each type obtained from 1B and 2B by reversing the 2-cycle of $\Gamma^{(1)}$.)

Now let us consider orientations of type 1A, 1B, 2A, and 2B in which $z_0^{-2} = z_1^{2} = x$, say, so that $z_1^{2} \in E_1$ (not E_2, as it would be in $D_0^{(1)}$); let us call these *flexible* orientations. Then $z_{i-1}^{-2} = z_i^{2}$ for every odd i, by the argument of Lemma 4. By the same argument, if $\mu_e = 1$ then there are no flexible orientations;

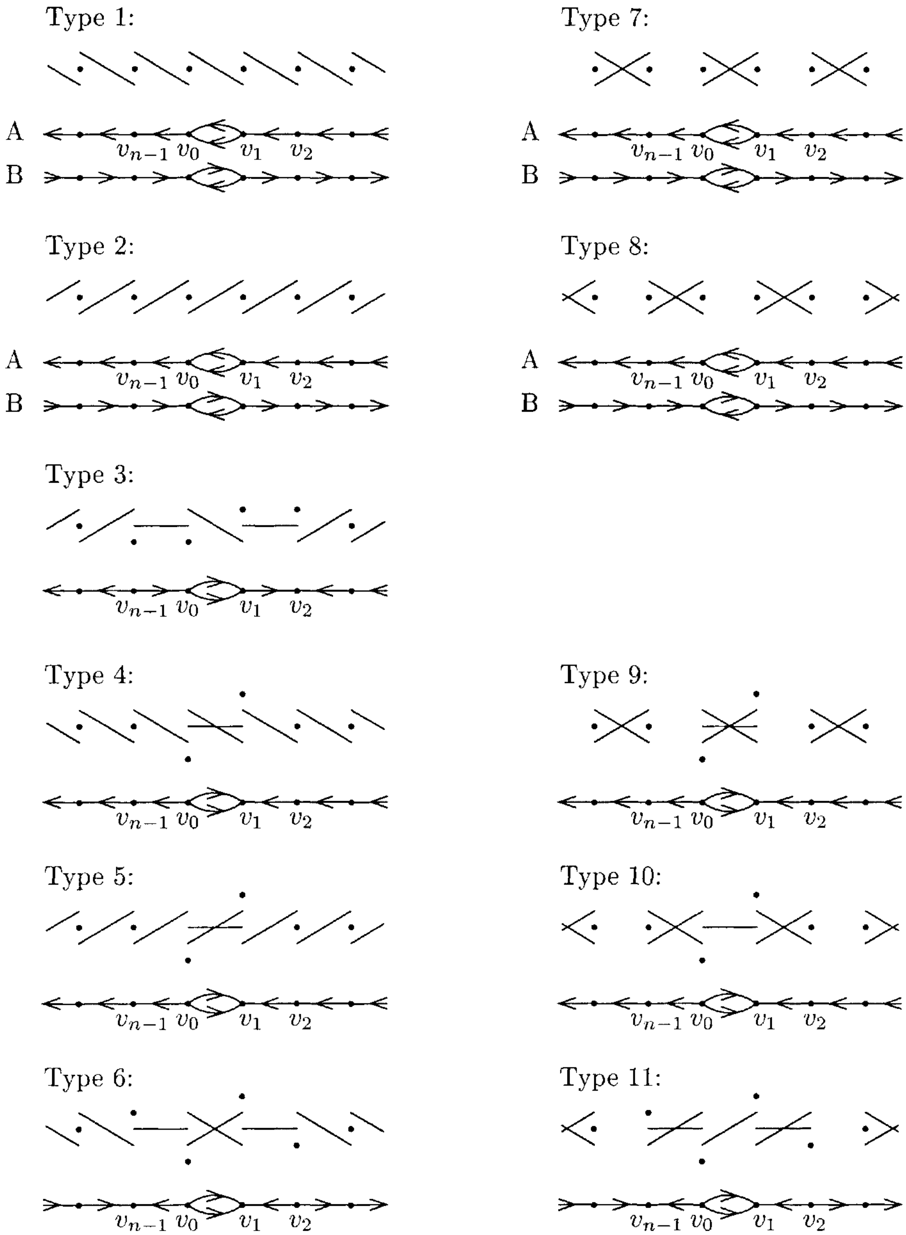

FIGURE 3. Reduced orientations of $G^{(1)}$.

this is obvious in any case since if $\mu_e = 1$ then (4) implies $\mu_1 = 1$, so that e_1 is the unique element of E_1. Otherwise, by the same argument as before, the flexible orientations of type 1A or B correspond to partitions of the set $\{-\mu_e, \ldots, -1, 1, \ldots, \mu_o\}$ into two disjoint subsets A, B with $|A| = \mu_e$ and $|B| = \mu_o$ such that $-1 \in A$, $1 \in B$, and $-2 \in B$. The flexible orientations of type 2A or B are similar but with $1 \in A$, $-1 \in B$, and $-2 \in B$. In each case, the number of such partitions is

$$\binom{\mu_e + \mu_o - 3}{\mu_o - 2} = \frac{\mu_o - 1}{\mu_e + \mu_o - 2} \binom{\mu_e + \mu_o - 2}{\mu_o - 1} \le \frac{1}{2} \binom{\mu_e + \mu_o - 2}{\mu_o - 1} \qquad (5)$$

by (4).

We can convert a flexible orientation of type 1A into an orientation of type 4 by reversing a 3-cycle $v_1 v_0 x$ and a 5-cycle $v_1 v_0 e_0 x e_2$. We can convert a flexible orientation of type 2A into an orientation of type 5 by reversing two 3-cycles $v_1 v_0 x$ and $v_1 v_0 e_1$ and the 2-cycle xe_1. And we can convert a flexible orientation of type 1B into one of type 6 by reversing the 3-cycles $v_{n-1} v_0 e_0$, $v_1 v_0 x$, and $v_1 v_2 e_2$. Moreover, these mappings define bijections, which show that the number of reduced patterns of each of types 4, 5 and 6 is $\binom{\mu_e + \mu_o - 3}{\mu_o - 2}$, those of types 4 and 5 having the same sign as $D_0^{(1)}$ and those of type 6 having the opposite sign.

