A List Analogue of Equitable Coloring

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Received December 10, 2001; Revised May 23, 2003

DOI 10.1002/jgt.10137

Abstract: Given lists of available colors assigned to the vertices of a graph G, a *list coloring* is a proper coloring of G such that the color on each vertex is chosen from its list. If the lists all have size k, then a list coloring is *equitable* if each color appears on at most $\lceil n(G)/k \rceil$ vertices. A graph is *equitably k-choosable* if such a coloring exists whenever the lists all have size k. We prove that G is equitably k-choosable when $k \geq \max\{\Delta(G), n(G)/$

Kostochka and West Contract grant sponsor: NSF; Contract grant number: DMS-0099608; Kostochka Contract grant sponsor: Russian Foundation for Basic Research; Contract grant numbers: 99-01-00581, 00-01-00916. West Contract grant sponsor: NSA; Contract grant number: MDA904-03-1-0037, which requires the disclaimer that any opinions, findings, and conclusions or recommendations expressed in this publication are those of the author(s) and do not necessarily reflect the views of the NSA.

2} unless G contains K_{k+1} or k is odd and $G = K_{k,k}$. For forests, the threshold improves to $k \geq 1 + \Delta(G)/2$. If G is a 2-degenerate graph (given $k \geq 5$) or a connected interval graph (other than K_{k+1}), then G is equitably k-choosable when $k \geq \Delta(G)$.

Keywords: *graph coloring; equitable coloring; outerplanar graph; tree; list coloring*

1. INTRODUCTION

In many applications of graph coloring, it is desirable that the color classes not be very large. When scheduling jobs, for example, the number of jobs that can be run at the same time may be limited by the number of processors (see [12] for such an application).

This restriction has a simple formalization. A proper vertex coloring of a graph is *m-bounded* if each color appears on at most m vertices, as discussed in [1,6]. There is also a more restrictive (and more common) notion: a proper vertex coloring of a graph is an *equitable* coloring if the sizes of the color classes differ by at most 1. Every equitable k-coloring of a graph G is $\lceil n(G)/k \rceil$-bounded, where $n(G)$ denotes the number of vertices of G and a k-coloring is a coloring with k color classes.

A graph may have an equitable k-coloring but not have an equitable $(k+1)$-coloring. For example, the complete bipartite graph $K_{7,7}$ has equitable k-colorings when $k \in \{2, 4, 6, 8\}$, but it has no equitable k-coloring when $k \in \{3, 5, 7\}$. In the positive direction, Meyer [8] proved that every tree T has an equitable k-coloring whenever $k = 1 + \lceil \Delta(T)/2 \rceil$, where $\Delta(G)$ denotes the maximum vertex degree for a graph G. Later, Guy [4] reported that Eggleton (unpublished) observed that the conclusion holds whenever $k \geq 1 + \lceil \Delta(T)/2 \rceil$.

Hajnál and Szemerédi [5] proved that a graph G has an equitable k-coloring whenever $k > \Delta(G)$; this answered a question of Paul Erdős. Chen, Lih, and Wu [2] conjectured that for $D \geq 3$, every connected graph with maximum degree D other than K_D and $K_{D,D}$ also has an equitable D-coloring. This has been proved for interval graphs, trees, and several other classes (see [7] for a survey). Yap and Zhang [14] proved it for outerplanar graphs.

Here we introduce a list analogue of equitable coloring. A *list assignment* L for a graph G assigns to each vertex $v \in V(G)$ a set $L(v)$ of allowable colors. An L-coloring of G is a proper vertex coloring such that for every $v \in V(G)$ the color on v belongs to $L(v)$. For example, when colors represent time periods and vertices are jobs, the list model incorporates restrictions that not all time periods are suitable for all jobs. A list assignment L for G is *k-uniform* if $|L(v)| = k$ for all $v \in V(G)$.

Given a k-uniform list assignment L for an n-vertex graph G, we say that G is *equitably L-colorable* if G has an $\lceil n/k \rceil$-bounded L-coloring of G. A graph G is *equitably list k-colorable* or *equitably k-choosable* if G is equitably L-colorable whenever L is a k-uniform list assignment for G.

