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# Estimating the Minimal Number of Colors in Acyclic $k$-Strong Colorings of Maps on Surfaces

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**Abstract**—A coloring of the vertices of a graph is called *acyclic* if the ends of each edge are colored in distinct colors, and there are no two-colored cycles. Suppose each face of rank $k$, $k \geq 4$, in a map on a surface $S^N$ is replaced by the clique having the same number of vertices. It is proved in [1] that the resulting pseudograph admits an acyclic coloring with the number of colors depending linearly on $N$ and $k$. In the present paper we prove a sharper estimate $55(-Nk)^{4/7}$ for the number of colors provided that $k \geq 1$ and $-N \geq 5^7 k^{4/3}$.

Key words: *graphs on surfaces, acyclic colorings, k-strong colorings.*

## 1. INTRODUCTION

A coloring of a graph embedded in a surface is called a *k-strong coloring* if any two vertices belonging to the boundary of the same face of rank not greater than $k$ have different colors.

A coloring of graph vertices is called *acyclic* if it is regular, i.e., the ends of each edge are colored in distinct colors, and there are no two-colored cycles. Acyclic colorings have a number of applications to other coloring problems (see [2–6]).

A pseudograph embedded in a surface without faces of rank 1 and 2 is called a *map.* Note that a loop generates an edge with the same colors of the ends, and a double edge generates a two-colored cycle. We consider a coloring of vertices of a map on an arbitrary surface acyclic if the ends of each edge $e$ which is not a loop have distinct colors and there are no two-colored cycles of length greater than 2. In [7] Alon, Mohar, and Sanders established the following estimate.

**Theorem 1.** *Every graph which admits an acyclic embedding in a surface $S^N$ is acyclically $50(2-N)^{4/7}$-colorable, and this estimate is sharp up to a factor of the form $c(\log(3-N))^{1/7}$.*

The proof of this statement was based on the following result due to Alon, MacDiarmid, and Reed [8].

**Theorem 2** [8]**.** *Each graph with degree of vertices not exceeding $\Delta$ admits an acyclic coloring in $50\Delta^{4/3}$ colors.*

In [1], colorings which are both acyclic and $k$-cyclic were discussed. Namely, we studied an acyclic coloring of the pseudograph $G^{(k)}$ obtained by replacing each face of rank not greater than $k$ of a map $G$ with the clique having the same set of vertices. This means that each face of rank not greater than $k$ acquires all "invisible diagonals." For $k = 3$ this coloring coincides with the acyclic coloring.

The main result of [1] is as follows.

**Theorem 3.** *Each map on a surface $S^N$ admits an acyclic $k$-strong coloring in $c_N k + d_N$ colors for all $k \geq 4$ and $N \leq 2$, where $c_N = \max\{999, 117 - 471N\}$ and $d_N = 39 - 156N$.*

Denote by $F(k, N)$ the minimal number of colors sufficient for an acyclic $k$-strong coloring of any map on the surface $S^N$. By Theorem 3, the function $F(k, N)$ grows in $k$ and in $-N$ not faster than linearly. It is easy to show that $F(k, N)$ grows in $k$ not slower than linearly. But the estimate with respect to $-N$ can be improved. The main goal of the present paper is the proof of the following fact.

**Theorem 4.** *Suppose that $k \geq 4$ and*

$$-N \geq 5^7 k^{4/3}. \tag{1}$$

*Then each map on the surface $S^N$ admits an acyclic $k$-strong coloring in $54(-Nk)^{4/7}$ colors.*

Theorem 1 implies that, for a given $k$, if this estimate can be sharpened, then only by a multiplicative factor not greater than $c_k(\log(3 - N))^{1/7}$.

The proof of Theorem 4 almost repeats that of Theorem 3 in [1] word for word, and at the end we apply Theorem 2. In order to avoid this long repetition we shall take the paper [1] for granted. In the next section, we begin by suggesting (small) changes in the proof of Theorem 3 and then describe the end of the proof in more detail. That is why we suggest that the reader should have the paper [1] at hand when reading the next section.

## 2. PROOF OF THEOREM 4

We repeat the proof of Theorem 3 in [1] up to Eq. (4) and skip Eq. (5). Then we define $\cosh(f)$ as in [1] for each face $f$ and set $\cosh(v) = d(v) - 4 - f_{3^*}(v)/3$ for each vertex $v$. Then, instead of Eq. (6), we have

$$\sum_{v \in V_{3+}} \cosh(v) + \sum_{f \in F} \cosh(f) < -4N + 1. \tag{2}$$

Then, forgetting Lemma 4, we note that with the new definition of $\cosh(v)$ Lemma 5 is valid already for $d(v) \geq 6$. Now we set $\varepsilon = \xi = 0$ and redistribute the contributions exactly in the same way as in [1]. Lemmas 6–10 are proved in the same way as in [1], taking into account the fact that (according to (1)) we have $54(-Nk)^{4/7} \geq 54 \cdot 5^4 k$ colors at our disposal. These lemmas imply, in particular, that $\cosh^*(x) \geq 0$ for any face or vertex $x$. Moreover, to the statement of Lemma 9 we can add that $\cosh^*(v) \geq d(v)/4$ for an arbitrary vertex $v$ with $d(v) \geq 48$.

Let $L = \frac{17}{30}(-N)^{3/7} k^{-4/7}$, and let $B$ be the set of vertices of valency greater than $L$. Let us show that

$$|B| \leq \lfloor 30(-Nk)^{4/7} \rfloor. \tag{3}$$

Indeed, if $|B| > 30(-Nk)^{4/7}$, then, by Eq. (2) and the new version of Lemma 9, using the inequality

$$L = \frac{17}{30}(-N)^{3/7} k^{-4/7} \geq \frac{17}{30}(5^7 k^{4/3})^{3/7} k^{-4/7} > 48,$$

we obtain

$$\begin{aligned} -4N + 1 &> \sum_{v \in V_{3+}} \cosh(v) \geq \sum_{v \in B} \cosh^*(v) \geq \sum_{v \in B} \frac{d(v)}{4} \geq |B| \frac{L}{4} \\ &> 30(-N)^{4/7} \frac{17}{4 \cdot 30} (-N)^{3/7} \geq \frac{17}{4}(-N). \end{aligned}$$

This is impossible for $-N \geq 4$, while Eq. (1) implies that $-N \geq 5^7$. The contradiction thus obtained proves Eq. (3).

By the definition of the set $B$, the maximal valency in the graph $G^{(k)} - B$ is not greater than $Lk$. Therefore, by Theorem 2, the vertices of $G^{(k)} - B$ admit an acyclic coloring in

$$50\left(\frac{17}{30}(-Nk)^{3/7}\right)^{4/3} \le 24(-Nk)^{4/7}$$

colors. Let us color each vertex from $B$ into its own color. Then not more than $|B| + 24(-Nk)^{4/7}$ colors will be used. By Eq. (3), this sum is not greater than $54(-Nk)^{4/7}$. The theorem is proved.

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