It is relatively straightforward to calculate the sum of the signs of all the patterns that we can obtain from each reduced pattern by reverse 3-switchings, making due allowance for the fact that when the two edges of Γ between v_0 and v_1 have opposite orientations, it makes a difference which way round they are. We tabulate the results in Table I, using the function $l(n)$ from Lemma 1. For clarity, we write signs in tables as $+$ and $-$ rather than as 1 and -1; the signs given in Table I for odd n need careful interpretation, and are discussed in Section 6.

If we write $f_1(n)$ for the sum of the contributions of types 1A, 1B, 2A, 2B and 3, and $f_2(n)$ for the sum of the contributions of types 4, 5 and 6, we thus have the following values, obtained using Lemma 1.

$$\begin{array}{rcrrl} n & \equiv & 0 & 2 & 4 \pmod 6 \\ f_1(n) = 2 + 3l(n-1) - 3l(n-3) + l(n-5) & = & 8 & -2 & 0 \\ f_2(n) = 1 + l(n-3) - l(n-5) & = & 0 & 3 & 0 \end{array}$$

TABLE I. Contributions to $\rho(D_0^{(1)})$

Type	Sign (n even)	Sign (n odd)	Contribution
1A	$+$	$+$	1
1B	$+$	$-$	$2l(n-1) - l(n-3)$
2A	$+$	$-$	$l(n-1) - 2l(n-3)$
2B	$+$	$+$	$2 - 1 = 1$
3	$+$	$-$	$l(n-5)$
4	$+$	$-$	1
5	$+$	$+$	$l(n-3)$
6	$-$	$-$	$l(n-5)$

For $n \equiv 0 \pmod 6$, the result has already been proved in the previous section; and for $n \equiv 4 \pmod 6$, we can deduce nothing because $\sigma(D_0^{(1)})$ is clearly zero. But for $n \equiv 2 \pmod 6$, we can deduce that

$$\rho(D_0^{(1)}) = \left(-2\binom{\mu_{\mathrm{e}} + \mu_{\mathrm{o}} - 2}{\mu_{\mathrm{o}} - 1} + 3\binom{\mu_{\mathrm{e}} + \mu_{\mathrm{o}} - 3}{\mu_{\mathrm{o}} - 2}\right) < 0$$

by (5). Since $\sigma(D_0^{(1)}) = \left(\prod_{i=0}^{n-1} \mu_i!\right)\rho(D_0^{(1)}) \neq 0$, Theorem 2 is proved for all multicircuits of order $n \equiv 2 \pmod 6$.

5. CASE $n \equiv 4 \pmod 6$

It is again not surprising that the argument of the previous section failed to prove the result for $n \equiv 4 \pmod 6$, since we did not assume there that $n > 10$ or $\Delta > 3$, and if $(n, \Delta) = (10, 3)$ then it is not true that $\mathrm{ch}''(C) \leq \Delta + 1$. To prove the result for $n \equiv 4 \pmod 6$ will actually require three subcases.

The basic idea is to add two extra edges to G instead of one. Form $G^{(2)}$ from G and $\Gamma^{(2)}$ from Γ by adding two extra edges, one between v_0 and v_1 and the other between v_p and v_{p+1}, where p will be specified later, and let $q := n - p$. Form the reference orientation $D_0^{(2)}$ from D_0 by orienting these extra edges anticlockwise, in the same direction as the existing edges between the same pairs of vertices. Let a proper orientation D of $G^{(2)}$ be called *good* if every vertex v has the same outdegree in D as in $D_0^{(2)}$, so that D makes nonzero contribution to $\sigma(D_0^{(2)})$. *Basic* and *reduced* good orientations are defined as before and Lemmas 2–5 still hold.

Provided that p and q are not too small, many of the reduced good orientations can be categorized by a pair of the types given in Fig. 3 for $D_0^{(1)}$, e.g., type $\{1\mathrm{A}, 4\}$, the two types in the pair describing the structure of the orientation in the neighborhoods of E_1 and E_{p+1}. We refer to these as *standard* types. In addition to these there are also four types that cannot be so described, which we refer to as *exotic* types. Two of these are shown in Fig. 4 in case $p = 4$; note that they exist only if p is even, and make zero contribution to $\sigma(D_0^{(2)})$ if $p \geq 6$, because of the presence of the cross ⋊⋉ above a single edge of Γ. The other two exotic types, which we shall call E1′ and E2′, are obtained from E1 and E2 by interchanging the roles of E_{p+1} and E_1; they exist only if q is even, and make zero contribution to $\sigma(D_0^{(2)})$ if $q \geq 6$.

The sums of the signs of all the patterns that we can obtain from each standard type of reduced pattern by reverse 3-switchings are shown in Table II. Here we use the notation $t(r) := l(p - r)l(q - r)$. All the standard types listed in Table II, even types $\{3, 3\}$ and $\{6, 6\}$, exist if $p \geq 4$ and $q \geq 4$, and the contributions in Table II are then correct if we define $l(-1) := l(0) := 1$; this definition preserves the important property that $l(r)$ and $t(r)$ both depend only on the congruence class

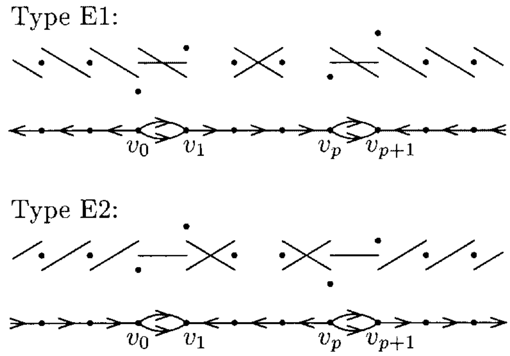

FIGURE 4. Exotic reduced orientations of $G^{(2)}$.

of r (mod 6). (However, if $p = 4$ or $q = 4$ then there are also contributions from exotic types, which are not listed in Table II.) We count a reduced good orientation as being of type $\{a, b\}$ if (i) it has the structure of type a at E_1 and the structure of type b at E_{p+1}, or (ii) it has the structure of type b at E_1 and the structure of type a at E_{p+1}; these two ordered subtypes, (i) and (ii), make equal contributions to $\sigma(D_0^{(2)})$, and so each standard type $\{a, b\}$ with $a \neq b$ is shown in Table II as making twice the contribution of an ordered subtype. As in Table I, the signs given in Table II for odd n need careful interpretation and are discussed in Section 6.