There are several reasons for using the term ‘‘equitable’’ for something that initially sounds like a list analogue of bounded coloring. First, ordinary equitable coloring can be expressed in terms of bounded coloring. Let $m = \lceil n(G)/k \rceil$, so $n(G) = mk - r$ with $r < k$. Now G has an equitable k-coloring if and only if the graph that is the disjoint union of G and K_r has an m-bounded k-coloring.

Second, the word ‘‘bounded’’ can mean many things, while ‘‘equitable’’ captures the notion that no color is used excessively often. Third, the concept of ordinary equitable coloring does not generalize to the list context. Consider a list assignment in which every vertex except one has the list $\{1, \ldots, k\}$ and the remaining vertex v has list $\{k+1, \ldots, 2k\}$. In every proper coloring, some colors are omitted, the color on v appears once, and some other color is used at least $\lceil (n(G) - 1)/k \rceil$ times. This proper coloring cannot be equitable in the classical sense unless $n(G) \le k + 1$.

Because we cannot ensure using each color, the techniques previously used for ordinary equitable colorings don't work well for equitable list colorings. Nevertheless, we propose the following analogues of the Hajnál–Szemerédi Theorem and the Chen–Lih–Wu Conjecture.

Conjecture 1.1. *Every graph G is equitably k-choosable whenever $k > \Delta(G)$.*

Conjecture 1.2. *If G is a connected graph with maximum degree at least 3, then G is equitably $\Delta(G)$-choosable, unless G is a complete graph or is $K_{D,D}$ for some odd D.*

The conclusion of Conjecture 1.2 holds easily when $\Delta(G) = 2$. Conjecture 1.1 has been proved for $\Delta(G) \le 3$ independently in [11] and [13]. In this paper, we provide the following partial results toward these conjectures. (Always k denotes an integer.)

Theorem 1.1. *If G is a graph and $k \ge \max\{\Delta(G), n(G)/2\}$, then G is equitably k-choosable unless G contains K_{k+1} or is $K_{k,k}$ (with k odd in the latter case).*

Theorem 1.2. *If G is a forest and $k \ge 1 + \Delta(G)/2$, then G is equitably k-choosable. Also, for all D there is a tree with maximum degree at most D that is not equitably $\lceil D/2 \rceil$-choosable.*

Theorem 1.3. *If G is a connected interval graph and $k \ge \Delta(G)$, then G is equitably k-choosable unless $G = K_{k+1}$.*

Theorem 1.4. *If G is a 2-degenerate graph and $k \ge \max\{\Delta(G), 5\}$, then G is equitably k-choosable.*

A graph G is *d-degenerate* if every subgraph has a vertex of degree at most d. The 1-degenerate graphs are just the forests, and every outerplanar graph is 2-degenerate. Thus, Theorem 1.4 implies that for $k \ge 5$, every outerplanar graph G is equitably k-choosable if $k \ge \Delta(G)$. Pelsmajer [10] obtained this conclusion for every $k \ge 3$.

Theorem 1.1 is a counterpart of a similar result in [2] for ordinary equitable coloring. Theorem 1.2 generalizes the result of Meyer [8] mentioned earlier. The counterpart of Theorem 1.3 for ordinary coloring was proved by Chen, Lih, and Yan [3]. The counterpart of Theorem 1.4 for ordinary coloring (for $k \geq 8$) was proved by Nakprasit [9].

In the next section we prove Theorem 1.1. In Section 3, we describe an inductive approach (introduced in Pelsmajer [10]) to proving equitable k-choosability. The idea is to select k vertices such that distinct colors can always be chosen from their lists to properly extend an equitable list coloring of the remaining graph. Since the colors chosen for the new vertices are distinct, the full coloring is equitable. In Sections 4 and 5, we use this approach to prove the other results.

2. SMALL GRAPHS

In this section we prove Theorem 1.1, showing that a graph with order at most $2k$ and maximum degree at most k is equitably k-choosable unless it contains K_{k+1} or is $K_{k,k}$ (with k odd).

Every graph with at most k vertices is equitably k-choosable, since we can color the vertices in an arbitrary order and always choose a previously unused color from each list. Hence we may assume that $k+1 \leq n(G) \leq 2k$, and thus $\lceil n(G)/k \rceil = 2$. Therefore, we want to prove that G has a 2-bounded L-coloring whenever L is a k-uniform list assignment, unless G is one of the exceptional graphs.

Suppose that Theorem 1.1 does not hold, and consider a counterexample G with fewest vertices. Let L be a k-uniform list assignment such that every L-coloring of G uses some color at least three times. By the minimality of G, for every $y \in V(G)$ there is a 2-bounded L-coloring of $G - y$. Among all vertices in G and all L-colorings of the resulting subgraphs, choose a vertex y and a 2-bounded coloring f of $G - y$ to maximize the number of color classes of size two in f.

Let $U = \{f(v)\colon v \in V(G) - \{y\}\}$; these are the colors used by f. For $i \in \{1, 2\}$, let U_i be the set of colors used exactly i times by f. Since $n \leq 2k$,

$$2|U_2| + |U_1| \leq 2k - 1. \tag{1}$$

We use $N(v)$ for $\{u\colon uv \in E(G)\}$ and $N[v]$ for $N(v) \cup \{v\}$. For a vertex x other than y, *switching on* x means giving color $f(x)$ to y and making x uncolored. A vertex x is *switchable* if switching on x yields a 2-bounded L-coloring of $G - x$.

Note that x is switchable if and only if $f(x) \in L(y)$ and y has no neighbor other than x with color $f(x)$. Let $Z = \{x \in V(G)\colon f(x) \in U_1 \cap L(y)\}$; every vertex of Z is switchable.

Claim 2.1. *If a vertex x is switchable, then $L(x) \subseteq U$. Also $Z \subseteq N(y)$.*

Proof. If $L(x) \nsubseteq U$, then we switch on x and use any color in $L(x) - U$ on x to obtain a 2-bounded L-coloring of G. If some vertex of Z is not adjacent to y, then we extend f by using the color of that vertex on y. ■

Claim 2.2. *If u and v are vertices such that u is switchable and $v \in Z$, then $uv \in E(G)$. In particular, $Z \subseteq N[u]$.*

Proof. Suppose that $uv \notin E(G)$. Consider first the case where $L(u)$ and $L(v)$ have a common color c that appears in U_1, and let z be the vertex such that $f(z) = c$. When $z = v$, we switch on u and then give color c to u to obtain a 2-bounded L-coloring of G. The subcase $z = u$ is similar, since every vertex of Z is switchable; just switch on v and then give color c to v.

We may therefore assume that $z \notin \{u, v\}$. In this subcase, we uncolor z and switch on u, and then we recolor u and v with c. The one uncolored vertex is now z, the number of vertices with color $f(u)$ is unchanged, and two colors in U_1 (on z and v) have been replaced by a single color on u and v. We have constructed a 2-bounded L-coloring of $G - z$ having more color classes of size 2 than f does, which contradicts the choice of f and y.

It remains only to consider the case where $L(u) \cap L(v)$ is disjoint from U_1. Since u and v are switchable, $L(u) \cup L(v) \subseteq U$, by Claim 2.1. Hence, we conclude that $L(u) \cap L(v) \subseteq U_2$. Now

$$2k = |L(u)| + |L(v)| = |L(u) \cup L(v)| + |L(u) \cap L(v)| \leq |U_1 \cup U_2| + |U_2|,$$

which contradicts (1). ■

Let $Z' = \{x \in V(G) : f(x) \in U_2 \cap L(y)\}$. Note that $L(y) \subseteq U$, since otherwise we extend f by using a color from $L(y) - U$ on y.

Claim 2.3. *The set Z is nonempty. If $z \in Z$, then $d(z) = k$, and $x \in N(z) - \{y\}$ if and only if x is switchable. Also $d(y) = k$ and $f(N(y)) \subseteq L(y)$, and $|N(y) \cap Z'| = |Z'|/2$.*

Proof. Since $L(y) \subseteq U$, we have $2(k - |Z|) = |Z'| \leq n(G) - 1 \leq 2k - 1$. Thus $Z \neq \emptyset$. Choose $z \in Z$.