TABLE II. Standard Types of Contribution to $\rho(D_0^{(2)})$

	Sign		
Type	(n even)	(n odd)	Contribution
{1A, 1A}	+	+	1
{1B, 1B}	+	−	$4t(1) - 4t(2) + t(3)$
{2A, 2A}	+	−	$t(1) - 4t(2) + 4t(3)$
{2A, 3}	+	−	$2[t(3) - 2t(4)]$
{3, 3}	+	−	$t(5)$
{2B, 2B}	+	+	$4 - 4 + 1 = 1$
{1A, 4}	+	−	$2[1] = 2$
{2A, 5}	+	+	$2[t(2) - 2t(3)]$
{3, 5}	+	+	$2[t(4)]$
{1B, 6}	−	−	$2[2t(3) - t(4)]$
{4, 4}	+	−	1
{5, 5}	+	+	$t(3)$
{6, 6}	+	+	$t(5)$

Let us write g_1 for the sum of the contributions of the first six types in Table II, g_2 for the sum of the contributions of the next four types, and g_3 for the sum of the contributions of the last three types. Then (for even n)

$$g_1 = 2 + 5t(1) - 8t(2) + 7t(3) - 4t(4) + t(5),$$
$$g_2 = 2[1 + t(2) - 4t(3) + 2t(4)],$$
$$g_3 = 1 + t(3) + t(5).$$

For the congruence classes of p and q (mod 6) that we are particularly interested in, we have the following values:

(p,q)	$t(1)$	$t(2)$	$t(3)$	$t(4)$	$t(5)$	g_1	g_2	g_3
$(6,4)$	-1	0	0	-1	0	1	-2	1
$(5,5)$	0	1	1	0	1	2	-4	3

(6)

We now consider three subcases.

Subcase 5.1. $n \geq 16$. In this case we choose $p = 6$. To apply the above results, we need to find an index i such that $\mu_i < \Delta - 1$ and $\mu_{i+6} < \Delta - 1$. Call an index i *good* if $\mu_i < \Delta - 1$ and *bad* if $\mu_i = \Delta - 1$. If there is no index i as required, then $i + 6$ is bad whenever i is good; thus there are at least as many bad indices as good ones. But two consecutive indices cannot both be bad, since we are assuming that $\Delta \geq 3$ and so the indices must alternate between good and bad; but then we can find good indices i and $i + 6$. Thus there must exist an i as required; without loss of generality, $\mu_1 < \Delta - 1$ and $\mu_7 < \Delta - 1$. Then, in $D_0^{(2)}$, v_1 has outdegree $\mu_1 + 2 \leq \Delta$ and v_7 has outdegree $\mu_7 + 2 \leq \Delta$.

Let us say that a reduced orientation D of type $\{1A, 1A\}$, $\{1B, 1B\}$, $\{2A, 2A\}$, or $\{2B, 2B\}$ is *flexible at i* if $z_{i-1}^{-2} = z_i^2$, so that $z_i^2 \in E_i$ (not E_{i+1}, as it would be in $D_0^{(2)}$). By the argument of Lemma 4, if D is flexible at i and i is odd then $2 \leq \mu_e$ and D is flexible at i for *every* odd i. Thus D is flexible at 1 if and only if D is flexible at 7. In the previous section, we described a number of bijections, e.g., one between reduced orientations of $G^{(1)}$ of type 4 and flexible orientations of type 1A. There are analogous bijections between reduced orientations of $G^{(2)}$ of type $\{4, 4\}$ or of type $\{1A, 4\}$ and orientations of type $\{1A, 1A\}$ that are flexible (at 1, or at 7, it is the same). So for each of the first six standard types of reduced good pattern listed in Table II, there are $\binom{\mu_e+\mu_o-2}{\mu_o-1}$ patterns conforming to this type, and for each of the last seven types, there are $\binom{\mu_e+\mu_o-3}{\mu_o-2}$ patterns.

By (6), each non-flexible pattern makes a contribution of $g_1 = 1$ to $\rho(D_0^{(2)})$ and each flexible pattern makes a contribution of $g_1 + g_2 + g_3 = 0$. Since there exist non-flexible patterns (the pattern of the reference orientation $D_0^{(2)}$ itself, for example), it follows that the total contribution is positive and $\rho(D_0^{(2)}) \neq 0$.

This shows that the result of Theorem 2 holds for all multicircuits of order $n \equiv 4 \pmod 6$, $n \geq 16$. The argument fails if $n = 10$ because then

$q = n - p = 10 - 6 = 4$, and the exotic orientations of type E2′ make a nonzero contribution which we have ignored. Of course, the argument *must* fail if $n = 10$, since we have allowed $\Delta = 3$ and the result is false if $(n, \Delta) = (10, 3)$.

Subcase 5.2. $n = 10$ and C is regular with $\mu_o = 1$ and $\mu_e = \Delta - 1$. This is the case that requires $\Delta \geq 4$; it is postponed until Section 8.

Subcase 5.3. $n = 10$ and Subcase 5.2 does not apply. In this case we choose $p = 5$. To apply the above results, we need to find an index i such that $\mu_i < \Delta - 1$ and $\mu_{i+5} < \Delta - 1$. Suppose that this is not possible. By the argument in Subcase 5.1, we quickly deduce that $\mu_i = 1$ for all i of one parity, say for all odd i, and $\mu_i = \Delta - 1$ for all even i. Thus unless C is regular with $\mu_o = 1$ and $\mu_e = \Delta - 1$ (when Subcase 5.2 applies), we can find an i as required; without loss of generality, $\mu_1 < \Delta - 1$ and $\mu_6 < \Delta - 1$. Then in $D_0^{(2)}$, v_1 has outdegree $\mu_1 + 2 \leq \Delta$ and v_6 has outdegree $\mu_6 + 2 \leq \Delta$.

Since 5 is odd, there may now be reduced patterns that are flexible at 1, or at 6, but not at both. These contribute $g_1 + \frac{1}{2}g_2 = 0$ to $\rho(D_0^{(2)})$, by (6). Patterns that are flexible at 1 and at 6 contribute $g_1 + g_2 + g_3 = 1$, while those that are flexible at neither contribute $g_1 = 2$. The total contribution is nonzero and so $\rho(D_0^{(2)}) \neq 0$. This shows that the result of Theorem 2 holds for all multicircuits of even order except for those of order 10 that are regular with $\mu_o = 1$ and $\mu_e = \Delta - 1$, which are dealt with in Section 8.

6. ODD *n* AND $\mu \leq 2$

For odd values of n, we shall need a new argument (in Section 7) when $\mu \geq 3$, but for $\mu = 1$ or 2 we can adapt the arguments we have used for even n. We shall suppose throughout this section that C is a semiregular multicircuit of order $n \geq 7$ with edge-multiplicities μ and $\Delta - \mu$, labeled so that $\mu_1 = \mu_3 = \mu_4 = \mu_6 = \cdots = \mu_{n-1} = \mu$.

The signs given in the tables for odd n refer to orientations obtained from the reference orientation by reversing specified cycles. For example, the reference orientation $D_0^{(1)}$ of $G^{(1)}$ is of type 1A and is positive by definition (see Table I). By reversing a $2n$-cycle $v_0e_0v_{n-1}e_{n-1}\cdots v_1e_1$ and an n-cycle $e_0e_{n-1}\cdots e_1$, we obtain an orientation of type 2A, which is therefore negative. By reversing an n-cycle $v_0v_{n-1}\cdots v_1$ in either of these orientations we obtain orientations of types 1B and 2B, which are therefore negative and positive respectively. By reversing two odd cycles in the orientation of type 2A, as described in Section 4, we can obtain a negative orientation of type 3.

We call an orientation *flexible* if $z_{i-1}^{-2} = z_i^2$ for every i. None of the five orientations described so far is flexible: if $\mu = 1$ then for some values of i the element z_i^2 does not exist, while if $\mu \geq 2$ then $z_i^2 \in E_{i+1}$ and so equals z_{i+1}^{-2}, not z_{i-1}^{-2}. But if $\mu \geq 2$ and $z_i^2 = z_{i+1}^{-2} = f_i$, say, for each i, then by reversing the orientations of the $6n$ edges $z_i^2z_i^h$ and $z_i^hz_i^{-2}$ $(0 \leq i \leq n-1, -1 \leq h \leq 1)$ and the n edges of the

n-cycle $f_0 f_{n-1} \cdots f_1$, we obtain a flexible orientation of the same type and the opposite sign. Now we can obtain orientations of types 4, 5, and 6 by reversing cycles in the flexible orientations of types 1A, 2A, and 1B as described in Section 4.

Similar remarks apply to orientations of $G^{(2)}$ as summarized in Table II. Notice that the edge-reversal that converts type 1A into type 2A, say, in Table I is "global," and so converts type $\{1A, 1A\}$ into type $\{2A, 2A\}$ in Table II, whereas the edge-reversal that converts the flexible type 1B into type 6 in Table I is "local," and so must be carried out twice in order to convert the flexible type $\{1B, 1B\}$ (which is positive, since the inflexible type $\{1B, 1B\}$ is negative) into type $\{6, 6\}$ in Table II. This accounts for the signs in the tables.

Let us concentrate on Table II. Recall from Lemma 4 that there are 2^μ basic patterns when n is odd. If $\mu = 1$, then one of these (the pattern of the reference orientation) is of type 1 and the other is of type 2. There is thus one full basic pattern of each of the six types in the top six lines of Table II. Let the sum of the contributions to $\rho(D_0^{(2)})$ of these six patterns be h_1. Since none of these patterns is flexible, there are no patterns of any of the types in the bottom seven lines of the table.

If $\mu = 2$, then there are two full basic patterns of each of the first six types, one flexible and one inflexible with opposite sign and the contributions of all these patterns therefore cancel out. However, from the flexible patterns, we can derive one basic pattern of each of the last seven types. Let the sum of the contributions of these seven patterns be h_2.

If $\mu \geq 3$ then there are $2^{\mu-1}$ full basic patterns of each of the first six types and $2^{\mu-2}$ of each of the last seven types, and in each case half the patterns are positive and half are negative, and so everything cancels out. The present method therefore only works when $\mu \leq 2$.

From Table II we find that (for odd n)

$$h_1 = 2 - 5t(1) + 8t(2) - 7t(3) + 4t(4) - t(5),$$
$$h_2 = -3 + 2t(2) - 7t(3) + 4t(4) + t(5).$$

We shall take $p = 5$, $q = n - 5$, noting that provided $\Delta \geq 3$ then $\mu_1 = \mu_6 = \mu \leq \Delta - 2$, and so v_1 and v_6 both have outdegree equal to $\mu + 2 \leq \Delta$ in $D_0^{(2)}$. There is no contribution from exotic types provided that $n \geq 11$, so that $q \geq 6$. The values of h_1 and h_2 depend only on the congruence classes of p and q (mod 6), as shown for representative values as follows:

(p,q)	$t(1)$	$t(2)$	$t(3)$	$t(4)$	$t(5)$	h_1	h_2
$(5,6)$	0	0	1	0	0	-5	-10
$(5,8)$	0	-1	-1	0	-1	2	1
$(5,10)$	0	1	0	0	1	9	0

(7)

Since five of these six values of h_1 and h_2 are nonzero, it follows that $\mathrm{ch}''(C) \leq \Delta + 1$ whenever n is odd, $n \geq 11$, $\Delta \geq 3$, and $\mu \leq 2$; except when $\mu = 2$ and $n \equiv 3 \pmod 6$. There remain several special subcases to consider.