A vertex of Z' is switchable if and only if the other vertex with its color is not adjacent to y. This implies that if y has s neighbors in Z', then exactly $|Z'| - s$ vertices in Z' are switchable. By Claim 2.2, $N(z)$ contains all these vertices, along with $Z - \{z\}$ (and y, by Claim 2.1). Hence $d(z) \geq |Z'| - s + |Z| = 2k - s - |Z|$.

However, we have assumed that $\Delta(G) \leq k$. From $k \geq d(z) \geq 2k - s - |Z|$ we obtain $|Z| + s \geq k$. With $k \geq d(y) \geq |Z| + s$, this yields $d(y) = k = |Z| + s$ and thus also $d(z) = k$, and z has no non-switchable neighbors other than y. Also, $N(y) \subseteq Z \cup Z'$, so $f(N(y)) \subseteq L(y)$. Finally, $k - |Z| = s$ yields $|Z'| = 2s$, so y is adjacent to exactly half of Z'. ■

Let a color class in Z' be *type i* if it has i switchable vertices. It will be type 2 if the vertices are both outside $N(y)$, type 0 if both are in $N(y)$, and type 1 otherwise. Since exactly half of Z' is in $N(y)$, the numbers of classes of types 2 and 0 are equal. Figure 1 shows the current situation. The switchable vertices are those in the shaded regions. The vertices with colors not in U, indicated by S, are not neighbors of y.

When we switch on a switchable vertex, the sizes of color classes do not change. We can, therefore, apply Claims 1.1–1.3 to the resulting coloring.

Claim 2.4. *If x is a switchable vertex of Z', then $d(x) = k$, and $f(N(x) - \{y\}) \subseteq L(x)$, and $N[x]$ contains all switchable vertices in $Z' \cap N(y)$.*

Proof. Switching on x yields a coloring f' to which we apply Claim 2.3. Since x plays the role of "y" in f', we have $d(x) = k$, and the colors on the neighbors of x other than y (which have not changed) lie in $L(x)$.

Let x' be a switchable vertex in $Z' \cap N(y)$ other than x, and choose $z \in Z$. By Claim 2.2, $x, x' \in N(z)$. By Claim 2.3, z is not adjacent to the other vertex with color $f(x')$. Now consider f', with x in the role of "y." Since $z \in N(x)$, we have $f'(z) \in L(x)$, and hence z serves as a vertex in the new "Z." Since z is adjacent to x' but not to the other vertex with color $f'(x')$, Claim 2.3 implies that $f(x')$ is a type 1 color and x' is switchable with respect to f'. This yields $x' \in N(x)$. ∎

The proof of Theorem 1.1 is now completed by considering three cases.

Case 2.1. *All color classes in Z' are type 1.* This puts all switchable vertices in $N(y)$. By Claim 2.3, the set of switchable vertices is exactly $N(y)$, and there are k of them. By Claim 2.4, these form a copy of K_{k+1} with y (it is a component of G).

Case 2.2. *There are color classes in Z' of types 1 and 2.* Let $\{x, x'\}$ be a color class of type 1 (with $x \in N(y)$), and let $\{w, w'\}$ be a color class of type 2. Choose $z \in Z$. By Claim 2.3 for f, we have $x, w, w' \in N(z)$, but $x' \notin N(z)$.

Switching on x or on w yields colorings f_x and f_w, respectively. Claim 2.4 for f implies that z serves as a vertex of "Z" in both f_x and f_w. Now Claim 2.3 for f_x implies that w and w' are switchable for f_x, and hence their color class is type 2

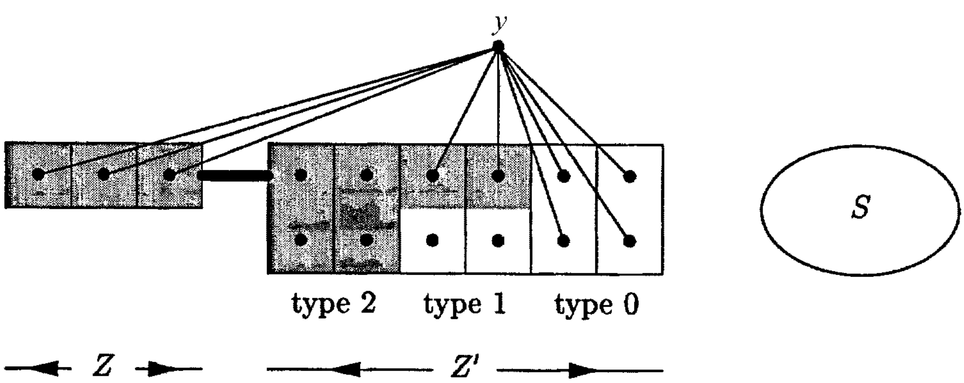

FIGURE 1. A 2-bounded L-coloring of $G-y$.