Subcase 6.1. $\mu = 2$ and $n \equiv 3 \pmod 6$. In this case, instead of taking $p = 5$ and $q = n - 5$ we take $p = n - 4$ and $q = 4$. The value of h_2 is unchanged by this and so equals 0 from the bottom line of (7). However, there is now a nonzero contribution from exotic type E2′. There is exactly one negative pattern of type E2′, which is obtained from the (positive) inflexible pattern of type $\{2\mathrm{B}, 2\mathrm{B}\}$, with $z^2_{n-1} = z^{-2}_0 = x$ and $z^2_{n-3} = z^{-2}_{n-2} = y$, say, by reversing a 7-cycle $v_{n-3}v_{n-2}v_{n-1}v_0xe_{n-1}y$ and two 3-cycles $v_1v_0e_1$ and $v_{n-3}v_{n-4}e_{n-3}$. Thus $\rho(D_0^{(2)}) = h_2 - 1 = -1$. The fact that this is nonzero proves the result in this case.

Subcase 6.2. $\mu = 1$ and $n \equiv 3 \pmod 6$. In this case, we have proved the result when $\Delta \geq 3$ and $n > 9$ but we have still to prove it when $\Delta = 2$ or $n = 9$. We do this from the original graph G as in Section 3. There is one positive full basic pattern of each of types 1A and 2B, and one negative full basic pattern of each of types 1B and 2A. The sum of their contributions to $\rho(D_0)$ is $2(1 - c(n))$. For odd values of n, we see from Lemma 1 that $1 - c(n) = 3$ if $n \equiv 3 \pmod 6$ and $1 - c(n) = 0$, otherwise. Thus the result holds when $n \equiv 3 \pmod 6$ and this argument works even when $\Delta = 2$ and/or $n = 9$.

Subcase 6.3. $n = 7$. This case is postponed to Section 8.

7. ODD n AND $\mu \geq 3$

In order to prove the result for semiregular multicircuits with odd values of n and $\mu \geq 3$, we shall look at different orientations of our original graph G from Sections 2 and 3. Let D be an arbitrary proper orientation of G satisfying (3). If $j > 0$ then, by definition, position j exists in $D(C_i)$ if and only if position $-j$ exists in $D(C_{i+1})$. Also, if v_i is not in position j in $D(C_i)$, then position j in $D(C_i)$ is occupied by an element $x \in E_i \cup E_{i+1}$ which is also in position $-j$ in C_{i-1} or C_{i+1}; and the same holds with j and $-j$ interchanged. Thus the number of vertices v_i that are in position j is the same as the number that are in position $-j$; call this number $p_j(D)$.

Let $\mathcal{D}_j$ be the set of orientations D such that $p_j(D) = 0$ $(1 \leq j \leq \mu)$. Since n is odd, for each $D \in \mathcal{D}_j$ either $z^j_i = z^{-j}_{i+1}$ for every i, or $z^j_i = z^{-j}_{i-1}$ for every i; call D *j-inflexible* in the first case and *j-flexible* in the second. Suppose that $D \in \mathcal{D}_j$ and we form $\bar{D}^j$ from D by interchanging the vertices z^j_i and z^{-j}_i in every $D(C_i)$. This has the effect of reversing the orientations of the $2n(2j-1)$ edges $z^j_i z^h_i$ and $z^h_i z^{-j}_i$ $(0 \leq i \leq n-1,\ -j < h < j)$ and the n edges of the n-cycle $z^j_0 z^j_{n-1} \cdots z^j_1$, an odd number in total. Moreover, $\bar{D}^j \in \mathcal{D}_j$ and every vertex has the same outdegree in $\bar{D}^j$ that it has in D. In fact, the mapping $D \mapsto \bar{D}^j$ pairs off all the elements of $\mathcal{D}_j$. So given any reference orientation D_0 of G satisfying (3), there are equally many positive and negative good orientations in $\mathcal{D}_j$, which therefore cancel out, so that the net contribution of all these orientations to $\sigma(D_0)$ is zero.

Now let us specify the outdegrees of two different classes of orientations of G, which for convenience we describe as *narrow* and *wide*. In each case, (1) holds except for the value of $d^+(v_i)$ for certain values of i. For convenience, let us write $\delta_i := d^+(v_i) - \mu_i - 1$, so that $\delta_i = 0$ if (1) holds. The narrow and wide orientations are defined so that, for $3 \le j \le \mu$, $\delta_{2j-1} = j - 1$ and $\delta_{2j} = -(j-1)$. The only other variation from (1) is that, in the class of narrow orientations, $\delta_2 = 1$ and $\delta_3 = -1$; whereas, in the class of wide orientations, $\delta_1 = 1$ and $\delta_2 = -1$. When working with the narrow class, we label C so that $\mu_0 = \mu_2 = \mu_5 = \mu_7 = \cdots = \mu_{n-2} = \mu$ (and either $\mu_3 = \mu$ or $\mu_4 = \mu$, it does not matter which), and when working with the wide class, we label C so that $\mu_1 = \mu_3 = \mu_5 = \cdots = \mu_{n-2} = \mu$ (and either $\mu_{n-1} = \mu$ or $\mu_0 = \mu$); then every vertex v_i has outdegree $d^+(v_i) \le d^+(v_{2\mu-1}) = 2\mu \le \Delta$, as required. These orientations exist provided that $n \ge 2\mu + 1$. Some types of reduced orientations of these two classes with $\mu = 3$ are illustrated in Fig. 5; in each case, we take as the reference orientation D_0 an arbitrary orientation of type 1. Note that Lemmas 2 and 3 still hold.

Let us consider which reduced good orientations in these two classes make a nonzero contribution to $\sigma(D_0)$. Reverse 3-switchings can move vertices from position 0 into position 1 and -1, but not into position j or $-j$ for any $j \ge 2$. It follows that if a reduced good orientation D is in $\mathcal{D}_j$ for some j ($2 \le j \le \mu$), then

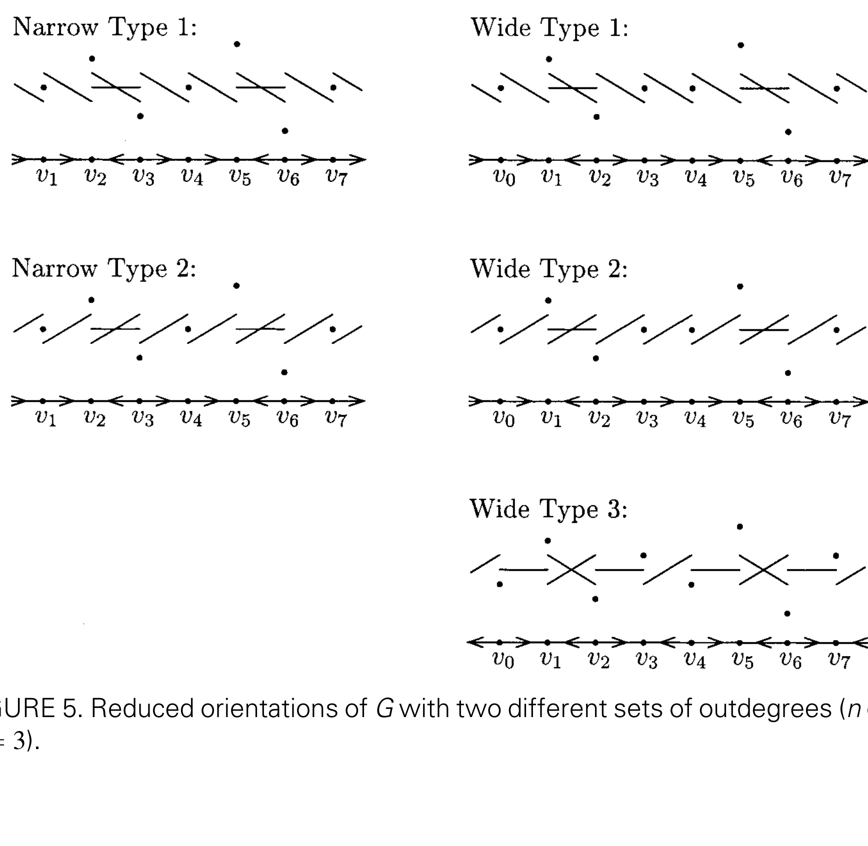

FIGURE 5. Reduced orientations of G with two different sets of outdegrees (n odd, $\mu = 3$).

the contributions to $\sigma(D_0)$ of all the good orientations that can be obtained from D by reverse 3-switchings exactly cancel out the contributions of all those that can be obtained similarly from $\bar{D}^j$. We may therefore confine our attention to reduced orientations D such that $p_j(D) > 0$ $(2 \leq j \leq \mu)$.

We can now describe a typical reduced good orientation of G in these two classes. The only vertices that can be in position μ or $-\mu$ are $v_{2\mu-1}$ and $v_{2\mu}$, and then only if $v_{2\mu-1}$ has outdegree 0 and $v_{2\mu}$ has outdegree 2 in $D(\Gamma)$; so we may assume that this is the case. (Note that $v_{2\mu-1}$ then has outdegree $\mu_{2\mu-1} + \mu = 2\mu$ in the induced acyclic orientation of $C_{2\mu-1}$.) Given that this is so, the only vertices that can be in position $\mu - 1$ or $-\mu + 1$ are $v_{2\mu-3}$ and $v_{2\mu-2}$, and then only if $v_{2\mu-3}$ has outdegree 0 and $v_{2\mu-2}$ has outdegree 2 in $D(\Gamma)$. Continuing in this way, we may suppose that v_{2j-1} has outdegree 0 and v_{2j} has outdegree 2 in $D(\Gamma)$ for each j $(3 \leq j \leq \mu)$, and that, in the narrow class, v_2 has outdegree 0 and v_3 has outdegree 2 in $D(\Gamma)$, while in the wide class v_1 has outdegree 0 and v_2 has outdegree 2.

In the narrow class, the only edges of Γ whose orientations are not now determined are those of the path $v_{2\mu+1}v_{2\mu+2}\cdots v_0v_1$, which must be oriented all clockwise or all anticlockwise, since all but the first and last vertices in this path must be in position 0, by the standard argument of Lemma 6 for reduced orientations. In fact, since v_4 is in position 0, v_1 must be in position 0 as well, since there is not room for two horizontal edges in Fig. 5 between v_0 and v_4. Since v_2 has outdegree 0, the path must be oriented clockwise. Thus the only two reduced orientations that contribute to $\sigma(D_0)$ are narrow types 1 and 2 shown (in case $\mu = 3$) in Fig. 5.

In the wide class, the edges of Γ whose orientations are not uniquely determined are those of the path $v_{2\mu+1}v_{2\mu+2}\cdots v_{n-1}v_0$ and the edge v_3v_4. There is no orientation in which the path is oriented anticlockwise and the edge clockwise, since if v_3 is in position 0 then v_0 must be in position 0 as well, by the same reasoning as before. There do exist orientations in which the path is oriented clockwise and the edge anticlockwise, but none of them are reduced. As in the narrow class, there are two types of reduced orientations, wide types 1 and 2, with all these edges oriented clockwise; but unlike in the narrow class, there is also a wide type 3 with all these edges oriented anticlockwise, although it exists only if $n \geq 2\mu + 3$.