and $w, w' \notin N(x)$. On the other hand, Claim 2.3 for f_w implies that x is switchable and x' is not for f_w, and hence their color class is type 1 and $x \in N(w)$. This contradiction about the pair $\{x, w\}$ eliminates this case.

Case 2.3. *No color class in Z' is type 1.* If $Z' = \emptyset$, then $Z \cup \{y\}$ induces a copy of K_{k+1}. Therefore, we may assume that Z' contains a type 2 class $\{x, x'\}$. Suppose first that Z has distinct vertices z and z'. By Claim 2.3, $\{x, x'\} \subset N[z] = N[z']$. However, $x, x \notin N(y)$. Switching on z' yields a new coloring f'. In f', both y and z belong to the new "Z." This contradicts Claim 2.3, since $N[y] \neq N[z]$.

By Claim 2.3, $Z \neq \emptyset$. Thus Z consists of one vertex, say z. Let X be the union of $\{z\}$ and all color classes of type 0 in Z'. Let Y be the union of $\{y\}$ and all color classes of type 2 in Z'. We argue that $X \cup Y = V(G)$. Recall that all colors on $N(y)$ appear in $L(y)$ (by Claim 2.3). Hence $N(y) \subseteq Z \cup Z'$. Since $d(y) = k$ (by Claim 2.2) and $|Z| = 1$, the vertex y has $k-1$ neighbors in Z'. Since $N(y) \cap Z'$ consists of the vertices in the type 0 classes, and the number of type 2 classes is the same (by Claim 2.3), we have $|X| = |Y| = k$. Thus $X \cup Y$ has $2k$ vertices. By hypothesis, $|V(G)| \leq 2k$.

We conclude that $X \cup Y = V(G)$. Furthermore, since the neighbors of y in Z' come in pairs, k is odd.

Since $d(z) = k$ and all of $N(z) - \{y\}$ is switchable (by Claim 2.3), we have $N(z) = Y$. Consider $w \in Y - \{y\}$. Switching on w yields a new coloring f' in which the new "Z" is again $\{z\}$, because the sizes of color classes have not changed. Since $N(z)$ has not changed and w now plays the role of "y," Claim 2.3 implies that the vertices of $X - \{z\}$ still form the type 0 classes, and hence $X \subseteq N(w) = X$. Applying this for each $w \in Y - \{y\}$ yields a copy of $K_{k,k}$ with partite sets X and Y. Since $\Delta(G) \leq k$, we conclude that $G = K_{k,k}$.

We have shown that G fails to have a 2-bounded L-coloring only in the exceptional cases. This completes the proof of Theorem 1.1.

3. THE KEY LEMMA

The remaining theorems are proved inductively using the next lemma.

Consider a graph G with a k-uniform list assignment L, and let $S = \{x_1, \ldots, x_k\}$ be a set of k vertices in G. If $G - S$ has an equitable L-coloring f that extends to an L-coloring f' of G by giving the vertices in S different colors, then f' is an equitable L-coloring of G, since the extension augments each color class at most once. We prove a simple sufficient condition for the existence of such an extension.

Lemma 3.1. *Let G be a graph with a k-uniform list assignment L. Let $S = \{x_1, \ldots, x_k\}$, where $\{x_1, \ldots, x_k\}$ are distinct vertices in G. If $G - S$ has an equitable L-coloring, and*

$$|N_G(x_i) - S| \leq k - i \tag{2}$$

for $1 \leq i \leq k$, then G has an equitable L-coloring.