We now calculate the contributions of these types of pattern to $\rho(D_0)$. Notice that the placing of the pairs of vertices in position j and $-j$ $(2 \leq j \leq \mu)$ forces $z_i^j = z_{i+1}^{-j}$ for all but one value of i, and $z_i^j = v_i$ and $z_{i+1}^{-j} = v_{i+1}$ for the exceptional value of i. Thus there is exactly one reduced pattern of each type shown in Fig. 5. The pattern of type 1 is positive in each case, by definition. The pattern of type 2 is therefore negative in each case, by the argument in the second paragraph of this section. If the sloping edge of each set E_i in wide type 2 in Fig. 5 is called e_i, and the horizontal edge $(i = 2, 6, 8, \ldots, 2\mu)$ is f_i, then wide type 3 is obtained from wide type 2 by reversing the $(n+2)$-cycle

$$v_{2\mu+1}v_{2\mu+2}\cdots v_{n-1}v_0e_1f_2e_3v_3v_4e_5f_6e_7\cdots e_{2\mu+1};$$

thus wide type 3 is positive.

The pattern of type 2 makes a contribution of -1 to $\rho(D_0)$ in each case, since no reverse 3-switchings are possible. Narrow type 1 makes a contribution of $l(n-2\mu)$, wide type 1 a contribution of $l(1)l(n-2\mu-1)=0$, and wide type 3 a contribution of $l(n-2\mu-3)$. Thus $\rho(D_0)=l(n-2\mu)-1$ in the narrow case and $\rho(D_0)=l(n-2\mu-3)-1$ in the wide case (unless $n=2\mu+1$, in which case wide type 3 does not exist and $\rho(D_0)=-1$). From Lemma 1, $l(n-2\mu)-1=-1$ if $n-2\mu\equiv 1 \pmod 6$ and $=-2$ if $n-2\mu\equiv 3 \pmod 6$, while $l(n-2\mu-3)-1=-1$ if $n-2\mu\equiv 1 \pmod 6$ and $=-2$ if $n-2\mu\equiv 5 \pmod 6$. Thus if $n\equiv 2\mu+1$ or $2\mu+3 \pmod 6$, we can get the result by looking at narrow orientations, and if $n\equiv 2\mu+1$ or $2\mu+5 \pmod 6$ we can get the result from wide orientations. In all cases we have the result, and this completes the proof of Theorem 2 for odd n and $\mu\geq 3$.

8. THREE EXCEPTIONAL CASES

It remains to prove Theorem 2 in the cases postponed from earlier, when C is regular or semiregular with $\Delta\geq\mu+3$ and $(n,\mu)=(7,1)$ or $(7,2)$ or $(10,1)$. In these cases, as in Section 7, we shall again look at different orientations of our original graph G. In each case, we label C so that $\mu_i=\mu$ if i is even and $\mu_i=\Delta-\mu\geq 3$ if i is odd ($0\leq i\leq n-1$; note the interchange of even and odd compared with Subcase 5.2). Let $\delta_i := d^+(v_i)-\mu_i-1$ as before, so that $\delta_i=0$ if (1) holds. We distinguish two subcases.

Subcase 8.1. $\mu=1$. In this case we consider the orientations that differ from (1) only in that $\delta_1=-2$, $\delta_2=-1$, $\delta_4=2$, and $\delta_6=1$. We shall show first that there is exactly one reduced pattern satisfying these conditions in which v_1 is in position -1 in $D(C_1)$, as shown in Fig. 6. We use the fact that v_2 cannot be in position -2 in $D(C_2)$, since $\mu_2=1$ and so there is no such position; and therefore, since neither v_1 nor v_2 is in position -2 or -3, we know that no vertex v_i is in position 2 or 3, since the number of vertices v_i in position j must equal the

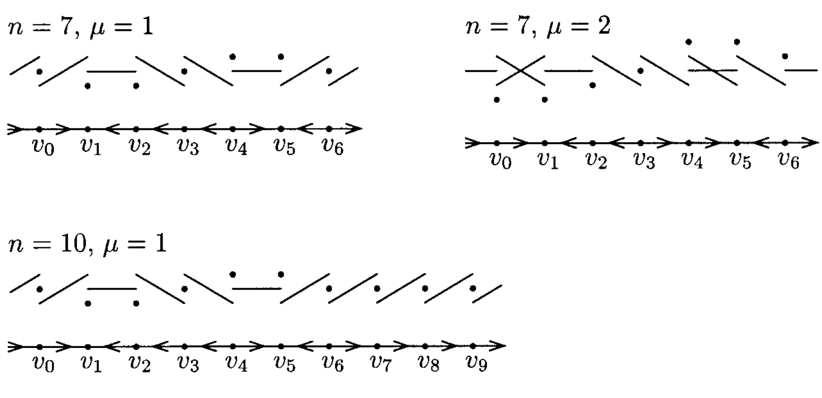

FIGURE 6. Reduced orientations of G in the three exceptional cases.

number in position $-j$ as explained at the start of the previous section; therefore v_4 is in position 1. We can set out the implications as follows.

$$
\begin{aligned}
v_1 = z_1^{-1} &\implies \overrightarrow{v_2v_1} \in D(\Gamma) \text{ and } z_2^1 \neq z_1^{-1} \implies v_2 \neq z_2^0\\
v_2 = z_2^{-1} &\implies \overrightarrow{v_3v_2} \in D(\Gamma) \text{ and } z_2^1 = z_3^{-1} \implies v_3 \neq z_3^1 \text{ or } z_3^{-1}\\
v_3 = z_3^0 &\implies \overrightarrow{v_4v_3} \in D(\Gamma) \text{ and } z_3^1 = z_4^{-1}\\
v_4 = z_4^1 &\implies \overrightarrow{v_4v_5} \in D(\Gamma) \text{ and } z_4^0 = z_5^0 \implies v_5 \neq z_5^{-1} \text{ or } z_5^0\\
v_5 = z_5^1 &\implies \overrightarrow{v_6v_5} \in D(\Gamma) \text{ and } z_5^{-1} = z_6^1 \implies v_6 \neq z_6^1\\
v_i = z_i^0 &\implies \overrightarrow{v_iv_{i+1}} \in D(\Gamma) \text{ and } z_i^{-1} = z_{i+1}^1\\
&\implies v_{i+1} \neq z_{i+1}^{-1} \text{ or } z_{i+1}^1 \ (6 \leq i \leq n-1).
\end{aligned}
$$

The rest of the diagram is now easy to fill in. Note that z_i^j is in E_{i+1} and so equals z_{i+1}^{-j} for every even i ($0 \leq i \leq n-1$) and every $j \geq 2$, since the unique element of E_i is already accounted for. Let D_0 be any orientation conforming to this pattern.

If $n = 7$, then there is no element z_6^{-2} or z_6^{-3}, and so necessarily $z_0^2 = z_1^{-2}$, $z_0^3 = z_1^{-3}$ and $v_1 = z_1^{-1}$. If $n = 10$, it is possible that $v_1 = z_1^{-j}$ where $j = 2$ or 3. However, this implies that $z_0^j = z_9^{-j}$ and $z_8^j = z_7^{-j}$. Since $\mu_0 = \mu_8 = 1$, these are the only elements of E_0 and E_8, respectively. But in a reduced good orientation, v_8 and v_9 are both in position 0 (by the argument of Lemma 6), and so $z_8^1 = z_9^{-1}$ and $z_8^{-1} = z_9^1$. Thus the diagram of such an orientation contains the cross $\times$, and so orientations of these types make zero contribution to $\sigma(D_0)$ and can be ignored.

Thus to determine $\sigma(D_0)$ it suffices, in each case, to consider the one reduced pattern shown in Fig. 6, in which no reverse 3-switchings are possible. It follows that $\rho(D_0) = 1$ and not 0, which completes the proof of Theorem 2 when $(n, \mu) = (7, 1)$ or $(10, 1)$.

Subcase 8.2. $\mu = 2$. In this case, we consider the orientations that differ from (1) only in that $(\delta_0, \delta_1, \ldots, \delta_6) = (-2, -3, -1, 1, 2, 1, 2)$. We shall show that there is exactly one reduced pattern satisfying these conditions, as shown in Fig. 6. Note that v_0 cannot be in position -3 in $D(C_0)$, since $\mu_0 = 2$ and so there is no such position; and v_1 cannot be in position -3 or -4 in $D(C_1)$, since there is no element z_6^{-3} or z_6^{-4} and so $z_0^3 = z_1^{-3}$ and (if $\mu_1 \geq 4$) $z_0^4 = z_1^{-4}$; thus v_1 is in position -2, which implies the orientations $\overrightarrow{v_0v_1}$ and $\overrightarrow{v_2v_1}$ in $D(\Gamma)$. Now v_0 is not in position -1 nor v_2 in position 0, and so v_0 is in position -2. Since $z_1^2 \neq z_0^{-2} = v_0$, therefore $z_1^2 = z_2^{-2}$, which means that v_2 must be in position -1. Of the four vertices v_3, v_4, v_5, v_6, therefore, one (either v_3 or v_5) is in position 0, one is in position 1, and two are in position 2. Whichever vertex is in position 0, we must have $z_6^0 = z_0^0$ and $z_1^0 = z_2^0$, and since $\mu_2 = 2$ there are no other elements of E_2 (other than z_2^0 and z_2^{-2}, that is), and so $z_1^1 = z_0^{-1}$, $z_1^{-1} = z_0^1$ and $z_2^1 = z_3^{-1}$. If v_5 is in position 0 then v_3 is in position 1 (it cannot be in position 2 because of the orientation $\overrightarrow{v_3v_2}$ implied by the position of v_2) and v_4 and v_6 are in position 2; thus z_6^1, which cannot equal z_0^{-1}, equals z_5^{-1}, and z_4^1 cannot equal anything. This contradiction shows that v_3 is in position 0 and $z_4^0 = z_5^0$. This implies the orientation $\overrightarrow{v_3v_4}$ so that v_4 cannot be in position 1 and must be in position 2, and also

$z_3^1 = z_4^{-1}$ and $z_4^1 = z_5^{-1}$. If v_5 is in position 1 then z_6^{-1} has nothing to pair with, and so v_5 is in position 2, v_6 in position 1, and $z_5^1 = z_6^{-1}$.

Thus there is exactly one reduced pattern of the type we are considering. Again, no reverse 3-switchings are possible. It follows that if D_0 is any orientation conforming to this pattern, then $\rho(D_0) = 1$ and not 0, and this proves Theorem 2 when $(n, \mu) = (7, 2)$.

This finally completes the proof of Theorem 2. ■

REFERENCES

[1] N. Alon, Restricted colorings of graphs, Surveys in Combinatorics, 1993, London Math Soc Lecture Note Series 187, K. Walker (Editor), Cambridge University Press, Cambridge, UK, 1993, pp. 1–33.

[2] N. Alon and M. Tarsi, Colorings and orientations of graphs, Combinatorica 12 (1992), 125–134.

[3] G. P. Egorychev, Integral representation and counting combinatorial sums (Nauka, Novosibirsk, 1977) (in Russian).

[4] M. Eisen, Elementary Combinatorial Analysis, Gordon and Breach, New York, 1969.

[5] M. N. Ellingham and L. Goddyn, List edge colourings of some 1-factorable multigraphs, Combinatorica 16 (1996), 343–352.

[6] H. Fleischner and M. Stiebitz, A solution to a colouring problem of P. Erdős, Discrete Math 101 (1992), 39–48.

[7] H. W. Gould, Combinatorial identities, a standardized set of tables listing 500 binomial coefficient summations, West Virginia University, Morgantown, 1972.

[8] R. Häggkvist and J. Janssen, New bounds on the list-chromatic index of the complete graph and other simple graphs, Combin Probab Comput 6 (1997), 295–313.

[9] A. V. Kostochka and D. R. Woodall, Choosability conjectures and multicircuits, Discrete Math 240 (2001), 123–143.

[10] A. V. Kostochka and D. R. Woodall, Total choosability of multicircuits I, J Graph Theory 40 (2002), 26–43.

[11] L. Lovász, Combinatorial Problems and Exercises, North-Holland, 1993.

[12] I. Tomescu, Problems in Combinatorics and Graph Theory, Wiley, 1985.