Proof. Let $G_i = G - \{x_{i+1}, \ldots, x_k\}$, so that $G - S = G_0$ and $G = G_k$. Let f_0 be an equitable L-coloring of G_0. For $1 \le i \le k$, extend f_{i-1} to an L-coloring f_i of G_i by giving x_i a color in $L(x_i)$ different from the colors that f_i has used on neighbors of x_i and on the vertices $x_1, \ldots, x_{i-1}$. Condition (2) guarantees that this is possible. By construction, the colors used on S are distinct, and hence f_k is an equitable L-coloring of G. ■

Lemma 3.1 is our main tool in the next two sections.

The technique suggested by Lemma 3.1 enables us to relate equitable k-choosability to equitable k-colorability. The slight discrepancy between the definitions of equitability for colorings and choice functions allows the possibility that an equitably k-choosable graph is not equitably k-colorable.

Nevertheless, for fixed k, suppose that $\mathcal{G}$ is a hereditary family of graphs G for which a set S satisfying (2) exists in $V(G)$ whenever $n(G) \ge k$. For a graph $G \in \mathcal{G}$, let G' be the graph with $k\lceil n(G)/k \rceil$ vertices obtained from G by adding a component that is a complete graph, and let $r = n(G') - n(G)$. Taking $x_{r+1}, \ldots, x_k$ from the set S in $V(G)$ and following them by the r vertices of the added complete graph yields an indexed set satisfying (2) for G'. Therefore, iteratively applying Lemma 3.1 yields that G' is equitably k-choosable. By the comment in the Introduction, G is thus equitably k-colorable. Hence, graphs in a family $\mathcal{G}$ that are shown to be equitably k-choosable by the method of Lemma 3.1 will also be equitably k-colorable.

4. FORESTS AND INTERVAL GRAPHS

In this section, we prove Theorems 1.2 and 1.3 about forests and interval graphs.

Theorem 1.2. *If G is a forest and $k \ge 1 + \Delta(G)/2$, then G is equitably k-choosable. Also, for all D there is a tree with maximum degree at most D that is not equitably $\lceil D/2 \rceil$-choosable.*

Proof. The second statement addresses the sharpness of the first. Given a natural number D, let d be the greatest odd integer bounded by D. Let $k = (d+1)/2 = \lceil D/2 \rceil$. Consider the star $K_{1,d}$; its maximum degree is at most D. Given identical color lists of size k, some color appears only on the center and another on at least $\lceil d/(k-1) \rceil$ leaves. Thus, some color is used at least three times. Since $K_{1,d}$ has $d+1$ vertices, equitable k-choosability requires the usage of each color to be at most $\lceil (d+1)/k \rceil$, which is only 2.

To prove the main statement, we use induction on the number of vertices, n; deleting vertices cannot increase the maximum degree. Let G be a forest with maximum degree at most D. If $n \le k$, then we color all vertices using different colors from their lists. Suppose now that $n > k$ and that every forest with maximum degree at most D and fewer than n vertices is equitably k-choosable.

Let L be a k-uniform list assignment for G. We consider several cases. In each case, we construct a set S consisting of vertices we call $x_1, \ldots, x_k$, and then Lemma 3.1 applies to complete the proof.

If G has no edges, then choose $x_1, \ldots, x_k$ arbitrarily.

Otherwise, let w be a neighbor of a leaf on a longest path in G (a *leaf* is a vertex of degree 1). Note that w has at most one non-leaf neighbor. If $d(w) \geq k$, then let $x_1 = w$, and let $x_2, \ldots, x_k$ be leaf neighbors of w.

If $d(w) = m \leq k - 1$, then let $x_k, x_{k-1}, \ldots, x_{k-m+1}$ be the leaf neighbors of w, and let $x_{k-m} = w$. Choose $x_{k-m-1}, \ldots, x_1$ successively by letting x_i be a vertex of degree at most one in $G - \{x_{i+1}, \ldots, x_k\}$. Such a vertex exists since G is a forest.

Inequality (2) holds for $i = 1$ when $d(w) \geq k$ since $k \geq 1 + r/2$, and in all cases for $i = k$ since x_k has no neighbors outside S. In all remaining instances, x_i has at most one neighbor outside S. Hence (2) holds for every i in all cases. By Lemma 3.1, G is equitably L-colorable. ∎

For sharpness in Theorem 1.2, we used $K_{1,D}$ for odd D and $K_{1,D-1}$ for even D. Among forests with maximum degree *equal* to D, the threshold $k \geq 1 + D/2$ for equitable k-choosability possibly is not sharp when D is even. Using $K_{1,D}$, we know that if D is divisible by 2 but not 6, then the threshold cannot be lowered to $\lceil D/3 \rceil$, and if D is divisible by 6 but not 12, then the threshold cannot be lowered to $\lceil D/4 \rceil$.

To prove Theorem 1.3, we again use induction and Lemma 3.1. Recall that an *interval graph* is a graph whose vertices can be assigned intervals on the real line so that vertices are adjacent if and only if their assigned intervals intersect. Such an assignment is an *interval representation* of the graph. In an interval representation of an interval graph, we use $l(v)$ and $r(v)$ to denote the left and right endpoints, respectively, of the interval $I(v)$ assigned to vertex v. We may assume that no two intervals have a common endpoint. Lemma 4.1 implies the statement, not immediately obvious, that the only regular interval graphs are disjoint unions of cliques.

Lemma 4.1. *Let G be an interval graph with maximum degree D. If $K_{1+D} \not\subseteq G$, then G has an interval representation such that the degree of the vertex whose interval extends farthest left is less than D.*

Proof. Given an interval representation of G, let x be the vertex whose interval has the leftmost right endpoint. For each $y \in N(x)$, the interval $I(y)$ contains $r(x)$. Hence $N[x]$ is a clique, and therefore $d(x) < D$.

Since $r(x)$ is the leftmost right endpoint, extending $I(x)$ to the left does not change the neighborhood of x. Therefore, extending $I(x)$ leftward past all other left endpoints yields the desired representation. ∎

Theorem 1.3. *If G is a connected interval graph and $k \geq \Delta(G)$, then G is equitably k-choosable unless $G = K_{k+1}$.*

Proof. Again we use induction on n, the number of vertices of G, since again deleting vertices cannot increase the maximum degree. If $n \leq k$, then we color the vertices with different colors from their lists. Suppose now that $n > k$ and that the claim holds for interval graphs with fewer than n vertices. Let L be an arbitrary k-uniform list assignment for G. Consider an interval representation of G as guaranteed by Lemma 4.1, and index the vertices so that $l(v_1) < \cdots < l(v_n)$; note that $d(v_1) < \Delta(G)$. Form the set S by letting $x_i = v_{n-k+i}$ for $1 \leq i \leq k$ (these are the k vertices with rightmost left endpoints).

Consider an index i with $1 \leq i \leq k$. If $\{x_1, \ldots, x_{i-1}\} \subseteq N(x_i)$, then all intervals for vertices in $(N(x_i) - S) \cup \{x_1, \ldots, x_{i-1}\}$ contain $l(x_i)$, as does $I(x_i)$. Since $K_{k+1} \not\subseteq G$, this yields $|N(x_i) - S| \leq k - i$.

If $\{x_1, \ldots, x_{i-1}\} \not\subseteq N(x_i)$, then let y be the neighbor of x_i whose interval extends farthest left. We have $N(x_i) - \{x_{i+1}, \ldots, x_k\} \in N(y)$. If $y = v_1$, then $d(y) < \Delta(G)$. If $y \neq v_1$, then y has a neighbor z with $l(z) < l(y)$, since G is connected, and $z \notin N(x_i)$ by the choice of y. In both cases, we obtain $|N(x_i) - \{x_{i+1}, \ldots, x_k\}| < k$. Thus (2) holds, and Lemma 3.1 implies that G is equitably L-colorable. ∎

5. 2-DEGENERATE GRAPHS

Let a *minor vertex* be a vertex of degree at most two. Every 2-degenerate graph has a minor vertex, but we need a bit more. Recall that a *trail* in a simple graph can be specified as a list of vertices (not necessarily distinct) such that any two successive vertices in the list are adjacent in the graph.

Lemma 5.1. *If G is a 2-degenerate graph, then G has at least one of the following:*

(a) *an isolated vertex,*
(b) *two adjacent minor vertices,*
(c) *two minor vertices x, y with a common neighbor z,*
(d) *a path with vertices x, y, z such that x is minor and $d(y) = d(z) = 3$,*
(e) *a trail with vertices x, y, z, w such that x and w are minor and $d(y) = 3$,*
(f) *a trail with vertices x, y, z, w, u such that x and u are minor and $d(y) = d(w) = 3$.*

Proof. Let G be a 2-degenerate graph violating (a), (b), and (c). Let M_1 be the set of minor vertices in G, let M_2 be the set of minor vertices in $G - M_1$, and let M_3 be the set of minor vertices in $G - M_1 - M_2$. Since (a) and (b) fail, $M_2 \neq \emptyset$. Since (c) fails, each vertex v in $G - M_1$ has at least $d_G(v) - 1$ neighbors in $G - M_1$. Since vertices of M_2 become minor only upon deletion of M_1, we conclude that every vertex of M_2 has degree 3 in G and has one neighbor in M_1.

If M_2 is not independent, then (d) holds. If M_2 is independent, then every vertex of M_2 has two neighbors outside $M_1 \cup M_2$. Hence $M_3 \neq \emptyset$. Consider $z \in M_3$; note that z has at least $d_G(z) - 2$ neighbors in $M_1 \cup M_2$.

Since $z \notin M_2$, z has a neighbor y in M_2, and y has a neighbor in M_1. If $d_G(z) = 3$, then (d) holds. If z has a neighbor $w \in M_1$, then (e) holds. Finally, if $d_G(z) \geq 4$ and z has no neighbors in M_1, then z has another neighbor w in M_2, and (f) holds.

Note that the trails in (e) and (f) may be closed. ■

Theorem 1.4. *If G is a 2-degenerate graph and $k \geq \max\{\Delta(G), 5\}$, then G is equitably k-choosable.*

Proof. If there is a counterexample, then let G be one with most edges among counterexamples with fewest vertices. Since $k \geq 5$, $n(G) \geq 6$. Adding an edge cannot produce an equitably k-choosable graph from one that is not. Therefore, addition of any missing edge gives degree $k+1$ to some vertex or destroys 2-degeneracy by forming a subgraph with minimum degree 3.

Let v be a vertex with degree at most 2 in G, and let u be a vertex with degree at most 2 in $G - N[v]$. Since $d_G(u) \leq 4$, adding the edge uv does not give degree $k+1$ to any vertex. Therefore, $G + uv$ has a subgraph with minimum degree 3 that contains uv. Thus $d_G(v) = 2$, and both neighbors of v have degree at least 3 in G. We conclude that G has no vertices of degree at most 1 and that no two vertices of degree 2 are adjacent.

By Lemma 5.1, G now has one of the structures listed there as (c)–(f). Let x, y, z, etc., be the vertices as labeled in these cases of Lemma 5.1. In each case, we construct $x_1, \ldots, x_k$ to be an ordered set S satisfing (2). We always let $x_k = x$ and $x_{k-1} = y$ and $x_2 = z$. We let x_1 be the remaining neighbor of x if it has not already been placed in S (if it has, then x_1 remains unspecified).

We have ensured that $N(x) \subseteq S$ and $|N(y) - S| \leq 1$. If we have specified x_1, then it has a neighbor in S and hence $|N(x_1) - S| \leq k - 1$. Since $x_2 = z$, we seek $|N(z) - S| \leq k - 2$. In Case (c) we have enforced this; in Case (d) it holds since $|N(z) - S| \leq 2$ and $k \geq 4$.

In Case (e) and Case (f), G has a trail via x, y, z, w such that $d(x) = 2$, $d(y) = 3$, and $d(w) \leq 3$ (possibly $x = w$ in Case (e)). Now we ensure that z has two neighbors in S by letting $x_{k-2} = w$ if $w \neq x$. The construction is well-defined since $k \geq 5$. Since $d(w) \leq 3$ and $z \in S$, (2) holds also for $i = k - 2$ in these cases. We fill the remaining unspecified positions in S from highest to lowest indices by choosing at each step a minor vertex in the graph obtained from G by deleting the vertices thus far chosen for S. Such a vertex always exists because G is 2-degenerate. Each such vertex has at most two neighbors outside S, which satisfies (2) for these vertices because they become x_i with $i \leq k - 2$.

This completes the verification of (2) in all cases, and hence Lemma 3.1 always applies to complete the induction step. ■

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